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THE
REORGANIZATION OF MATHEMATICS
IN
SECONDARY EDUCATION

A Report by

THE NATIONAL COMMITTEE
ON
MATHEMATICAL REQUIREMENTS

under the auspices of

THE MATHEMATICAL ASSOCIATION
OF AMERICA, INC.

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1923

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The National Committee on Mathematical Requirements

(Under the auspices of The Mathematical Association of America, Inc.)

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[1] Prof. Moore took the place vacated in 1918 by the resignation of Oswald Veblen, Princeton University.

[2] Mr. Downey took the place vacated in 1919 by the resignation of G. W. Evans, Charlestown High School, Boston, Mass.

[3] Until July, 1921, of the Crane Technical High School, Chicago, Ill.

PREFACE

The National Committee on Mathematical Requirements was organized in the late summer of 1916 under the auspices of The Mathematical Association of America for the purpose of giving national expression to the movement for reform in the teaching of mathematics, which had gained considerable headway in various parts of the country, but which lacked the power that coördination and united effort alone could give.

The original nucleus of the committee, appointed by E. R. Hedrick, then president of the association, consisted of the following: A. R. Crathorne, University of Illinois; E. H. Moore, University of Chicago; D. E. Smith, Columbia University; H. W. Tyler, Massachusetts Institute of Technology; Oswald Veblen, Princeton University; and J. W. Young, Dartmouth College, chairman. This committee was instructed to add to its membership so as to secure adequate representation of secondary school interests, and then to undertake a comprehensive study of the whole problem concerned with the improvement of mathematical education and to cover the field of secondary and collegiate mathematics.

This group held its first meeting in September, 1916, at Cambridge, Mass. At that meeting it was decided to ask each of the three large associations of secondary school teachers of mathematics (The Association of Teachers of Mathematics in New England, The Association of Teachers of Mathematics in the Middle States and Maryland, and The Central Association of Science and Mathematics Teachers) to appoint an official representative on the committee. At this time also a general plan for the work of the committee was outlined and agreed upon.

In response to the request above referred to the following were appointed by the respective associations: Miss Vevia Blair, Horace Mann School, New York City, representing the Middle States and Maryland Association; G. W. Evans, Charlestown High School, Boston, Mass., representing the New England Association;[1] and J. A. Foberg, Crane Technical High School, Chicago, Ill., representing the Central Association.

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[1] Mr. Evans resigned in the summer of 1919, owing to an extended trip abroad; his place was taken by W. F. Downey, English High School, Boston, Mass.

At later dates the following members were appointed: A. C. Olney, Commissioner of Secondary Education, Sacramento, Calif.; Raleigh Schorling, The Lincoln School, New York City; P. H. Underwood, Ball High School, Galveston, Tex.; and Miss Eula A. Weeks, Cleveland High School, St. Louis, Mo.

From the very beginning of its deliberations the committee felt that the work assigned to it could not be done effectively without adequate financial support. The wide geographical distribution of its membership made a full attendance at meetings of the committee difficult if not impossible without financial resources sufficient to defray the traveling expenses of members, the expenses of clerical assistance, etc. Above all, it was felt that, in order to give to the ultimate recommendations of the committee the authority and effectiveness which they should have, it was necessary to arouse the interest and secure the active coöperation of teachers, administrators, and organizations throughout the country —that the work of the committee should represent a coöperative effort on a truly national scale.

For over two years, owing in large part to the World War, attempts to secure adequate financial support proved unsuccessful. Inevitably also the war interfered with the committee's work. Several members were engaged in war work[2] and the others were carrying extra burdens on account of such work undertaken by their colleagues.

In the spring of 1919, however, and again in 1920, the committee was fortunate in securing generous appropriations from the General Education Board of New York City for the prosecution of its work.[3]

This made it possible greatly to extend the committee's activities. The work was planned on a large scale for the purpose of organizing a truly nation-wide discussion of the problems facing the committee, and J. W. Young and J. A. Foberg were selected to devote their whole time to the work of the committee. Suitable office space was secured and adequate stenographic and clerical help was employed.

The results of the committee's work and deliberations are presented in the following report. A word as to the methods employed may, however, be of interest at this point. The committee attempted to establish working contact with all organizations of teachers and others interested in its problems and to secure their active assistance. Nearly 100 such organizations have taken part in this work. A list of these organizations will be found in the appendix to this report (p. 632). Provisional reports on various phases of the problem were submitted to these coöperating organizations in advance of publication, and criticisms, comments, and suggestions for improvement were invited from individuals and special coöperating committees. The reports previously published for the committee by the United States Bureau of Education[4] and in The Mathematics Teacher[5] and designated as "preliminary" are the result of this kind of coöperation. The value of such assistance can hardly be overestimated and the committee desires to express to all individuals, organizations, and educational journals that have taken part its hearty appreciation and thanks. The committee believes it is safe to say, in view of the methods used in formulating them, that the recommendations of this final report have the approval of the great majority of progressive teachers throughout the country.

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[2] Professor Veblen resigned in 1917 on account of the pressure of his war duties. His place was taken on the committee by Professor C. N. Moore, University of Cincinnati.

[3] Again in November, 1921, the General Education Board made appropriations to cover the expense of publishing and distributing the present report and to enable the committee to carry on certain phases of its work during the years 1922-1923.

No attempt has been made in this report to trace the origin and history of the various proposals and movements for reform nor to give credit either to individuals or organizations for initiating them. A convenient starting point for the history of the modern movement in this country may be found in E. H. Moore's presidential address before the American Mathematical Society in 1902.[6] But the movement here is only one manifestation of a movement that is world-wide and in which very many individuals and organizations have played a prominent part. The student interested in this phase of the subject is referred to the extensive publications of the International Commission on the Teaching of Mathematics, to the Bibliography of the Teaching of Mathematics, 1900-1912, by D. E. Smith and C. Goldziher (U. S. Bureau of Education, Bulletin, 1912, No. 29) and to the bibliography (since 1912) to be found in this report (Chap. XVI, p. 539*f*).

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[4] The Reorganization of the First Courses in Secondary School Mathematics, U. S. Bureau of Education, Secondary School Circular, No. 5, February, 1920. 11pp. Junior High School Mathematics, U. S. Bureau of Education, Secondary School Circular, No. 6, July, 1920. 10 pp. The Function Concept in Secondary School Mathematics. Secondary School Circular, No. 8, June, 1921. 10 pp.

[5] Terms and Symbols in Elementary Mathematics. The Mathematics Teacher, Vol. 14 (March, 1921), pp. 107-118. Elective Courses in Mathematics for Secondary Schools. The Mathematics Teacher, vol. 14 (April, 1921), pp. 161-170. College Entrance Requirements in Mathematics. The Mathematics Teacher, vol. 14 (May, 1921), pp. 224-245.

[6] E. H. Moore: On the Foundations of Mathematics, Bulletin of the American Mathematical Society, vol. 9 (1902-3), p. 402; Science, vol. 17, p. 401.

The National Committee expects to maintain its office, with a certain amount of clerical help, during the year 1922-23 and perhaps for a longer period. It is hoped that in this way it may continue to serve as a clearing house for all activities looking to the improvement of the teaching of mathematics in this country, and to assist in bringing about the effective adoption in practice of the recommendations made in the following report, with such modifications of them as continued study and experimentation may show to be desirable.

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PART I

GENERAL PRINCIPLES AND RECOMMENDATIONS

PART I. GENERAL PRINCIPLES AND RECOMMENDATIONS

CHAPTER I

A Brief Outline of the Report

The present chapter gives a brief general outline of the contents of this report for the purpose of orienting the reader and making it possible for him to gain quickly an understanding of its scope and the problems which it considers.

The valid aims and purposes of instruction in mathematics are considered in Chapter II. A formulation of such aims and a statement of general principles governing the committee's work is necessary as a basis for the later specific recommendations. Here will be found the reasons for including mathematics in the course of study for all secondary school pupils.

To the end that all pupils in the period of secondary education shall gain early a broad view of the whole field of elementary mathematics, and, in particular, in order to insure contact with this important element in secondary education on the part of the very large number of pupils who, for one reason or another, drop out of school by the end of the ninth year, the National Committee recommends emphatically that the course of study in mathematics during the seventh, eighth, and ninth years contain the fundamental notions of arithmetic, of algebra, of intuitive geometry, of numerical trigonometry, and at least an introduction to demonstrative geometry, and that this body of material be required of all secondary school pupils.

A detailed account of this material is given in Chapter III. Careful study of the later years of our elementary schools, and comparison with European schools, have shown the vital need of reorganization of mathematical instruction, especially in the seventh and eighth years. The very strong tendency now evident to consider elementary education as ceasing at the end of the sixth school year, and to consider the years from the seventh to the twelfth inclusive as comprising years of secondary education, gives impetus to the movement for reform of the teaching of mathematics at this stage.

While Chapter III is devoted to a consideration of the body of materials of instruction in mathematics that is regarded as of sufficient importance to form part of the course of study for all secondary school pupils, Chapter IV is devoted to consideration of the types of material that properly enter into courses of study for pupils who continue their study of mathematics beyond the minimum regarded as essential for all pupils. Here will be found recommendations concerning the traditional subject matter of the tenth, eleventh, and twelfth school years, and also certain material that heretofore has been looked upon in this country as belonging rather to college courses of study; as, for instance, the elementary ideas and processes of the calculus.

Chapter V is devoted to a study of the types of secondary school instruction in mathematics that may be looked upon as furnishing the best preparation for successful work in college. This study leads to the conclusion that there is no conflict between the needs of those pupils who ultimately go to college and those who do not. Certain very definite recommendations are made as to changes that appear desirable in the statement of college entrance requirements and in the type of college entrance examination.

Chapter VI contains lists of propositions and constructions in plane and in solid geometry. The propositions are classified in such a way as to separate from others of less importance those which are regarded as so fundamental that they should form the common minimum of any standard course in the subject.

The statement previously made in preliminary reports of the National Committee and repeated in Chapter II, that the function concept should serve as a unifying element running throughout the instruction in the mathematics of the secondary school, has brought many requests for a more precise definition of the rôle of the function concept in secondary school mathematics. Chapter VII is intended to meet this demand.

Recommendations as to the adoption and use of terms and symbols in elementary mathematics are contained in Chapter VIII. It is intended to present a norm embodying agreement as to best current practice.

The remaining chapters (Part II) give for the most part the results of special investigations conducted for the National Committee. The contents of these chapters are indicated sufficiently in the general table of contents and in the tables of contents preceding many of the chapters in question.

CHAPTER II

Aims of Mathematical Instruction—General Principles

I. Introduction

A discussion of mathematical education, and of ways and means of enhancing its value, must be approached first of all on the basis of a precise and comprehensive formulation of the valid aims and purposes of such education.[1] Only on such a basis can we approach intelligently the problems relating to the selection and organization of material, the methods of teaching and the point of view which should govern the instruction, and the qualifications and training of the teachers who impart it. Such aims and purposes of the teaching of mathematics, moreover, must be sought in the nature of the subject, the rôle it plays in the practical, intellectual, and spiritual life of the world, and in the interests and capacities of the students.

Before proceeding with the formulation of these aims, however, we may properly limit to some extent the field of our enquiry. We are concerned primarily with the period of secondary education—comprising, in the modern junior and senior high schools, the period beginning with the seventh and ending with the twelfth school year, and concerning itself with pupils ranging in age normally from 12 to 18 years. References to the mathematics of the grades below the seventh (mainly arithmetic) and beyond the senior high school will be only incidental.

Furthermore, we are primarily concerned at this point with what may be described as "general" aims, that is to say aims which are valid for large sections of the school population and which may properly be thought of as contributing to a general education as distinguished from the specific needs of vocational, technical, or professional education.

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[1] Reference may here be made to the formulation of the principal aims in education to be found in the Cardinal Principles of Secondary Education, published by the U. S. Bureau of Education as Bulletin No. 35, 1918. The main objectives of education are there stated to be: 1. Health; 2. Command of fundamental processes; 3. Worthy home membership; 4. Vocation; 5. Citizenship; 6. Worthy use of leisure; 7. Ethical character. These objectives are held to apply to all education—elementary, secondary, and higher—and all subjects of instruction are to contribute to their achievement.

II. The Aims of Mathematical Instruction

With these limitations in mind we may now approach the problem of formulating the more important aims that the teaching of mathematics should serve. It has been customary to distinguish three classes of aims: (1) Practical or utilitarian, (2) disciplinary, (3) cultural; and such a classification is indeed a convenient one. It should be kept clearly in mind, however, that the three classes mentioned are not mutually exclusive and that convenience of discussion rather than logical necessity often assigns a given aim to one or the other of these classes. Indeed, any truly disciplinary aim is practical and, in a broad sense, the same is true of cultural aims.

Practical aims.—By a practical or utilitarian aim, in the narrower sense, we mean then the immediate or direct usefulness in life of a fact, method, or process in mathematics.

1. The immediate and undisputed utility of the *fundamental processes of arithmetic* in the life of every individual demands our first attention. The first instruction in these processes, it is true, falls outside the period of instruction which we are considering. By the end of the sixth grade the child should be able to carry out the four fundamental operations with integers and with common and decimal fractions accurately and with a fair degree of speed. This goal can be reached in all schools—as it is being reached in many—if the work is done under properly qualified teachers and if drill is confined to the simpler cases which alone are of importance in the practical life of the great majority. (See more specifically, Chapter III, p. 21.) Accuracy and facility in numerical computation are of such vital importance, however, to every individual that effective drill in this subject should be continued throughout the secondary school period, not in general as a separate topic, but in connection with the numerical problems arising in other work. In this numerical work, besides accuracy and speed, the following aims are of the greatest importance:

(*a*) A progressive increase in the pupil's understanding of the nature of the fundamental operations and power to apply them in new situations. The fundamental laws of algebra are a potent influence in this direction. (See 3, below.)

(*b*) Exercise of common sense and judgment in computing from approximate data, familiarity with the effect of small errors in measurements, the determination of the number of figures to be used in computing and to be retained in the result, and the like.

(*c*) The development of self-reliance in the handling of numerical problems through the consistent use of checks on all numerical work.

2. ~~Of almost~~ equal importance to every educated person is *an understanding of the language of algebra* and the ability to use this language intelligently and readily in the expression of such simple quantitative relations as occur in every-day life and in the normal reading of the educated person.

Appreciation of the significance of formulas and ability to work out simple problems by setting up and solving the necessary equations must nowadays be included among the minimum requirements of any program of universal education.

3. The development of the ability to understand and to use such elementary algebraic methods involves a study of the *fundamental laws of algebra* and at least a certain minimum of drill in algebraic technique, which, when properly taught, will furnish the foundation for an understanding of the significance of the processes of arithmetic already referred to. The essence of algebra as distinguished from arithmetic lies in the fact that algebra concerns itself with the operations upon numbers *in general*, while arithmetic confines itself to operations on *particular* numbers.

4. The ability to understand and interpret correctly *graphic representations* of various kinds, such as nowadays abound in popular discussions of current scientific, social, industrial, and political problems, will also be recognized as one of the necessary aims in the education of every individual. This applies to the representation of statistical data which are becoming increasingly important in the consideration of our daily problems, as well as to the representation and understanding of various sorts of dependence of one variable quantity upon another.

5. Finally, among the practical aims to be served by the study of mathematics should be listed familiarity with the *geometric forms* common in nature, industry, and life; the elementary properties and relations of these forms, including their *mensuration;* the development of *space-perception;* and the exercise of *spatial imagination.*

This involves acquaintance with such fundamental ideas as congruence and similarity and with such fundamental facts as those concerning the sum of the angles of a triangle, the pythagorean proposition, and the areas and volumes of the common geometric forms.

Among directly practical aims should also be included the acquisition of the ideas and concepts in terms of which the quantitative thinking of the world is done, and of ability to think clearly in terms of those concepts. It seems more convenient, however, to discuss this and, in a broad sense, the same is true of cultural aims.

Disciplinary aims.—We would include here those aims which relate to mental training, as distinguished from the acquisition of certain specific skills discussed in the preceding section. Such training involves the development of certain more or less general characteristics and the formation of certain mental habits which, besides being directly applicable in the setting in which they are developed or formed, are expected to operate also in more or less closely related fields—that is, to "transfer" to other situations.

The subject of the transfer of training has for a number of years been a very controversial one. Only recently has there been any evidence of agreement among the body of educational psychologists. We need not at this point go into detail as to the present status of disciplinary values since this forms the subject of a separate chapter (Chap. IX; see also Chap. X). It is sufficient for our present purpose to call attention to the fact that most psychologists have abandoned two extreme positions as to transfer of training. The first asserted that a pupil trained to reason well in geometry would thereby be trained to reason equally well in any other subject; the second denied the possibility of any transfer and hence the possibility of any general mental training. That the effects of training do transfer from one field of learning to another is now, however, recognized. The amount of transfer in any given case depends upon a number of conditions. If these conditions are favorable, there may be considerable transfer, but in any case the amount of transfer is difficult to measure. Training in connection with certain attitudes, ideals, and ideas is now almost universally admitted by psychologists to have general value. It may, therefore, be said that, with proper restrictions, general mental discipline is a valid aim in education.

The aims which we are discussing are so important in the restricted domain of quantitative and spatial (i. e., mathematical or partly mathematical) thinking which every educated individual is called upon to perform that we do not need for the sake of our argument to raise the question as to the extent of transfer to less mathematical situations.

In formulating the disciplinary aims of the study of mathematics the following should be mentioned:

(1) The acquisition, in precise form, of those *ideas or concepts in terms of which the quantitative thinking of the world is done.* Among these ideas and concepts may be mentioned ratio and measurement (lengths, areas, volumes, weights, velocities, and rates in general, etc.), proportionality and similarity, positive and negative numbers, and the dependence of one quantity upon another.

(2) The development of *ability to think clearly in terms of such ideas and concepts.* This ability involves training in—

(*a*) Analysis of a complex situation into simpler parts. This includes the recognition of essential factors and the rejection of the irrelevant.

(*b*) The recognition of logical relations between interdependent factors and the understanding and, if possible, the expression of such relations in precise form.

(*c*) Generalization; that is, the discovery and formulation of a general law and an understanding of its properties and applications.

(3) The acquisition of *mental habits and attitudes* which will make the above training effective in the life of the individual. Among such habitual reactions are the following: a seeking for relations and their precise expression; an attitude of enquiry; a desire to understand, to get to the bottom of a situation; concentration and persistence; a love for precision, accuracy, thoroughness, and clearness, and a distaste for vagueness and incompleteness; a desire for orderly and logical organization as an aid to understanding and memory.

(4) Many of these disciplinary aims are included in the broad sense of *the idea of relationship or dependence*—in what the mathematician in his technical vocabulary refers to as a "function" of one or more variables. Training in "functional thinking," that is thinking in terms of and about relationships, is one of the most fundamental disciplinary aims of the teaching of mathematics.

Cultural aims.—By cultural aims we mean those somewhat less tangible but none the less real and important intellectual, ethical, esthetic or spiritual aims that are involved in the development of appreciation and insight and the formation of ideals of perfection. As will be at once apparent the realization of some of these aims must await the later stages of instruction, but some of them may and should operate at the very beginning.

More specifically we may mention the development or acquisition of—

(1) *Appreciation of beauty* in the geometrical forms of nature, art, and industry.

(2) *Ideals of perfection* as to logical structure, precision of statement and of thought, logical reasoning (as exemplified in the geometric demonstration), discrimination between the true and the false, etc.

(3) *Appreciation of the power of mathematics*—of what Byron expressively called "the power of thought, the magic of the mind"[2]—and the rôle that mathematics and abstract thinking, in general, have played in the development of civilization; in particular in science, in industry, and in philosophy. In this connection mention should be made of the religious effect, in the broad sense, which the study of the infinite and of the permanence of laws in mathematics tends to establish.[3]

III. The Point of View Governing Instruction

The practical aims enumerated above, in spite of their vital importance, may without danger be given a secondary position in seeking to formulate the general point of view which should govern the teacher, provided only that they receive due recognition in the selection of material and that the necessary minimum of technical drill is insisted upon.

The primary purposes of the teaching of mathematics should be to develop those powers of understanding and of analyzing relations of quantity and of space which are necessary to an insight into and control

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[2] D. E. Smith: Mathematics in the Training for Citizenship, Teachers College Record, vol. 18 (May, 1917), p. 6.

[3] For an elaboration of the ideas here presented in the barest outline, the reader is referred to the article by D. E. Smith already mentioned and to his presidential address before the Mathematical Association of America. "Religio Matematici", American Mathematical Monthly, vol. 28 (Oct., 1921), pp. 339-349, also published in The Mathematics Teacher, vol. 14 (Dec., 1921), pp. 413-426.

over our environment and to an appreciation of the progress of civilization in its various aspects, and to develop those habits of thought and of action which will make these powers effective in the life of the individual.

All topics, processes, and drill in technique which do not directly contribute to the development of the powers mentioned should be eliminated from the curriculum. It is recognized that in the earlier periods of instruction the strictly logical organization of subject-matter[4] is of less importance than the acquisition, on the part of the pupil, of experience as to facts and methods of attack on significant problems, of the power to see relations, and of training in accurate thinking in terms of such relations. Care must be taken, however, through the dominance of the course by certain general ideas that it does not become a collection of isolated and unrelated details.

Continued emphasis throughout the course must be placed on the development of ability to grasp and to utilize ideas, processes, and principles in the solution of concrete problems rather than on the acquisition of mere facility or skill in manipulation. The excessive emphasis now commonly placed on manipulation is one of the main obstacles to intelligent progress. On the side of algebra, the ability to understand its language and to use it intelligently, the ability to analyze a problem, to formulate it mathematically, and to interpret the result must be dominant aims. *Drill in algebraic manipulation should be limited to those processes and to the degree of complexity required for a thorough understanding of principles and for probable applications either in common life or in subsequent courses which a substantial proportion of the pupils will take.* It must be conceived throughout as a means to an end, not as an end in itself. Within these limits, skill in algebraic manipulation is important, and drill in this subject should be extended far enough to enable students to carry out the essential processes accurately and expeditiously.

On the side of geometry the formal demonstrative work should be preceded by a reasonable amount of informal work of an intuitive, experimental, and constructive character. Such work is of great value in itself; it is needed also to provide the necessary familiarity with geometric ideas, forms, and relations, on the basis of which alone intelligent appreciation of formal demonstrative work is possible.

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[4] "The logical from the standpoint of subject matter represents the goal, the last term of training, not the point of departure." Dewey, "How We Think," p. 62.

The one great idea which is best adapted to unify the course is that of the *functional relation.* The concept of a variable and of the dependence of one variable upon another is of fundamental importance to everyone. It is true that the general and abstract form of these concepts can become significant to the pupil only as a result of very considerable mathematical experience and training. There is nothing in either concept, however, which prevents the presentation of specific concrete examples and illustrations of dependence even in the early parts of the course. Means to this end will be found in connection with the tabulation of data and the study of the formula and of the graph and of their uses.

The primary and underlying principle of the course should be the idea of relationship between variables, including the methods of determining and expressing such relationship. The teacher should have this idea constantly in mind, and the pupil's advancement should be consciously directed along the lines which will present first one and then another of the ideas upon which finally the formation of the general concept of functionality depends. (For a more detailed discussion of these ideas see Chapter VII).

The general ideas which appear more explicitly in the course and under the dominance of one or another of which all topics should be brought are: (1) The formula, (2) graphic representation, (3) the equation, (4) measurement and computation, (5) congruence and similarity, (6) demonstration. These are considered in more detail in a later section of the report (Chaps. III and IV).

IV. The Organization of Subject Matter

"General" courses.—We have already called attention to the fact that, in the earlier periods of instruction especially, logical principles of organization are of less importance than psychological and pedagogical principles. In recent years there has developed among many progressive teachers a very significant movement away from the older rigid division into "subjects" such as arithmetic, algebra, and geometry, each of which shall be "completed" before another is begun, and toward a rational breaking down of the barriers separating these subjects, in the interest of an organization of subject-matter that will offer a psychologically and pedagogically more effective approach to the study of mathematics.

There has thus developed the movement toward what are variously called "composite," "correlated," "unified," or "general" courses. The advocates of this new method of organization base their claims on the obvious and important interrelations between arithmetic, algebra, and geometry (mainly intuitive), which the student must grasp before he can gain any real insight into mathematical methods and which are inevitably obscured by a strict adherence to the conception of separate "subjects." The movement has gained considerable new impetus by the growth of the junior high school, and there can be little question that the results already achieved by those who are experimenting with the new methods of organization warrant the abandonment of the extreme "water-tight compartment" method of presentation.

The newer method of organization enables the pupil to gain a broad view of the whole field of elementary mathematics early in his high-school course. In view of the very large number of pupils who drop out of school at the end of the eighth or the ninth school year or who for other reasons then cease their study of mathematics, this fact offers a weighty advantage over the older type of organization under which the pupils studied algebra alone during the ninth school year, to the complete exclusion of all contact with geometry.

It should be noted, however, that the specific recommendations as to content given in the next two chapters do not necessarily imply the adoption of a different type of organization of the materials of instruction. A large number of high schools will for some time continue to find it desirable to organize their courses of study in mathematics by subjects—algebra, plane geometry, etc. Such schools are urged to adopt the recommendations made with reference to the content of the separate subjects. These, in the main, constitute an essential simplification as compared with present practice. The economy of time that will result in courses in ninth-year algebra, for instance, will permit of the introduction of the newer type of material, including intuitive geometry and numerical trigonometry, and thus the way will be prepared for the gradual adoption in larger measure of the recommendations of this report.

At the present time it is not possible to designate any particular order of topics or any organization of the materials of instruction as being the best or as calculated most effectively to realize the aims and purposes here set forth. More extensive and careful experimental work must be done by teachers and administrators before any such designation can be made that shall avoid undesirable extremes and that shall bear the stamp of general approval. This experimental work will prove successful in proportion to the skill and insight exercised in adapting the aims and purposes of instruction to the interests and capacities of the pupils. One of the greatest weaknesses of the traditional courses is the fact that both the interests and the capacities of pupils have received insufficient consideration and study. For a detailed account of courses in mathematics at a number of the most successful experimental schools the reader is referred to Chapter XII of this report.

Required courses.—The National Committee believes that the material described in the next chapter should be required of all pupils and that under favorable conditions this minimum of work can be completed by the end of the ninth school year. In the junior high school, comprising grades seven, eight, and nine, the course for these three years should be planned as a unit *with the purpose of giving each pupil the most valuable mathematical training he is capable of receiving in those years, with little reference to courses which he may or may not take in succeeding years.* In particular, college entrance requirements should, during these years, receive no specific consideration. Fortunately there appears to be no conflict of interest during this period between those pupils who ultimately go to college and those who do not; a course planned in accordance with the principle just enunciated will form a desirable foundation for college preparation. (See Chapter V; also, the experience of the schools described in Chapter XII.)

Similarly, in case of the at present more prevalent 8-4 school organization, the mathematical material of the seventh and eighth grades should be selected and organized as a unit with the same purpose; the same applies to the work of the first year (ninth grade) of the standard four-year high school, and to later years in which mathematics may be a required subject.

In the case of some elective courses the principle needs to be modified so as to meet whatever specific vocational or technical purposes the courses may have. (See Chap. IV.)

The movement toward correlation of the work in mathematics with other courses in the curriculum, notably those in science, is as yet in its infancy. The results of such efforts will be watched with the keenest interest.

The junior high-school movement.—Reference has several times been made to the junior high school. The National Committee adopted the following resolution on April 24, 1920:

> The National Committee approves the junior high school form of organization, and urges its general adoption in the conviction that it will secure greater efficiency in the teaching of mathematics.

The committee on the reorganization of secondary education, appointed by the National Education Association, in its pamphlet on the "Cardinal Principles of Secondary School Education," issued in 1918 by the Bureau of Education, advocates an organization of the school system whereby the first six years shall be devoted to elementary education and the following six years to secondary education to be divided into two periods which may be designated as junior and senior periods.

To those interested in the study of the questions relating to the history and present status of the junior high school movement, the following books are recommended: Principles of Secondary Education, by Inglis (Houghton Mifflin & Co., 1918); The Junior High School, The Fifteenth Yearbook (Pt. III) of the National Society for the Study of Education (Public School Publishing Co., 1919); The Junior High School, by Bennett (Warwick & York, 1919); The Junior High School, by Briggs (Houghton Mifflin & Co., 1920); and The Junior High School, by Koos (Harcourt, Brace & Howe, 1920).

V. The Training of Teachers

While the greater part of this report concerns itself with the content of courses in mathematics, their organization and the point of view which should govern the instruction, and investigations relating thereto, the National Committee must emphasize strongly its conviction that even more fundamental is the problem of the teacher—his qualifications and training, his personality, skill, and enthusiasm.

The greater part of the failure of mathematics is due to poor teaching. Good teachers have in the past succeeded, and will continue to succeed, in achieving highly satisfactory results with the traditional material; poor teachers will not succeed even with the newer and better material.

The United States is far behind Europe in the scientific and professional training required of its secondary school teachers (see Chap. XIV). The equivalent of two or three years of graduate and professional training in addition to a general college course is the normal requirement for secondary school teachers in most European countries. Moreover, the recognized position of the teacher in the community must be such as to attract men and women of the highest ability into the profession. This means not only higher salaries but smaller classes and more leisure for continued study and professional advancement. It will doubtless require a considerable time before the public can be educated to realize the wisdom of taxing itself sufficiently to bring about the desired result. But if this ideal is continually advanced and supported by sound argument there is every reason to hope that in time the goal may be reached.

In the meantime everything possible should be done to improve the present situation. One of the most vicious and widespread practices consists in assigning a class in mathematics to a teacher who has had no special training in the subject and whose interests lie elsewhere because in the construction of the time schedule he or she happens to have a vacant period at the time. This is done on the principle, apparently, that "anybody can teach mathematics" by simply following a textbook and devoting 90 per cent of the time to drill in algebraic manipulation or to the recitation of the memorized demonstration of theorems in geometry.

It will be apparent from the study of this report that a successful teacher of mathematics must not only be highly trained in his subject and have a genuine enthusiasm for it but must have also peculiar attributes of personality and above all insight of a high order into the psychology of the learning process as related to the higher mental activities. Administrators should never lose sight of the fact that while mathematics if properly taught is one of the most important, interesting, and valuable subjects of the curriculum, it is also one of the most difficult to teach successfully.

Standards for teachers.—It is necessary at the outset to make a fundamental distinction between standards in the sense of requirements for appointment to teaching positions, and standards of scientific attainment which shall determine the curricula of colleges and normal schools aiming to give candidates the best practicable preparation. The former requirements should be high enough to insure competent teaching, but they must not be so high as to form a serious obstacle to admission to the profession even for candidates who have chosen it relatively late. The main factors determining the level of these requirements are the available facilities for preparation, the needs of the pupils, and the economic or salary conditions.

Relatively few young people deliberately choose before entering college the teaching of secondary mathematics as a life work. In the more frequent or more typical case, the college student who will ultimately become a teacher of secondary mathematics makes the choice gradually, perhaps unconsciously, late in the college course or even after its completion, perhaps after some trial of teaching in other fields. The possible supply of young people who have the real desire to become teachers of mathematics is so meager in comparison with the almost unlimited needs of the country that every effort should be made to develop and maintain that desire and all possible encouragement given those who manifest it. If, as will usually be the case, the desire is associated with the necessary mathematial capacity, it will not be wise to hamper the candidate by requiring too high attainments, though as a matter of course he will need guidance in continuing his preparation for a profession of exceptional difficulty and exceptional opportunity.

Another factor which must tend to restrict requirements of high mathematical attainment is the importance to the candidate of breadth of preparation. In college he may be in doubt as to becoming a teacher of mathematics or physics or some other subject. It is unwise to hasten the choice. In many cases the secondary teacher must be prepared in more than one field, and to the future teacher of mathematics preparation in physics and drawing, not to mention chemistry, engineering, etc., may be at least as valuable as purely mathematical college electives beyond the calculus.

In the second sense—of standards of scientific attainment to be held by the colleges and normal schools—these institutions should make every effort—

1. To awaken interest in the subject and the teaching of it in as many young people of the right sort as possible.
2. To give them the best possible opportunity for professional preparation and improvement, both before and after the beginning of teaching.

How the matter of requirements for appointment will actually work out in a given community will inevitably depend upon conditions of time and place, varying widely in character and degree. In many communities it is already practicable and customary to require not less than two years of college work in mathematics, including elementary calculus, with provision for additional electives. Such a requirement the committee would strongly recommend, recognizing, however, that in some localities it would be for the present too restrictive of the supply. In some cases preparation in the pedagogy, philosophy, and history of mathematics could be reasonably demanded or at least given weight; in other cases, any considerable time spent upon them would be of doubtful value. In all cases requirements should be carefully adjusted to local conditions with a view to recognizing the value both of broad and thorough training on the part of those entering the profession and of continued preparation and improvement by summer work and the like. Particular pains should be taken that such preparation is made accessible and attractive in the colleges and normal schools from which teachers are drawn.

It is naturally important that entrance to the profession should not be much delayed by needlessly high or extended requirements, and the danger of creating a teacher who may be too much a specialist for school work and too little for college teaching must be guarded against. There may naturally also be a wide difference between requirements in a strong school offering many electives and a weaker one or a junior high school. Practically, it may be fair to expect that the stronger schools will maintain their standards not by arbitrary or general requirements for entrance to the profession but often by recruiting from other schools teachers who have both high attainments and successful teaching experience.

Programs of courses for colleges and normal schools preparing teachers in secondary mathematics will be found in Chapter XIV, together with an account of existing conditions.

CHAPTER III

Mathematics for Years Seven, Eight, and Nine

I. Introduction

There is a well-marked tendency among school administrators to consider grades one to six, inclusive, as constituting the elementary school and to consider the seondary school period as commencing with the seventh grade and extending through the twelfth.[1] Conforming to this view, the content of the courses of study in mathematics for grades seven, eight, and nine are considered together. In the succeeding chapter the content for grades ten, eleven, and twelve is considered.

The committee is fully aware of the widespread desire on the part of teachers throughout the country for a detailed syllabus by years or half-years which shall give the best order of topics with specific time allotments for each. This desire can not be met at the present time for the simple reason that no one knows what is the best order of topics, nor how much time should be devoted to each in an ideal course. The committee feels that its recommendations should be so formulated as to give every encouragement to further experimentation rather than to restrict the teacher's freedom by a standardized syllabus.

However, certain suggestions as to desirable arrangements of the material are offered in a later section (Sec. III) of this chapter, and in Chapter XII there will be found detailed outlines giving the order of presentation and time allotments in actual operation in schools of various types. This material should be helpful to teachers and administrators in planning courses to fit their individual needs and conditions.

It is the opinion of the committee that the material included in this chapter should be required of all pupils. It includes mathematical knowledge and training which is likely to be needed by every citizen. Differentiation due to special needs should be made after and not before the completion of such a general minimum foundation. Such portions of the recommended content as have not been completed by the end of the ninth year should be required in the following year.

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[1] "We therefore recommend a reorganization of the school system whereby the first six years shall be devoted to elementary education designed to meet the needs of pupils of approximately 6 to 12 years of age; and the second six years to secondary education designed to meet the needs of approximately 12 to 18 years of age. * * * The six years to be devoted to secondary education may well be divided into two periods, which may be designated as the junior and senior periods." Cardinal Principles of Secondary Education, p. 18.

The general principles which have governed the selection of the material presented in the next section and which should govern the point of view of the teaching have already been stated (Chap. II). At this point it seems desirable to recall specifically what was then said concerning principles governing the organization of material, the importance to be attached to the development of insight and understanding and of ability to think clearly in terms of relationships (dependence), and the limitations imposed on drill in algebraic manipulation. In addition we would call attention to the following considerations:

It is assumed that at the end of the sixth school year the pupil will be able to perform with accuracy and with a fair degree of speed the fundamental operations with integers and with common and decimal fractions. The fractions here referred to are such simple ones in common use as are set forth in detail under A(*c*) in the following section. It may be pointed out that the standard of attainment here implied is met in a large number of schools, as is shown by various tests now in use (see Chap. XIII), and can easily be met generally if time is not wasted on the relatively unimportant parts of the subject.

In adapting instruction in mathematics to the mental traits of pupils care should be taken to maintain the mental growth too often stunted by secondary school materials and methods, and an effort should be made to associate with inquisitiveness, the desire to experiment, the wish to know "how and why" and the like, the satisfaction of these needs.

In the years under consideration it is also especially important to give the pupils as broad an outlook over the various fields of mathematics as is consistent with sound scholarship. These years especially are the ones in which the pupil should have the opportunity to find himself, to test his abilities and aptitudes, and to secure information and experience which will help him choose wisely his later courses and ultimately his life work.

II. Material for Grades Seven, Eight, and Nine

In the material outlined in the following pages no attempt is made to indicate the most desirable order of presentation. Stated by topics rather than years the mathematics of grades seven, eight, and nine may properly be expected to include the following:

A. Arithmetic.—(*a*) The fundamental operations of arithmetic.

(*b*) Tables of weights and measures in general practical use, including the most common metric units (meter, centimeter, millimeter, kilometer, gram, kilogram, liter). The meaning of such foreign monetary units as pound, franc, and Mark.

(*c*) Such simple fractions as $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{4}$, $\frac{3}{4}$, $\frac{1}{5}$, $\frac{1}{8}$; others than these to have less attention.

(*d*) Facility and accuracy in the four fundamental operations; time tests, taking care to avoid subordinating the teaching to the tests or to use the tests as measures of the teacher's efficiency. (See Chap. XIII.)

(*e*) Such simple short cuts in multiplication and division as that of replacing multiplication by 25 by multiplying by 100 and dividing by 4.

(*f*) Percentage. Interchanging common fractions and per cents; finding any per cent of a number; finding what per cent one number is of another; finding a number when a certain per cent of it is known; and such applications of percentage as come within the student's experience.

(*g*) Line, bar, and circle graphs, wherever they can be used to advantage.

(*h*) Arithmetic of the home: household accounts, thrift, simple bookkeeping, methods of sending money, parcel post.

Arithmetic of the community: property and personal insurance, taxes.

Arithmetic of banking: savings accounts, checking accounts.

Arithmetic of investment: real estate, elementary notions of stocks and bonds, postal savings.

(*i*) Statistics: fundamental concepts, statistical tables and graphs; pictograms; graphs showing simple frequency distributions.

It will be seen that the material listed above includes some material of earlier instruction. This does not mean that this material is to be made the direct object of study but that drill in it shall be given in connection with the new work. It is felt that this shift in emphasis will make the arithmetic processes here involved much more effective and will also result in a great saving of time.

The amount of time devoted to arithmetic as a distinct subject should be greatly reduced from what is at present customary. This does not mean a lessening of emphasis on drill in arithmetic processes for the purpose of securing accuracy and speed. The need for continued arithmetic work and numerical computation throughout the secondary school period is recognized elsewhere in this report. (Chap. II.)

The applications of arithmetic to business should be continued late enough in the course to bring to their study the pupil's greatest maturity, experience, and mathematical knowledge, and to insure real significance of this study in the business and industrial life which many of the pupils will enter at the close of the eighth or ninth school year. (See I, below.) In this connection care should be taken that the business practices taught in the schools are in accord with the best actual usage. Arithmetic should not be completed before the pupil has acquired the power of using algebra as an aid.

B. Intuitive geometry.—*(a)* The direct measurement of distances and angles by means of a linear scale and protractor. The approximate character of measurement. An understanding of what is meant by the degree of precision as expressed by the number of "significant" figures.

(b) Areas of the square, rectangle, parallelogram, triangle, and trapezoid; circumference and area of a circle; surfaces and volumes of solids of corresponding importance; the construction of the corresponding formulas.

(c) Practice in numerical computation with due regard to the number of figures used or retained.

(d) Indirect measurement by means of drawings to scale; use of square ruled paper.

(e) Geometry of appreciation; geometric forms in nature, architecture, manufacture, and industry.

(f) Simple geometric constructions with ruler and compasses, T-square, and triangle, such as that of the perpendicular bisector, the bisector of an angle, and parallel lines.

(g) Familiarity with such forms as the equilateral triangle, the 30°–60° right triangle, and the isosceles right triangle; symmetry; a knowledge of such facts as those concerning the sum of the angles of a triangle and the pythagorean relation; simple cases of geometric loci in the plane and in space.

(h) Informal introduction to the idea of similarity.

The work in intuitive geometry should make the pupil familiar with the elementary ideas concerning geometric forms in the plane and in space with respect to shape, size, and position. Much opportunity should be provided for exercising space perception and imagination. The simpler geometric ideas and relations in the plane may properly be extended to three dimensions. The work should, moreover, be carefully planned so as so bring out geometric relations and logical connections. Before the end of this intuitive work the pupil should have definitely begun to make inferences and to draw valid conclusions from the relations discovered. In other words, this informal work in geometry should be so organized as to make it a gradual approach to, and provide a foundation for, the subsequent work in demonstrative geometry.

C. Algebra.—1. The formula—its construction, meaning, and use (*a*) as a concise language; (*b*) as a shorthand rule for computation; (*c*) as a general solution; (*d*) as an expression of the dependence of one variable upon another.

The pupil will already have met the formula in connection with intuitive geometry. The work should now include translation from English into algebraic language, and vice versa, and special care should be taken to make sure that the new language is understood and used intelligently. The nature of the dependence of one variable in a formula upon another should be examined and analyzed, with a view to seeing "how the formula works." (See Chap. VII.)

2. Graphs and graphic representations in general—their construction and interpretation in (*a*) representing facts (statistical, etc.); (*b*) representing dependence; (*c*) solving problems.

After the necessary technique has been adequately presented graphic representation should not be considered as a separate topic but should be used throughout, whenever helpful, as an illustrative and interpretative instrument.

3. Positive and negative numbers—their meaning and use: *(a)* as expressing both magnitude and one of two opposite directions or senses; (*b*) their graphic representation; (*c*) the fundamental operations applied to them.

4. The equation—its use in solving problems:

(*a*) Linear equations in one unknown—their solution and applications.

(*b*) Simple cases of quadratic equations when arising in connection with formulas and problems.

(*c*) Equations in two unknowns, with numerous concrete illustrations.

(*d*) Various simple applications of ratio and proportion in cases in which they are generally used in problems of similarity and in other problems of ordinary life. In view of the usefulness of the ideas and training involved, this subject may also properly include simple cases of variation.

5. Algebraic technique: (*a*) The fundamental operations.

Their connection with the rules of arithmetic should be clearly brought out and made to illuminate numerical processes. Drill in these operations should be limited strictly in accordance with the principle mentioned in Chapter II, page 11. In particular, "nests" of parentheses should be avoided, and multiplication and division should not involve much beyond monomial and binomial multipliers, divisors, and quotients.

(*b*) Factoring. The only cases that need be considered are (i) common factors of the terms of a polynomial; (ii) the difference of two squares; (iii) trinomials of the second degree that can be easily factored by trial.

(c) Fractions. Here again the intimate connection with the corresponding processes of arithmetic should be made clear and should serve to illuminate such processes. The four fundamental operations with fractions should be considered only in connection with simple cases and should be applied constantly throughout the course so as to gain the necessary accuracy and facility.

(*d*) Exponents and radicals. The work done on exponents and radicals should be confined to the simplest material required for the treatment of formulas. The laws for positive integral exponents should be included. The consideration of radicals should be confined to transformations of the following types: $\sqrt{a^2b} = a\sqrt{b}$, $\sqrt{a/b} = \frac{1}{b}\sqrt{ab}$ and $\sqrt{a/b} = \sqrt{a}/\sqrt{b}$, and to the numerical evaluation of simple expressions involving the radical sign. A process for finding the square root of a number should be included, but not for finding the square root of a polynomial.

(*e*) Stress should be laid upon the need for checking solutions.

D. Numerical trigonometry.—(*a*) Definition of sine, cosine, and tangent.

(*b*) Their elementary properties as functions.

(*c*) Their use in solving problems involving right triangles.

(*d*) The use of tables of these functions (to three or four places).

The introduction of the elementary notions of trigonometry into the earlier courses in mathematics has not been as general in the United States as in foreign countries. (See Chap. XI.) Among the reasons for an early introduction of this topic are these: Its practical usefulness for many citizens; the insight it gives into the nature of mathematical methods, particularly those concerned with indirect measurement, and into the rôle that mathematics plays in the life of the world; the fact that it is not difficult and that it offers wide opportunity for concrete and significant application; and the interest it arouses in the pupils. It should be based upon the work in intuitive geometry, with which it has intimate contacts (see B, *d*, *h*, above), and should be confined to the simplest material needed for the numerical treatment of the problems indicated. Relations between the trigonometric functions need not be considered.

E. Demonstrative geometry.—The demonstration of a limited number of propositions, with no attempt to limit the number of fundamental assumptions, the principal purpose being to show to the pupil what "demonstration" means.

Many of the geometric facts previously inferred intuitively may be used as the basis upon which the demonstrative work is built. This is not intended to preclude the possibility of giving at a later time rigorous proofs of some of the facts inferred intuitionally. It should be noted that from the strictly logical point of view the attempt to reduce to a minimum the list of axioms, postulates, or assumptions is not at all necessary, and from a pedagogical point of view such an attempt in an elementary course is very undesirable. It is necessary, however, that those propositions which are to be used as the basis of subsequent formal proofs be explicitly listed and their logical significance recognized.

In regard to demonstrative geometry some teachers have objected to the introduction of such work below the tenth grade on the ground that with such immature pupils as are found in the ninth grade nothing worth while could be accomplished in the limited time available. These teachers may be right with regard to conditions prevailing or likely to prevail in the majority of schools in the immediate future. The committee has therefore in a later section of this chapter (Sec. III) made alternative provision for the omission of work in demonstrative geometry.

On the other hand, it is proper to call attention to the fact that certain teachers have successfully introduced a limited amount of work in demonstrative geometry into the ninth grade (see Chap. XII) and that it would seem desirable that others should make the experiment when conditions are favorable. Much of the opposition is probably due to a failure to realize the extent to which the work in intuitive geometry, if properly organized, will prepare the way for the more formal treatment, and to a misconception of the purposes and extent of the work in demonstrative geometry that is proposed. In reaching a decision on this question teachers should keep in mind that it is one of their important duties and obligations, in the grades under consideration, to show their pupils the nature, content, and possibilities of later courses in their subject and to give to each pupil an opportunity to determine his aptitudes and preferences therefor. The omission in the earlier courses of all work of a demonstrative nature in geometry would disregard one educationally important aspect of mathematics.

F. **History and biography.**—Teachers are advised to make themselves reasonably acquainted with the leading events in the history of mathematics, and thus to know that mathematics has developed in answer to human needs, intellectual as well as technical. They should use this material incidentally throughout their courses for the purpose of adding to the interest of the pupils by means of informal talks on the growth of mathematics and on the lives of the great makers of the science.

G. Optional topics.—Certain schools have been able to cover satisfactorily the work suggested in sections A to F before the end of the ninth grade (see Chap. XII). The committee looks with favor on the efforts, in such schools, to introduce earlier than is now customary certain topics and processes which are closely related to modern needs, such as the meaning and use of fractional and negative exponents, the use of the slide-rule, the use of logarithms and of other simple tables, and simple work in arithmetic and geometric progressions, with modern applications to such financial topics as interest and annuities and to such scientific topics as falling bodies and laws of growth.

H. Topics to be omitted or postponed.—In addition to the large amount of drill in algebraic technique already referred to, the following topics should, in accordance with our basic principles, be excluded from the work of grades seven, eight, and nine; some of them will properly be included in later courses (see Chap. IV):

Highest common factor and lowest common multiple, except the simplest cases involved in the addition of simple fractions.

The theorems on proportion relating to alternation, inversion, composition, and division.

Literal equations, except such as appear in common formulas, including the derivation of formulas and of geometric relations, or to show how needless computation may be avoided.

Radicals, except as indicated in a previous section.

Square root of polynomials.

Cube root.

Theory of exponents.

Simultaneous equations in more than two unknowns.

The binomial theorem.

Imaginary and complex numbers.

Radical equations except such as arise in dealing with elementary formulas.

I. Problems.—As already indicated, much of the emphasis now generally placed on the formal exercise should be shifted to the "concrete" or "verbal" problem. The selection of problem material is, therefore, of the highest importance.

The demand for "practical" problems should be fully met in so far as the maturity and previous experience of the pupil will permit. But above all, the problems must be "real" *to the pupil,* must connect with his ordinary thought, and must be within the world of his experience and interest. "The educational utility of problems is not to be measured by their commercial or scientific value, but by their degree of reality for the pupils. . . . They must exemplify those leading ideas which it is desired to impart, and they must do so through media which are real to those under instruction. The reality is found in the students, the utility in their acquisition of principles."[2]

There should be, moreover, a conscious effort through the selection of problems to correlate the work in mathematics with the other courses of the curriculum, especially in connection with courses in science. The introduction of courses in "general science" increases the opportunities in this direction.

J. Numerical computation, use of tables, etc.—The solution of problems should offer opportunity throughout the grades under consideration for considerable arithmetical and computational work. In this connection attention should be called to the importance of exercising common sense and judgment in the use of approximate data, keeping in mind the fact that all data secured from measurement are approximate. A pupil should be led to see the absurdity of giving the area of a circle to a thousandth of a square inch when the radius has been measured only to the nearest inch. He should understand the conception of "the number of significant figures" and should not retain more figures in his result than are warranted by the accuracy of his data. The ideals of accuracy and of self-reliance and the necessity of checking all numerical results should be emphasized. An insight into the nature of tables, including some elementary notions as to interpolation, is highly desirable. The use of tables of various kinds (such as squares and square roots, interest, and trigonometric functions) to facilitate computation and to develop the idea of dependence should be encouraged.

III. Suggested Arrangements of Material

In approaching the problem of arranging or organizing this material it is necessary to consider the different situations that may have to be met.

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[2] Carson: Mathematical Education, pp. 42-45.

1. **The junior high school.**—In view of the fact that under this form of school organization pupils may be expected to remain in school until the end of the junior high school period instead of leaving in large numbers at the end of the eighth school year, the mathematics of the three years of the junior high school should be planned as a unit, and should include the material recommended in the preceding section. There remains the question as to the order in which the various topics should be presented and the amount of time to be devoted to each. The committee has already stated its reasons for not attempting to answer this question (see Sec. I of this chapter). The following plans for the distribution of time are, however, suggested in the hope that they may be helpful; but no one of them is recommended as superior to the others, and only the large divisions of material are mentioned.

PLAN A.

First year: Applications of arithmetic, particularly in such lines as relate to the home, to thrift, and to the various school subjects; intuitive geometry.

Second year: Algebra; applied arithmetic, particularly in such lines as relate to commercial, industrial and social needs.

Third year: Algebra, trigonometry, demonstrative geometry.

By this plan the demonstrative geometry is introduced in the third year, and arithmetic is practically completed in the second year.

PLAN B.

First year: Applied arithmetic (as in plan A); intuitive geometry.

Second year: Algebra, intuitive geometry, trigonometry.

Third year: Applied arithmetic, algebra, trigonometry, demonstrative geometry.

By this plan trigonometry is taken up in two years, and the arithmetic is transferred from the second year to the third year.

PLAN C.

First year: Applied arithmetic (as in plan A), intuitive geometry, algebra

Second year: Algebra, intuitive geometry.

Third year: Trigonometry, demonstrative geometry, applied arithmetic.

By this plan Algebra is confined chiefly to the first two years.

PLAN D.

First year: Applied arithmetic (as in plan A), intuitive geometry.

Second year: Intuitive geometry, algebra.

Third year: Algebra, trigonometry, applied arithmetic.

By this plan Algebra is confined chiefly to the first two years.

PLAN E.

First year: Intuitive geometry, simple formulas, elementary principles of statistics, arithmetic (as in plan A).

Second year: Intuitive geometry, algebra, arithmetic.

Third year: Geometry, numerical trigonometry, arithmetic.

2. Schools organized on the 8-4 plan.—It cannot be too strongly emphasized that, in the case of the older and at present more prevalent plan of the 8-4 school organization, the work in mathematics of the seventh, eighth, and ninth grades should also be organized to include the material here suggested.

The prevailing practice of devoting the seventh and eighth grades almost exclusively to the study of arithmetic is generally recognized as a wasteful marking of time. It is mainly in these years that American children fall behind their European brothers and sisters. No essentially new arithmetic principles are taught in these years, and the attempt to apply the previously learned principles to new situations in the more advanced business and economic aspects of arithmetic is doomed to failure on account of the fact that the situations in question are not and cannot be made real and significant to pupils of this age. We need only refer to what has already been said in this chapter on the subject of problems (I, Sec. II).

The same principles should govern the selection and arrangement of material in mathematics for the seventh and eighth grades of a grade school as govern the selection for the corresponding grades of a junior high school, with this exception: Under the 8-4 form of organization many pupils will leave school at the end of the eighth year. This fact must receive due consideration. The work of the seventh and eighth years should be so planned as to give the pupils in these grades the most valuable mathematical information and training that they are capable of receiving in those years, with little reference to courses that they may take in later years. As to possibilities for arrangement, reference may be made to the plans given above for the first two years of the junior high school. When the work in mathematics of the seventh and eighth grades has been thus reorganized, the work of the first year of a standard four-year high school should complete the program suggested.

Finally, there must be considered the situation in those four-year high schools in which the pupils have not had the benefit of the reorganized instruction recommended for grades seven and eight. It may be hoped that this situation will be only temporary, although it must be recognized that, owing to a variety of possible reasons (lack of adequately prepared teachers in grades seven and eight, lack of suitable textbooks, administrative inertia, and the like), the new plans will not be immediately adopted and that, therefore, for some years many high schools will have to face the situation implied.

In planning the work of the ninth grade under these conditions teachers and administrative officers should again be guided by the principle of giving the pupils the most valuable mathematical information and training which they are capable of receiving in this year with little reference to future courses which the pupil may or may not take. It is to be assumed that the work of this year is to be required of all pupils. Since for many this will constitute the last of their mathematical instruction, it should be so planned as to give them the widest outlook consistent with sound scholarship.

Under these conditions it would seem desirable that the work of the ninth grade should contain both algebra and geometry. It is, therefore, recommended that about two-thirds of the time be devoted to the most useful parts of algebra, including the work on numerical trigonometry, and that about one-third of the time be devoted to geometry, including the necessary informal introduction and, if feasible, the first part of demonstrative geometry.

It should be clear that owing to the greater maturity of the pupils much less time need be devoted in the ninth grade to certain topics of intuitive geometry (such as direct measurement, for example) than is desirable when dealing with children in earlier grades. Even under the conditions presupposed, pupils will be acquainted with most of the fundamental geometric forms and with the mensuration of the most important plane and solid figures. The work in geometry in the ninth grade can then properly be made to center about indirect measurement and the idea of similarity (leading to the processes of numerical trigonometry), and such geometric relations as the sum of the angles of a triangle, the pythagorean proposition, congruence of triangles, parallel and perpendicular lines, quadrilaterals, and the more important simple constructions.

CHAPTER IV

Mathematics for Years Ten, Eleven, and Twelve

I. Introduction

The committee has in the preceding chapter expressed its judgment that the material there recommended for the seventh, eighth, and ninth years should be required of all pupils. In the tenth, eleventh, and twelfth years, however, the extent to which election of subjects is permitted will depend on so many factors of a general character that it seems unnecessary and inexpedient for the present committee to urge a positive requirement beyond the minimum one already referred to. The subject must, like others, stand or fall on its intrinsic merit or on the estimate of such merit by the authorities responsible at a given time and place. The committee believes nevertheless that every standard high school should not merely offer courses in mathematics for the tenth, eleventh, and twelfth years, but should encourage a large proportion of its pupils to take them. Apart from the intrinsic interest and great educational value of the study of mathematics, it will in general be necessary for those preparing to enter college or to engage in the numerous occupations involving the use of mathematics to extend their work beyond the minimum requirement.

The present chapter is intended to suggest for students in general courses the most valuable mathematical training that will appropriately follow the courses outlined in the previous chapter. Under present conditions most of this work will normally fall in the last three years of the high school; that is, in general, in the tenth, eleventh, and twelfth years.

The selection of material is based on the general principles formulated in Chapter II. At this point attention need be directed only to the following:

1. In the years under consideration it is proper that some attention be paid to the students' vocational or other later educational needs.

2. The material for these years should include as far as possible those mathematical ideas and processes that have the most important applications in the modern world. As a result, certain material will naturally be included that at present is not ordinarily given in secondary school courses; as, for instance, the material concerning the calculus. On the other hand, certain other material that is now included in college entrance requirements will be excluded. The results of an investigation made by the National Committee in connection with a study of these requirements indicates that modifications to meet these changes will be desirable from the standpoint of both college and secondary school (see Chap. V).

3. During the years now under consideration an increasing amount of attention should be paid to the logical organization of the material, with the purpose of developing habits of logical memory, appreciation of logical structure, and ability to organize material effectively.

It cannot be too strongly emphasized that the broadening of content of high school courses in mathematics suggested in the present and in previous chapters will materially increase the usefulness of these courses to those who pursue them. It is of prime importance that educational administrators and others charged with the advising of students should take careful account of this fact in estimating the relative importance of mathematical courses and their alternatives. The number of important applications of mathematics in the activities of the world is today very large and is increasing at a very rapid rate. This aspect of the progress of civilization has been noted by all observers who have combined a knowledge of mathematics with an alert interest in the newer developments in other fields. It was revealed in very illuminating fashion during the recent war by the insistent demand for persons with varying degrees of mathematical training for many war activities of the first moment. If the same effort were made in time of peace to secure the highest level of efficiency available for the specific tasks of modern life, the demand for those trained in mathematics would be no less insistent; for it is in no wise true that the applications of mathematics in modern warfare are relatively more important or more numerous than its applications in those fields of human endeavor which are of a constructive nature.

There is another important point to be kept in mind in considering the relative value to the average student of mathematical and various alternative courses. If the student who omits the mathematical courses has need of them later, it is almost invariably more difficult, and it is frequently impossible, for him to obtain the training in which he is deficient. In the case of a considerable number of alternative subjects a proper amount of reading in spare hours at a more mature age will ordinarily furnish him the approximate equivalent to that which he would have obtained in the way of information in a high school course in the same subject. It is not, however, possible to make up deficiencies in mathematical training in so simple a fashion. It requires systematic work under a competent teacher to master properly the technique of the subject, and any break in the continuity of the work is a handicap for which increased maturity rarely compensates. Moreover, when the individual discovers his need for further mathematical training it is usually difficult for him to take the time from his other activities for systematic work in elementary mathematics.

II. Recommendations for Elective Courses

The following topics are recommended for inclusion in the mathematical electives open to pupils who have satisfactorily completed the work outlined in the preceding chapter, comprising arithmetic, the elementary notions of algebra, intuitive geometry, numerical trigonometry, and a brief introduction to demonstrative geometry.

1. **Plane demonstrative geometry.**—The principal purposes of the instruction in this subject are: To exercise further the spatial imagination of the student, to make him familiar with the great basal propositions and their applications, to develop understanding and appreciation of a deductive proof and the ability to use this method of reasoning where it is applicable, and to form habits of precise and succinct statement, of the logical organization of ideas, and of logical memory. Enough time should be spent on this subject to accomplish these purposes.

The following is a suggested list of topics under which the work in demonstrative geometry may be organized:[1] (*a*) Congruent triangles, perpendicular bisectors, bisectors of angles; (*b*) arcs, angles, and chords in circles; (*c*) parallel lines and related angles, parallelograms; (*d*) the sum of the angles for triangle and polygon; (*e*) secants and tangents to circles with related angles, regular polygons; (*f*) similar triangles, similar figures; (*g*) areas; numerical computation of lengths and areas, based upon geometric theorems already established.

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[1] It is not intended that the order here given should imply anything as to the order of presentation. (See also Chap. VI.)

Under these topics constructions, loci, originals and other exercises are to be included.

It is recommended that the formal theory of limits and of incommensurable cases be omitted, but that the ideas of limit and of incommensurable magnitudes receive informal treatment.

It is believed that a more frequent use of the idea of motion in the demonstration of theorems is desirable, both from the point of view of gaining greater insight and of saving time.[2]

If the great basal theorems are selected and effectively organized into a logical system, a considerable reduction (from 30 to 40 per cent) can be made in the number of theorems given either in the Harvard list or in the report of the Committee of Fifteen. Such a reduction is exhibited in the lists prepared by the committee and printed later in this report (Chap. VI). In this connection it may be suggested that more attention than is now customary may profitably be given to those methods of treatment which make consistent use of the idea of motion (already referred to), continuity (the tangent as the limit of a secant, etc.), symmetry, and the dependence of one geometric magnitude upon another.

If the student has had a satisfactory course in intuitive geometry and some work in demonstration before the tenth grade, he may find it possible to cover a minimum course in demonstrative geometry, giving the great basal theorems and constructions, together with exercises, in the 90 periods constituting a half year's work.

2. **Algebra.**—(*a*) *Simple functions of one variable:* Numerous illustrations and problems involving linear, quadratic, and other simple functions including formulas from science and common life. More difficult problems in variation than those included in the earlier course.

(*b*) *Equations in one unknown:* Various methods for solving a quadratic equation (such as factoring, completing the square, use of formula) should be given. In connection with the treatment of the quadratic a very brief discussion of complex numbers should be included. Simple cases of the graphic solution of equations of degree higher than the second should be discussed and applied.

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[2] Reference may here be made to the treatment given in recent French texts such as those by Bourlet and Méray.

(c) *Equations in two or three unknowns*: The algebraic solution of linear equations in two or three unknowns and the graphic solution of linear equations in two unknowns should be given. The graphic and algebraic solution of a linear and a quadratic equation and of two quadratics that contain no first degree term and no xy term should be included.

(d) *Exponents, radicals, and logarithms:* The definitions of negative, zero, and fractional exponents should be given, and it should be made clear that these definitions must be adopted if we wish such exponents to conform to the laws for positive integral exponents. Reduction of radical expressions to those involving fractional exponents should be given as well as the inverse transformation. The rules for performing the fundamental operations on expressions involving radicals, and such transformations as

$$\sqrt[n]{a/b} = \frac{1}{b}\sqrt[n]{ab^{n-1}}, \quad \sqrt[n]{a^n b} = a\sqrt[n]{b}, \quad \frac{a}{\sqrt{b}+\sqrt{c}} = \frac{a(\sqrt{b}-\sqrt{c})}{b-c}$$

should be included. In close connection with the work on exponents and radicals there should be given as much of the theory of logarithms as is involved in their application to computation and sufficient practice in their use in computation to impart a fair degree of facility.

(e) *Arithmetic and geometric progressions*: The formulas for the nth term and the sum of n terms should be derived and applied to significant problems.

(f) *Binomial theorem*: A proof for positive integral exponents should be given; it may also be stated that the formula applies to the case of negative and fractional exponents under suitable restrictions, and the problems may include the use of the formula in these cases as well as in the case of positive integral exponents.

3. **Solid geometry.**—The aim of the work in solid geometry should be to exercise further the spatial imagination of the student and to give him both a knowledge of the fundamental spatial relationships and the power to work with them. It is felt that the work in plane geometry gives enough training in logical demonstration to warrant a shifting of emphasis in the work on solid geometry away from this aspect of the subject and in the direction of developing greater facility

in visualizing spatial relations and figures, in representing such figures on paper, and in solving problems in mensuration.

For many of the practical applications of mathematics it is of fundamental importance to have accurate space perceptions. Hence it would seem wise to have at least some of the work in solid geometry come as early as possible in the mathematical courses, preferably not later than the beginning of the eleventh school year. Some schools will find it possible and desirable to introduce the more elementary notions of solid geometry in connection with related ideas of plane geometry.

The work in solid geometry should include numerous exercises in computation based on the formulas established. This will serve to correlate the work with arithmetic and algebra and to furnish practice in computation.

The following provisional outline of subject matter is submitted:

(*a*) Propositions relating to lines and planes, and to dihedral and trihedral angles.

(*b*) Mensuration of the prism, pyramid, and frustum; the (right circular) cylinder, cone and frustum, based on an informal treatment of limits; the sphere, and the spherical triangle.

(*c*) Spherical geometry.

(*d*) Similar solids.

Such theorems as are necessary as a basis for the topics here outlined should be studied in immediate connection with them.

Desirable simplification and generalization may be introduced into the treatment of mensuration theorems by employing such theorems as Cavalieri's and Simpson's, and the Prismoid Formula; but rigorous proofs or derivations of these need not be included.

Beyond the range of the mensuration topics indicated above, it seems preferable to employ the methods of the elementary calculus (see Section 6, below).

It should be possible to complete a minimum course covering the topics outlined above in not more than one-third of a year.

The list of propositions in solid geometry given in Chapter VI should be considered in connection with the general principles stated at the beginning of this section. By requiring formal proofs to a more limited extent than has been customary, time will be gained to attain the aims indicated and to extend the range of geometric information of the pupil. Care must be exercised to make sure that the pupil is thoroughly familiar with the facts, with the associated terminology, with all the necessary formulas, and that he secures the necessary practice in working with and applying the information acquired to concrete problems.

4. **Trigonometry.**—The work in elementary trigonometry begun in the earlier years should be completed by including the logarithmic solution of right and oblique triangles, radian measure, graphs of trigonometric functions, the derivation of the fundamental relations between the functions and their use in proving identities and in solving easy trigonometric equations. The use of the transit in connection with the simpler operations of surveying and of the sextant for some of the simpler astronomical observations, such as those involved in finding local time, is of value; but when no transit or sextant is available, simple apparatus for measuring angles roughly may and should be improvised. Drawings to scale should form an essential part of the numerical work in trigonometry. The use of the slide-rule in computations requiring only three-place accuracy and in checking other computations is also recommended.

5. **Elementary statistics.**—Continuation of the earlier work to include the meaning and use of fundamental concepts and simple frequency distributions with graphic representations of various kinds and measures of central tendency (average, mode, and median).

6. **Elementary calculus.**—The work should include:

(*a*) The general notion of a derivative as a limit indispensable for the accurate expression of such fundamental quantities as velocity of a moving body or slope of a curve.

(*b*) Applications of derivatives to easy problems in rates and in maxima and minima.

(*c*) Simple cases of inverse problems; e.g., finding distance from velocity, etc.

(*d*) Approximate methods of summation leading up to integration as a powerful method of summation.

(*e*) Applications to simple cases of motion, area, volume, and pressure.

Work in the calculus should be largely graphic and may be closely related to that in physics; the necessary technique should be reduced to a minimum by basing it wholly or mainly on algebraic polynomials. No formal study of analytic geometry need be presupposed beyond the plotting of simple graphs.

It is important to bear in mind that, while the elementary calculus is sufficiently easy, interesting, and valuable to justify its introduction, special pains should be taken to guard against any lack of thoroughness in the fundamentals of algebra and geometry; no possible gain could compensate for a real sacrifice of such thoroughness.

It should also be borne in mind that the suggestion of including elementary calculus is not intended for all schools nor for all teachers or all pupils in any school. It is not intended to connect in any direct way with college entrance requirements. The future college student will have ample opportunity for calculus later. The capable boy or girl who is not to have the college work ought not on that account to be prevented from learning something of the use of this powerful tool. The applications of elementary calculus to simple concrete problems are far more abundant and more interesting than those of algebra. The necessary technique is extremely simple. The subject is commonly taught in secondary schools in England, France, and Germany, and appropriate English texts are available.[3]

7. **History and biography.**—Historical and biographical material should be used throughout to make the work more interesting and significant.

8. **Additional electives.**—Additional electives such as *mathematics of investment, shop mathematics, surveying and navigation, descriptive* or *projective geometry* will appropriately be offered by schools which have special needs or conditions, but it seems unwise for the National Committee to attempt to define them pending the results of further experience on the part of these schools.

III. Plans for Arrangement of the Material

In the majority of high schools at the present time the topics suggested can probably be given most advantageously as separate units of a three-year program. However, the National Committee is of the opinion that methods of organization are being experimentally perfected whereby teachers will be enabled to present much of this material more effectively in combined courses unified by one or more of such central ideas as functionality and graphic representation.

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[3] Quotations and typical problems from one of these texts will be found in a supplementary note appended to this chapter.

As to the arrangement of the material the committee gives below four plans which may be suggestive and helpful to teachers in arranging their courses. No one of them is, however, recommended as superior to the others.

PLAN A.

Tenth year: Plane demonstrative geometry, algebra.
Eleventh year: Statistics, trigonometry, solid geometry.
Twelfth year: The calculus, other elective.

PLAN B.

Tenth year: Plane demonstrative geometry, solid geometry.
Eleventh year: Algebra, trigonometry, statistics.
Twelfth year: The calculus, other elective.

PLAN C.

Tenth year: Plane demonstrative geometry, trigonometry.
Eleventh year: Solid geometry, algebra, statistics.
Twelfth year: The calculus, other elective.

PLAN D.

Tenth year: Algebra, statistics, trigonometry.
Eleventh year: Plane and solid geometry.
Twelfth year: The calculus, other elective.

Additional information on ways of organizing this material will be found in Chapter XII.

Supplementary Note on the Calculus as a High School Subject.

In connection with the recommendations concerning the calculus, such questions as the following may arise: Why should a college subject like this be added to a high school program? How can it be expected that high school teachers will have the necessary training and attainments for teaching it? Will not the attempt to teach such a subject result in loss of thoroughness in earlier work? Will anything be gained beyond a mere smattering of the theory? Will the boy or girl ever use the information or training secured? The subsequent remarks are intended to answer such objections as these and to develop more fully the point of view of the committee in recommending the inclusion of elementary work in the calculus in the high school program.

By the calculus we mean for the present purpose a study of *rates of change.* In nature all things change. How much do they change in a given time? How fast do they change? Do they increase or decrease? When does a changing quantity become largest or smallest? How can rates of changing quantities be compared?

These are some of the questions which lead us to study the elementary calculus. Without its essential principles these questions cannot be answered with definiteness.

The following are a few of the specific replies that might be given in answer to the questions listed at the beginning of this note: The difficulties of the college calculus lie mainly outside the boundaries of the proposed work. The elements of the subject present less difficulty than many topics now offered in advanced algebra. It is not implied that in the near future many secondary school teachers will have any occasion to teach the elementary calculus. It is the culminating subject in a series which only relatively strong schools will complete and only then for a selected group of students. In such schools there should always be teachers competent to teach the elementary calculus here intended. No superficial study of calculus should be regarded as justifying any substantial sacrifice of thoroughness. In the judgment of the committee the introduction of elementary calculus necessarily includes sufficient algebra and geometry to compensate for whatever diversion of time from these subjects would be implied.

The calculus of the algebraic polynominal is so simple that a boy or girl who is capable of grasping the idea of limit, of slope, and of velocity, may in a brief time gain an outlook upon the field of mechanics and other exact sciences, and acquire a fair degree of facility in using one of the most powerful tools of mathematics, together with the capacity for solving a number of interesting problems. Moreover, the fundamental ideas involved, quite aside from their technical applications, will provide valuable training in understanding and analyzing quantitative relations—and such training is of value to everyone.

The following typical extracts from an English text intended for use in secondary schools may be quoted:

"It has been said that the calculus is that branch of mathematics which schoolboys understand and senior wranglers fail to comprehend. * * * So long as the graphic treatment and practical applications of the calculus are kept in view, the subject is an extremely easy and attractive one. Boys can be taught the subject early in their mathematical career, and there is no part of their mathematical training that they enjoy better or which opens up to them wider fields of useful exploration. * * * The phenomena must first be known practically and then studied philosophically. To reverse the order of these processes is impossible."

The text in question, after an interesting historical sketch, deals with such problems as the following:

A train is going at the rate of 40 miles on hour. Represent this graphically.

At what rate is the length of the daylight increasing or decreasing on December 31, March 26, etc.? (From tabular data.)

A cart going at the rate of 5 miles per hour passes a milestone, and 14 minutes afterwards a bicycle, going in the same direction at 12 miles an hour, passes the same milestone. Find when and where the bicycle will overtake the cart.

A man has 4 miles of fencing wire and wishes to fence in a rectangular piece of prairie land through which a straight river flows, the bank of the stream being utilized as one side of the enclosure. How can he do this so as to inclose as much land as possible?

A circular tin canister closed at both ends has a surface area of 100 square centimeters. Find the greatest volume it can contain.

Post-office regulations prescribe that the combined length and girth of a parcel must not exceed 6 feet. Find the maximum volume of a parcel whose shape is a prism with the ends square.

A pulley is fixed 15 feet above the ground, over which passes a rope 30 feet long with one end attached to a weight which can hang freely, and the other end is held by a man at a height of 3 feet from the ground. The man walks horizontally away from beneath the pulley at the rate of 3 feet per second. Find the rate at which the weight rises when it is 10 feet above the ground.

The pressure on the surface of a lake due to the atmosphere is known to be 14 pounds per square inch. The pressure in the liquid x inches below the surface is known to be given by the law $dp/dx = 0.036$. Find the pressure in the liquid at a depth of 10 feet.

The arch of a bridge is parabolic in form. It is 5 feet wide at the base and 5 feet high. Find the volume of water that passes through per second in a flood when the water is rushing at the rate of 10 feet per second.

A force of 20 tons compresses the spring buffer of a railway stop through 1 inch, and the force is always proportional to the compression produced. Find the work done by a train which compresses a pair of such stops through 6 inches.

These may illustrate the aims and point of view of the proposed work. It will be noted that not all of them involve calculus, but those that do not lead up to it.

CHAPTER V

College Entrance Requirements

The present chapter is concerned with a study of topics and training in elementary mathematics that will have most value as preparation for college work, and with recommendations of definitions of college entrance requirements in elementary algebra and plane geometry.

General considerations.—The primary purpose of college entrance requirements is to test the candidate's ability to benefit by college instruction. This ability depends, so far as our present inquiry is concerned, upon (1) general intelligence, intellectual maturity, and mental power; (2) specific knowledge and training required as preparation for the various courses of the college curriculum.

Mathematical ability appears to be a sufficient but not a necessary condition for general intelligence.[1] For this, as well as for other reasons, it would appear that *college entrance requirements in mathematics should be formulated primarily on the basis of the special knowledge and training required for the successful study of courses which the student will take in college.*

The separation of prospective college students from the others in the early years of the secondary school is neither feasible nor desirable. It is therefore obvious that secondary school courses in mathematics cannot be planned with specific reference to college entrance requirements. Fortunately there appears to be no real conflict of interest between those students who ultimately go to college and those who do not, so far as mathematics is concerned. It will be made clear in what follows that a course in this subject, coverning from two to two and one-half years in a standard four-year high school, and so planned as to give the most valuable mathematical training which the student is capable of receiving, will provide adequate preparation for college work.

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[1] A recent investigation made by the department of psychology at Dartmouth College showed that all students of high rank in mathematics had a high rating on general intelligence; the converse was not true, however.

Topics to be included in high-school courses.—In the selection of material of instruction for high school courses in mathematics, its value as preparation for college courses in mathematics need not be specifically considered. Not all college students study mathematics; it is therefore reasonable to expect college departments in this subject to adjust themselves to the previous preparation of their students. Nearly all college students do, however, study one or more of the physical sciences (astronomy, physics, chemistry) and one or more of the social sciences (history, economics, political science, sociology). Entrance requirements must therefore insure adequate mathematical preparation in these subjects. Moreover, it may be assumed that adequate preparation for these two groups of subjects will be sufficient for all other subjects for which the secondary shools may be expected to furnish the mathematical prerequisites.

The National Committee recently conducted an investigation for the purpose of securing information as to the content of high school courses of instruction most desirable from the point of view of preparation for college work. A number of college teachers, prominent in their respective fields, were asked to assign to each of the topics in the following table an estimate of its value as preparation for the elementary courses in their respective subjects. Table 1 gives a summary of the replies, arranged in two groups—"Physical sciences," including astronomy, physics, and chemistry; and "Social sciences," including history, economics, sociology, and political science.

The high value attached to the following topics is significant: Simple formulas—their meaning and use; the linear and quadratic functions and variation; numerical trigonometry; the use of logarithms and other topics relating to numerical computation; statistics. These all stand well above such standard requirements as arithmetic and geometric progression, binomial theorem, theory of exponents, simultaneous equations involving one or two quadratic equations, and literal equations.

These results would seem to indicate that a modification of present college entrance requirements in mathematics is desirable from the point of view of college teachers in departments other than mathematics. It is interesting to note how closely the modifications suggested by this inquiry correspond to the modifications in secondary school mathematics foreshadowed by the study of needs of the high school

TABLE 1.—*Value of topics as preparation for elementary college courses.*

[In the headings of the table, E=essential, C=of considerable value, S=of some value, O=of little or no value, N=number of replies received. The figures in the first four columns of each group are percentages of the number of replies received.]

	Physical sciences.					Social sciences.				
	E.	C.	S.	O.	N.	E.	C.	S.	O.	N.
Negative numbers—their meaning and use	79	5	10	5	39	45	17	22	17	18
Imaginary numbers—their meaning and use	23	21	25	31	39	13	13	37	37	16
Simple formulas—their meaning and use	93	5	2	—	41	47	26	21	5	19
Graphic representation of statistical data	57	25	15	3	40	57	24	14	5	21
Graphs (mathematical and empirical):										
(*a*) As a method of representing dependence	62	16	22	—	37	15	54	15	15	13
(*b*) As a method of solving problems	45	20	28	6	25	18	18	46	18	11
The linear function, $y=mx+b$	78	14	8	—	37	29	29	14	29	14
The quadratic function, $y=ax^2+bx+c$	59	21	17	3	34	8	8	33	50	12
Equations: Problems leading to—										
Linear equations in one unknown	98	2	—	—	41	40	7	20	33	15
Quadratic equations in one unknown	78	15	5	2	40	31	8	8	54	13
Simultaneous linear equations in 2 unknowns	71	24	3	3	38	33	8	—	58	12
Simultaneous linear equations in more than 2 unknowns	43	29	23	6	35	8	8	17	67	12
One quadratic and one linear equation in 2 unknowns	40	24	27	0	33	—	9	9	82	11
Two quadratic equations in 2 unknowns	31	19	28	22	32	—	9	—	91	11
Equations of higher degree than the second	10	32	32	26	31	—	—	9	91	11
Literal equations (other than formulas)	43	18	32	7	28	—	10	40	50	10
Ratio and proportion	84	8	3	5	39	37	26	32	5	19
Variation	50	13	20	17	30	17	33	25	25	12
Numerical computation:										
With approximate data—rational use of significant figures	61	36	—	3	39	40	27	20	13	12
Short-cut methods	27	38	24	10	37	29	35	23	12	17
Use of logarithms	62	29	7	2	42	12	29	29	29	17
Use of other tables to facilitate computation	24	45	26	5	38	18	29	41	12	17
Use of slide-rule	24	39	26	12	38	11	39	28	22	18
Theory of exponents	36	31	25	8	36	—	21	21	57	14
Theory of logarithms	34	26	21	18	38	7	13	20	60	15
Arithmetic progression	16	32	38	13	37	23	29	12	35	17
Geometric progression	19	27	40	14	37	23	25	18	35	17
Binomial theorem	35	32	18	13	37	13	20	27	40	15
Probability	9	32	41	19	32	20	35	35	10	20
Statistics:										
Meaning and use of elementary concepts	23	28	31	17	29	55	36	5	5	22
Frequency distributions and frequency curves	15	19	35	32	26	47	33	10	10	21
Correlation	11	18	39	32	28	33	47	14	5	21
Numerical trigonometry:										
Use of sine, cosine, and tangent in the solution of simple problems involving right triangles	68	21	3	8	38	—	—	25	75	12
Demonstrative geometry	68	15	12	6	34	—	21	43	36	14
Plane trigonometry (usual course)	57	27	11	5	37	8	23	31	38	13
Analytic geometry:										
Fundamental conceptions and methods in the plane	32	45	19	3	31	—	15	38	46	13
Systematic treatment of—										
Straight line	34	37	20	9	35	9	9	18	64	11
Circle	29	43	20	9	35	—	18	9	73	11
Conic sections	18	41	26	15	34	—	9	18	73	11
Polar coordinates	18	26	41	15	34	—	—	18	82	11
Empirical curves and fitting curves to observations	12	38	38	12	34	8	—	25	67	12

pupil irrespective of his possible future college attendance. The recommendations made in Chapter II that functional relationship be made the "underlying principle of the course," that the meaning and use of simple formulas be emphasized, that more attention be given to numerical computation (especially to the methods relating to approximate data), and that work on numerical trigonometry and statistics be included, have

TABLE 2.—*Topics in order of value as preparation for elementary college courses.*

[The figures in the column headed "E" are taken from Table 1, taking in each case the higher of the two "E" ratings there given. The column headed "E+C" gives in each case the sum of the two ratings for "E" and "C." An asterisk indicates that the topic in question is now included in the definitions of the College Entrance Examination Board.[2]]

	E.	E+C.
Linear equations in one unknown	98	100
Simple formulas—their meaning and use	93	98
*Ratio and proportion	84	92
*Negative numbers—their meaning and use	79	84
*Quadratic equations in one unknown	78	93
The linear function: $y=mx+b$	78	92
*Simultaneous linear equations in two unknowns	71	95
Numerical trigonometry—the use of the sine, cosine, and tangent in the solution of simple problems involving right triangles	68	89
*Demonstrative geometry	68	83
Use of logarithms in computation	62	91
*Graphs as a method of representing dependence	62	78
Computation with approximate data—rational use of significant figures	61	97
The quadratic function: $y=ax^2+bx+c$	59	80
Plane trigonometry—usual course	57	84
Graphic representation of statistical data	57	82
Statistics—meaning and use of elementary concepts	55	91
Variation	50	63
Statistics—frequency distributions and curves	47	80
*Graphic solution of problems	45	65
*Literal equations	43	61
*Simultaneous linear equations in more than 2 unknowns	43	72
*Simultaneous equations, one quadratic, one linear	40	64
*Theory of exponents	36	67
*Binomial theorem	35	67
Analytic geometry of the straight line	34	71
Theory of logarithms	34	60
Statistics—correlation	33	80
Analytic geometry—Fundamental Conceptions	32	77
*Simultaneous quadratic equations	31	50
Analytic geometry of the circle	29	72
Short-cut methods of computation	29	65
Use of tables in computation (other than logarithms)	24	69
Use of slide rule	24	63
*Arithmetic progression	23	52
*Geometric progression	23	48
Imaginary numbers	23	44
Probability	20	55
Conic sections	18	59
Polar coordinates	18	44
Empirical curves and fitting curves to observations	12	50
Equations of higher degree than the second	10	42

received widespread approval throughout the country. That they should be in such close accord with the desires of college teachers in the fields of physical and social sciences as to entrance requirements is striking. We find here the justification for the belief expressed earlier in this report that there is no real conflict between the needs of students who ultimately go to college and those who do not.

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[2] The list includes all the requirements of the college entrance examination board except those relating to algebraic technique. The topic of "Negative numbers" has also been given an asterisk, as it is clearly implied, though not explicitly mentioned in the C. E. E. B. definitions.

The attitude of the colleges.—Mathematical instruction in this country is at present in a period of transition. While a considerable number of our most progressive schools have for several years given courses embodying most of the recommendations contained in Chapters II, III, and IV of the present report, the large majority of schools are still continuing the older types of courses or are only just beginning to introduce modifications. The movement toward reorganization is strong, however, throughout the country, not only in the standard four-year high schools but also in the newer junior high schools.

During this period of transition it should be the policy of the colleges, while exerting a desirable steadying influence, to help the movement toward a sane reorganization. In particular, they should take care not to place obstacles in the way of changes which are clearly in the interest of more effective college preparation, as well as of better general education.

College entrance requirements will continue to exert a powerful influence on secondary school teaching. Unless they reflect the spirit of sound progressive tendencies, they will constitute a serious obstacle.

In the present chapter revised definitions of college-entrance requirements in plane geometry and elementary algebra are presented. So far as plane geometry is concerned, the problem of definition is comparatively simple. The proposed definition of the requirement in plane geometry does not differ from the one now in effect under the College Entrance Examination Board. A list of propositions and constructions has however been prepared and is given in the next chapter for the guidance of teachers and examiners.

In elementary algebra a certain amount of flexibility is obviously necessary both on account of the quantitative differences among colleges and of the special conditions attending a period of transition. The former differences are recognized by the proposal of a minor and a major requirement in elementary algebra. The second of these includes the first and is intended to correspond with the two-unit rating of the C. E. E. B.

In connection with this matter of units, the committee wishes particularly to disclaim any emphasis upon a special number of years or hours. The unit terminology is doubtless too well established to be entirely ignored in formulating college entrance requirements, but the standard definition of unit [3] has never been precise, and will now become much less so with the inclusion of the newer six-year program. A time allotment of 4 or 5 hours per week in the seventh year can certainly not have the same weight as the same number of hours in the twelfth year, and the disparity will vary with different subjects. *What is really important is the amount of subject matter and the quality of work done in it.* The "unit" can not be anything but a crude approximation to this. The distribution of time in the school program should not be determined by any arbitrary unit scale.

As a further means of securing reasonable flexibility, the committee recommends that for a limited time—say five years—the option be offered between examinations based on the old and on the new definitions, so far as differences between them may make this desirable.

In view of the changes taking place at the present time in mathematical courses in secondary schools, and the fact that college entrance requirements should so soon as possible reflect desirable changes and assist in their adoption, the National Committee recommends that either the American Mathematical Society or the Mathematical Association of America (or both) maintain a permanent committee on college entrance requirements in mathematics, such a committee to work in close coöperation with other agencies which are now or may in the future be concerned in a responsible way with the relations between colleges and secondary schools.

Proposed Definition of College Entrance Requirements

ELEMENTARY ALGEBRA.

Minor requirement.—The meaning, use, and evaluation (including the necessary transformations) of simple formulas involving ideas with which the student is familiar and the derivation of such formulas from rules expressed in words.

The dependence of one variable upon another. Numerous illustrations and problems involving the linear function $y=mx+b$. Illustrations and problems involving the quadratic function $y=kx^2$.

Graphs and graphic representations in general; their construction and interpretation, including the representation of statistical data and the use of the graph to exhibit dependence.

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[3] The following definition, formulated by the National Committee on Standards of Colleges and Secondary Schools, has been given the approval of the C. E. E. B. "A unit represents a year's study in any subject in a secondary school, constituting approximately a quarter of a full year's work. A four-year secondary school curriculum should be regarded as representing not more than 16 units of work."

Positive and negative numbers; their meaning and use.

Linear equations in one unknown quantity; their use in solving problems.

Sets of linear equations involving two unknown quantities; their use in solving problems.

Ratio, as a case of simple fractions; proportion without the theorems on alternation, etc.; and simple cases of variation.

The essentials of algebraic technique. This should include —

(*a*) The four fundamental operations.

(*b*) Factoring of the following types: Common factors of the terms of a polynomial; the difference of two squares; trinomials of the second degree (including the square of a binomial) that can be easily factored by trial.

(*c*) Fractions, including complex fractions of a simple type.

(*d*) Exponents and radicals. The laws for positive integral exponents; the meaning and use of fractional exponents, but not the formal theory. The consideration of radicals may be confined to the simplification of expressions of the form $\sqrt{a^2b}$ and $\sqrt{a/b}$ and to the evaluation of simple expressions involving the radical sign. A process for extracting the square root of a number should be included but not the process for extracting the square root of a polynomial.

Numerical trigonometry. The use of the sine, cosine, and tangent in solving right triangles. The use of three or four place tables of natural functions.

Major requirement.—In addition to the minor requirement as specified above, the following should be included:

Illustrations and problems involving the quadratic function $y = ax^2 + bx + c$.

Quadratic equations in one unknown; their use in solving problems.

Exponents and radicals. Zero and negative exponents, and more extended treatment of fractional exponents. Rationalizing denominators. Solution of simple types of radical equations.

The use of logarithmic tables in computation without the formal theory.

Elementary statistics, including a knowledge of the fundamental concepts and simple frequency distributions, with graphic representations of various kinds.

The binomial theorem for positive integral exponents less than 8; with such applications as compound interest.

The formula for the nth term, and the sum of n terms, of arithmetic and geometric progressions, with applications.

Simultaneous linear equations in three unknown quantities and simple cases of simultaneous equations involving one or two quadratic equations; their use in solving problems.

Drill in algebraic manipulation should be limited, particularly in the minor requirement, by the purpose of securing a thorough understanding of important principles and facility in carrying out those processes which are fundamental and of frequent occurrence either in common life or in the subsequent courses that a substantial proportion of the pupils will study. Skill in manipulation must be conceived of throughout as a means to an end, not as an end in itself. Within these limits, skill and accuracy in algebraic technique are of prime importance, and drill in this subject should be extended far enough to enable students to carry out the fundamentally essential processes accurately and with reasonable speed.

The consideration of literal equations, when they serve a significant purpose, such as the transformation of formulas, the derivation of a general solution (as of the quadratic equation), or the proof of a theorem, is important. As a means for drill in algebraic technique they should be used sparingly.

The solution of problems should offer opportunity throughout the course for considerable arithmetical and computational work. The conception of algebra as an extension of arithmetic should be made significant both in numerical applications and in elucidating algebraic principles. Emphasis should be placed upon the use of common sense and judgment in computing from approximate data, especially with regard to the number of figures retained, and on the necessity for checking the results. The use of tables to facilitate computation (such as tables of squares and square roots, of interest, and of trigonometric functions) should be encouraged.

PLANE GEOMETRY.

The usual theorems and constructions of good textbooks, including the general properties of plane rectilinear figures; the circle and the measurement of angles; similar polygons; areas; regular polygons and the measurement of the circle. The solution of numerous original exercises, including locus problems. Applications to the mensuration of lines and plane surfaces.

The scope of the required work in plane geometry is indicated by the List of Fundamental Propositions and Constructions, which is given in the next chapter. This list indicates in Section I the type of proposition which, in the opinion of the committee, may be assumed without proof or given informal treatment. Section II contains 52 propositions and 19 constructions which are regarded as so fundamental that they should constitute the common minimum of all standard courses in plane geometry. Section III gives a list of subsidiary theorems which suggests the type of additional propositions that should be included in such courses.

College entrance examinations.—College entrance examinations exert in many schools, and especially throughout the eastern section of the country, an influence on secondary school teaching which is very far-reaching. It is, therefore, well within the province of the National Committee to inquire whether the prevailing type of examination in mathematics serves the best interests of mathematical education and of college preparation.

The reason for the almost controlling influence of entrance examinations in the schools referred to is readily recognized. Schools sending students to such colleges for men as Harvard, Yale, and Princeton, to the larger colleges for women, or to any institution where examinations form the only or prevailing mode of admission, inevitably direct their instruction toward the entrance examination. This remains true even if only a small percentage of the class intends to take these examinations, the point being that the success of a teacher is often measured by the success of his or her students in these examinations.

In the judgment of the committee, the prevailing type of entrance examination in algebra is primarily a test of the candidate's skill in formal manipulation, and not an adequate test of his understanding or of his ability to apply the principles of the subject. Moreover, it is quite generally felt that the difficulty and complexity of the formal manipulative questions, which have appeared on recent papers set by colleges and by such agencies as the College Entrance Examination Board, has often been excessive. As a result, teachers preparing pupils for these examinations have inevitably been led to devote an excessive amount of time to drill in algebraic technique, without insuring an adequate understanding of the principles involved. Far from providing the desired facility, this practice has tended to impair it. For "practical skill, modes of effective technique, can be intelligently, non-mechanically used only when intelligence has played a part in their acquisition." (Dewey, *How We Think,* p. 52.)

Moreover, it must be noted that authors and publishers of textbooks are under strong pressure to make their content and distribution of emphasis conform to the prevailing type of entrance examination. Teachers in turn are too often unable to rise above the textbook. An improvement in the examinations in this respect will cause a corresponding improvement in textbooks and in teaching.

On the other hand, the makers of entrance examinations in algebra cannot be held solely responsible for the condition described. Theirs is a most difficult problem. Not only can they reply that as long as algebra is taught as it is, examinations must be largely on technique[4], but they can also claim with considerable force that technical facility is the only phase of algebra that can be fairly tested by an examination; that a candidate can rarely do himself justice amid unfamiliar surroundings and subject to a time limit on questions involving real thinking in applying principles to concrete situations; and that we must face here a real limitation on the power of an examination to test attainment. Many, and perhaps most, teachers will agree with this claim. Past experience is on their side; no generally accepted and effective "power test" in mathematics has as yet been devised and, if devised, it might not be suitable for use under conditions prevailing during an entrance examination.

But if it is true that the power of an examination is thus inevitably limited, the wisdom and fairness of using it as the sole means of admission to college is surely open to grave doubt. That many unqualified candidates are admitted under this system is not open to question. Is it not probable that many qualified candidates are at the same time excluded? If the entrance examination is a fair test of manipulative skill only, should not the colleges use additional means for learning something about the candidate's other abilities and qualifications?

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[4] The vicious circle is now complete. Algebra is taught mechanically because of the character of the entrance examination; the examination, in order to be fair, must conform to the character of the teaching.

Some teachers believe that an effective "power test" in mathematics is possible. Efforts to devise such a test should receive every encouragement.

In the meantime, certain desirable modifications of the prevailing type of entrance examination are possible. The College Entrance Examination Board recently appointed a committee to consider this question and a conference[5] on this subject was held by representatives of the College Entrance Examination Board, members of the National Committee, and others. The following recommendations are taken from the report of the committee just referred to:

> Fully one-third of the questions should be based on topics of such fundamental importance that they will have been thoroughly taught, carefully reviewed, and deeply impressed by effective drill. . . . They should be of such a degree of difficulty that any pupil of regular attendance, faithful application, and even moderate ability may be expected to answer them satisfactorily.
>
> There should be both simple and difficult questions testing the candidate's ability to apply the principles of the subject. The early ones of the easy questions should be really easy for the candidate of good average ability who can do a little thinking under the stress of an examination; but even these questions should have genuine scientific content.
>
> There should be a substantial question which will put the best candidates on their mettle, but which is not beyond the reach of a fair proportion of the really good candidates. This question should test the normal workings of a well-trained mind. It should be capable of being thought out in the limited time of the examination. It should be a test of the candidate's grasp and insight not a catch question or a question of unfamiliar character making extraordinary demands on the critical powers of the candidate or one the solution of which depends on an inspiration. Above all, this question should lie near to the heart of the subject as all well-prepared candidates understand the subject.
>
> As a rule, a question should consist of a single part and be framed to test one thing—not pieced together out of several unrelated and perhaps unequally important parts.
>
> Each question should be a substantial test on the topic or topics which it represents. It is, however, in the nature of the case impossible that all questions be of equal value.
>
> Care should be used that the examination be not too long. * * * The examiner should be content to ask questions on the important topics, so chosen that their answers will be fair to the candidate and instructive to the readers; and beyond this merely to sample the candidate's knowledge of the minor topics.

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[5] At this conference the following vote was unanimously passed: "Voted, that the results of examinations (of the College Entrance Examination Board), be reported by letters A, B, C, D, E and that the definition of the groups represented by these letters should be determined in each year by the distribution of ability in a standard group of papers representing widely both public and private schools."

The National Committee suggests the following additional principles: The examination as a whole should, as far as practicable, reflect the principle that algebraic technique is a means to an end, and not an end in itself.

Questions that require of the candidate skill in algebraic manipulation beyond the needs of actual application should be used very sparingly.

An effort should be made to devise questions which will fairly test the candidate's understanding of principles and his ability to apply them, while involving a minimum of manipulative complexity.

The examinations in geometry should be definitely constructed to test the candidate's ability to draw valid conclusions rather than his ability to memorize an argument.

A chapter on mathematical terms and symbols is included in this report.[6] It is hoped that examining bodies will be guided by the recommendations there made relative to the use of terms and symbols in elementary mathematics.

The College Entrance Examination Board, early in 1921, appointed a commission to recommend such revisions as might seem necessary in the definitions of the requirements in the various subjects of elementary mathematics. The recommendations contained in the present chapter have been laid before this commission. It is hoped that the commission's report, when it is finally made effective by action of the College Entrance Examination Board and the various colleges concerned, will give impetus to the reorganization of the teaching of elementary mathematics along the lines recommended in the report of the National Committee.

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[6] See Chap. VIII.

CHAPTER VI

List of Propositions in Plane and Solid Geometry

General basis of the selection of material.—The subcommittee appointed to prepare a list of basal propositions made a careful study of a number of widely used textbooks on geometry. The bases of selection of the propositions were two: (1) The extent to which the propositions and corollaries were used in subsequent proofs of important propositions and exercises; (2) the value of the propositions in completing important pieces of theory. Although the list of theorems and problems is substantially the same in nearly all textbooks in general use in this country, the wording, the sequence, and the methods of proof vary to such an extent as to render difficult a definite statement as to the number of times a proposition is used in the several books examined. A tentative table showed, however, less variation than might have been anticipated.

Classification of propositions.—The classification of propositions is not the same in plane geometry as in solid geometry. This is partly due to the fact that it is generally felt that the student should limit his construction work to figures in a plane and in which the compasses and straight edge are sufficient. The propositions have been divided as follows:

Plane geometry: I. Assumptions and theorems for informal treatment; II. Fundamental theorems and constructions: A. Theorems, B. Constructions; III. Subsidiary theorems.

Solid geometry: I. Fundamental theorems; II. Fundamental propositions in mensuration; III. Sudsidiary theorems; IV. Sudsidiary propositions in mensuration.

Plane Geometry

I. Assumptions and theorems for informal treatment.—This list contains propositions which may be assumed without proof (postulates) and theorems which it is permissible to treat informally.

Some of these propositions will appear as definitions in certain methods of treatment. Moreover, teachers should feel free to require formal proofs in certain cases, if they desire to do so. The precise wording given is not essential, nor is the order in which the propositions are here listed. The list should be taken as representative of the type of propositions which may be assumed, or treated informally, rather than as exhaustive.

1. Through two distinct points it is possible to draw one straight line, and only one.
2. A line segment may be produced to any desired length.
3. The shortest path between two points is the line segment joining them.
4. One and only one perpendicular can be drawn through a given point to a given straight line.
5. The shortest distance from a point to a line is the perpendicular distance from the point to the line.
6. From a given center and with a given radius one and only one circle can be described in a plane.
7. A straight line intersects a circle in at most two points.
8. Any figure may be moved from one place to another without changing its shape or size.
9. All right angles are equal.
10. If the sum of two adjacent angles equals a straight angle, their exterior sides form a straight line.
11. Equal angles have equal complements and equal supplements.
12. Vertical angles are equal.
13. Two lines perpendicular to the same line are parallel.
14. Through a given point not on a given straight line, one straight line, and only one, can be drawn parallel to the given line.
15. Two lines parallel to the same line are parallel to each other.
16. The area of a rectangle is equal to its base times its altitude.

II. Fundamental theorems and constructions.—It is recommended that theorems and constructions (other than originals) to be proved on college entrance examinations be chosen from the following list. Originals and other exercises should be capable of solution by direct reference to one or more of these propositions and constructions. It should be obvious that any course in geometry that is capable of giving adequate training must include considerable additional material. The order here given is not intended to signify anything as to the order of presentation. It should be clearly understood that certain of the statements contain two or more theorems, and that the precise wording is not essential. The committee favors entire freedom in statement and sequence.

A. Theorems.

1. Two triangles are congruent if[1] (*a*) two sides and the included angle of one are equal, respectively, to two sides and the included angle of the other; (*b*) two angles and a side of one are equal, respectively, to two angles and the corresponding side of the other; (*c*) the three sides of one are equal, respectively, to the three sides of the other.

2. Two right triangles are congruent if the hypotenuse and one other side of one are equal, respectively, to the hypotenuse and another side of the other.

3. If two sides of a triangle are equal, the angles opposite these sides are equal; and conversely.[2]

4. The locus of a point (in a plane) equidistant from two given points is the perpendicular bisector of the line segment joining them.

5. The locus of a point equidistant from two given intersecting lines is the pair of lines bisecting the angles formed by these lines.

6. When a transversal cuts two parallel lines, the alternate interior angles are equal; and conversely.

7. The sum of the angles of a triangle is two right angles.

8. A parallelogram is divided into congruent triangles by either diagonal.

9. Any (convex) quadrilateral is a parallelogram (*a*) if the opposite sides are equal; (*b*) if two sides are equal and parallel.

10. If a series of parallel lines cut off equal segments on one transversal they cut off equal segments on any transversal.

11. (*a*) The area of a parallelogram is equal to the base times the altitude.

(*b*) The area of a triangle is equal to one-half the base times the altitude.

(*c*) The area of a trapezoid is equal to half the sum of its bases times its altitude.

(*d*) The area of a regular polygon is equal to half the product of its apothem and perimeter.

12. (*a*) If a straight line is drawn through two sides of a triangle parallel to the third side it divides these sides proportionally.

(*b*) If a line divides two sides of a triangle proportionally it is parallel to the third side. (Proofs for commensurable cases only.)

(*c*) The segments cut off on two transversals by a series of parallels are proportional.

13. Two triangles are similar if (*a*) they have two angles of one equal, respectively, to two angles of the other; (b) they have an angle of one equal to an angle of the other and the including sides are proportional; (*c*) their sides are respectively proportional.

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[1] Teachers should feel free to separate this theorem into three distinct theorems and to use other phraseology for any such proposition. For example, in 1, "Two triangles are equal if * * *", "a triangle is determined by * * *", etc. Similarly in 2, the statement might read: "Two right triangles are congruent if, besides the right angles, any two parts (not both angles) in the one are equal to corresponding parts of the other."

[2] It should be understood that the converse of a theorem need not be treated in connection with the theorem itself, it being sometimes better to treat it later. Furthermore, a converse may occasionally be accepted as true in an elementary course, if the necessity for proof is made clear. The proof may then be given later.

14. If two chords intersect in a circle, the product of the segments of one is equal to the product of the segments of the other.

15. The perimeters of two similar polygons have the same ratio as any two corresponding sides.

16. Polygons are similar, if they can be decomposed into triangles which are similar and similarly placed; and conversely.

17. The bisector of an (interior or exterior) angle of a triangle divides the opposite side (produced if necessary) into segments proportional to the adjacent sides.

18. The areas of two similar triangles (or polygons) are to each other as the squares of any two corresponding sides.

19. In any right triangle the perpendicular from the vertex of the right angle on the hypotenuse divides the triangle into two triangles each similar to the given triangle.

20. In a right triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides.

21. In the same circle or in equal circles, if two arcs are equal, their central angles are equal; and conversely.

22. In any circle angles at the center are proportional to their intercepted arcs. (Proof for commensurable case only.)

23. In the same circle or in equal circles, if two chords are equal their corresponding arcs are equal; and conversely.

24. (*a*) A diameter perpendicular to a chord bisects the chord and the arcs of the chord. (*b*) A diameter which bisects a chord (that is not a diameter) is perpendicular to it.

25. The tangent to a circle at a given point is perpendicular to the radius at that point; and conversely.

26. In the same circle or in equal circles, equal chords are equally distant from the center; and conversely.

27. An angle inscribed in a circle is equal to half the central angle having the same arc.

28. Angles inscribed in the same segment are equal.

29. If a circle is divided into equal arcs, the chords of these arcs form a regular inscribed polygon and tangents at the points of division form a regular circumscribed polygon.

31.[3] The area of a circle is equal to πr^2. (Informal proof only.)

30. The circumference of a circle is equal to $2\pi r$. (Informal proof only.)

The treatment of the mensuration of the circle should be based upon related theorems concerning regular polygons, but it should be informal as to the limiting processes involved. The aim should be an understanding of the concepts involved, so far as the capacity of the pupil permits.

B. Constructions.

1. Bisect a line segment and draw the perpendicular bisector.
2. Bisect an angle.

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[3] The total number of theorems given in this list when separated, as will probably be found advantageous in teaching, including the converses indicated, is 52.

3. Construct a perpendicular to a given line through a given point.
4. Construct an angle equal to a given angle.
5. Through a given point draw a straight line parallel to a given straight line.
6. Construct a triangle, given (*a*) the three sides; (*b*) two sides and the included angle; (*c*) two angles and the included side.
7. Divide a line segment into parts proportional to given segments.
8. Given an arc of a circle, find its center.
9. Circumscribe a circle about a triangle.
10. Inscribe a circle in a triangle.
11. Construct a tangent to a circle through a given point.
12. Construct the fourth proportional to three given line segments.
13. Construct the mean proportional between two given line segments.
14. Construct a triangle (polygon) similar to a given triangle (polygon).
15. Construct a triangle equal to a given polygon.
16. Inscribe a square in a circle.
17. Inscribe a regular hexagon in a circle.

III. Subsidiary list of propositions.—The following list of propositions is intended to suggest some of the additional material referred to in the introductory paragraph of Section II. It is not intended, however, to be exhaustive; indeed, the committee feels that teachers should be allowed considerable freedom in the selection of such additional material, theorems, corollaries, originals, exercises, etc., in the hope that opportunity will thus be afforded for constructive work in the development of courses in geometry.

1. When two lines are cut by a transversal, if the corresponding angles are equal or if the interior angles on the same side of the transversal are supplementary, the lines are parallel.

2. When a transversal cuts two parallel lines, the corresponding angles are equal, and the interior angles on the same side of the transversal are supplementary.

3. A line perpendicular to one of two parallels is perpendicular to the other also.

4. If two angles have their sides respectively parallel or respectively perpendicular to each other, they are either equal or supplementary.

5. Any exterior angle of a triangle is equal to the sum of the two opposite interior angles.

6. The sum of the angles of a convex polygon of n sides is $2(n-2)$ right angles.

7. In any parallelogram (*a*) the opposite sides are equal; (*b*) the opposite angles are equal; (*c*) the diagonals bisect each other.

8. Any (convex) quadrilateral is a parallelogram, if (*a*) the opposite angles are equal; (*b*) the diagonals bisect each other.

9. The medians of a triangle intersect in a point which is two-thirds of the distance from a vertex to the mid-point of the opposite side.

10. The altitudes of a triangle meet in a point.

11. The perpendicular bisectors of the sides of a triangle meet in a point.

12. The bisectors of the angles of a triangle meet in a point.

13. The tangents to a circle from an external point are equal.

14.[4] (*a*) If two sides of a triangle are unequal, the greater side has the greater angle opposite it, and conversely.

(*b*) If two sides of one triangle are equal respectively to two sides of another triangle, but the included angle of the first is greater than the included angle of the second, then the third side of the first is greater than the third side of the second, and conversely.

(*c*) If two chords are unequal, the greater is at the less distance from the center, and conversely.

(*d*) The greater of two minor arcs has the greater chord, and conversely.

15. An angle inscribed in a semicircle is a right angle.

16. Parallel lines cutting a circle intercept equal arcs on the circle.

17. An angle formed by a tangent of a circle and a chord drawn through the point of contact is measured by half the intercepted arc.

18. An angle formed by two intersecting chords is measured by half the sum of the intercepted arcs.

19. An angle formed by two secants or by two tangents to a circle is measured by half the difference between the intercepted arcs.

20. If from a point without a circle a secant and a tangent are drawn, the tangent is the mean proportional between the whole secant and its external segment.

21. Parallelograms or triangles of equal bases and equal altitudes are equal.

22. The perimeters of two regular polygons of the same number of sides are to each other as their radii and also as their apothems.

Solid Geometry

In the following list the precise wording and the sequence are not considered:

I. Fundamental Theorems.

1. If two planes meet, they intersect in a straight line.

2. If a line is perpendicular to each of two intersecting lines at their point of intersection it is perpendicular to the plane of the two lines.

3. Every perpendicular to a given line at a given point lies in a plane perpendicular to the given line at the given point.

4. Through a given point (internal or external) there can pass one and only one line perpendicular to a plane.

5. Two lines perpendicular to the same plane are parallel.

6. If two lines are parallel, every plane containing one of the lines and only one is parallel to the other.

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[4] Such inequality theorems as these are of importance in developing the notion of dependence or functionality in geometry. The fact that they are placed in the "Subsidiary list of propositions" should not imply that they are considered of less educational value than those in List II. They are placed here because they are not "fundamental" in the same sense that the theorems of List II are fundamental.

7. Two planes perpendicular to the same line are parallel.

8. If two parallel planes are cut by a third plane, the lines of intersection are parallel.

9. If two angles not in the same plane have their sides respectively parallel in the same sense, they are equal and their planes are parallel.

10. If two planes are perpendicular to each other, a line drawn in one of them perpendicular to their intersection is perpendicular to the other.

11. If a line is perpendicular to a given plane, every plane which contains this line is perpendicular to the given plane.

12. If two intersecting planes are each perpendicular to a third plane, their intersection is also perpendicular to that plane.

13. The sections of a prism made by parallel planes cutting all the lateral edges are congruent polygons.

14. An oblique prism is equal to a right prism whose base is equal to a right section of the oblique prism and whose altitude is equal to a lateral edge of the oblique prism.

15. The opposite faces of a parallelopiped are congruent.

16. The plane passed through two diagonally opposite edges of a parallelopiped divides the parallelopiped into two equal triangular prisms.

17. If a pyramid (or a cone) is cut by a plane parallel to the base:

(*a*) The lateral edges (or elements) and the altitude are divided proportionally;

(*b*) The section is a figure similar to the base;

(*c*) The area of the section is to the area of the base as the square of the distance from the vertex is to the square of the altitude of the pyramid (or cone).

18. Two triangular pyramids having equal bases and equal altitudes are equal.

19. All points on a circle of a sphere are equidistant from either pole of the circle.

20. On any sphere a point which is at a quadrant's distance from each of two other points not the extremities of a diameter is a pole of the great circle passing through these two points.

21. If a plane is perpendicular to a radius at its extremity on a sphere, it is tangent to the sphere.

22. A sphere can be inscribed in or circumscribed about any tetrahedron.

23. If one spherical triangle is the polar of another, then reciprocally the second is the polar triangle of the first.

24. In two polar triangles each angle of either is the supplement of the opposite side of the other.

25. Two symmetric spherical triangles are equal.

II. Fundamental Propositions in Mensuration.

26. The lateral area of a prism or a circular cylinder is equal to the product of a lateral edge or element, respectively, by the perimeter (circumference) of a right section.

27. The volume of a prism (including any parallelopiped) or of a circular cylinder is equal to the product of its base by its altitude.

28. The lateral area of a regular pyramid or a right circular cone is equal to half the product of its slant height by the perimeter (circumference) of its base.

29. The volume of a pyramid or a cone is equal to one-third the product of its base by its altitude.

30. The area of a sphere.

31. The area of a spherical polygon.

32. The volume of a sphere.

III. Subsidiary Theorems.

33. If from an external point a perpendicular and obliques are drawn to a plane, (*a*) the perpendicular is shorter than any oblique; (*b*) obliques meeting the plane at equal distances from the foot of the perpendicular are equal; (*c*) of two obliques meeting the plane at unequal distances from the foot of the perpendicular, the more remote is the longer.

34. If two lines are cut by three parallel planes, their corresponding segments are proportional.

35. Between two lines not in the same plane there is one common perpendicular, and only one.

36. The bases of a cylinder are congruent.

37. If a plane intersects a sphere, the line of intersection is a circle.

38. The volume of two tetrahedrons that have a trihedral angle of one equal to a trihedral angle of the other are to each other as the products of the three edges of these trihedral angles.

39. In any polyhedron the number of edges increased by two is equal to the number of vertices increased by the number of faces.

40. Two similar polyhedrons can be separated into the same number of tetrahedrons similar each to each and similarly placed.

41. The volumes of two similar tetrahedrons are to each other as the cubes of any two corresponding edges.

42. The volumes of two similar polyhedrons are to each other as the cubes of any two corresponding edges.

43. If three face angles of one trihedral angle are equal, respectively, to the three face angles of another the trihedral angles are either congruent or symmetric.

44. Two spherical triangles on the same sphere are either congruent or symmetric if (*a*) two sides and the included angle of one are equal to the corresponding parts of the other; *(b)* two angles and the included side of one are equal to the corresponding parts of the other; *(c)* they are mutually equilateral; *(d)* they are mutually equiangular.

45. The sum of any two face angles of a trihedral angle is greater than the third face angle.

46. The sum of the face angles of any convex polyhedral angle is less than four right angles.

47. Each side of a spherical triangle is less than the sum of the other two sides.

48. The sum of the sides of a spherical polygon is less than 360°.

49. The sum of the angles of a spherical triangle is greater than 180° and less than 540°.

50. There can not be more than five regular polyhedrons.

51. The locus of points equidistant (*a*) from two given points; (*b*) from two given planes which intersect.

IV. Subsidiary Propositions in Mensuration

52. The volume of a frustum of *(a)* a pyramid or *(b)* a cone.

53. The lateral area of a frustum of *(a)* pyramid or *(b)* a cone of revolution.

54. The volume of a prismoid (without formal proof).

CHAPTER VII

The Function Concept in Secondary School Mathematics[1]

In Chapter II, and incidentally in later chapters, considerable emphasis has been placed on the function concept or, better, on the idea of relationship between variable quantities as one of the general ideas that should dominate instruction in elementary mathematics. Since this recommendation is peculiarly open to misunderstanding on the part of teachers, it seems desirable to devote a separate chapter to a rather detailed discussion of what the recommendation means and implies.

It will be seen in what follows that there is no disposition to advocate the teaching of any sort of function *theory*. A prime danger of misconception that should be removed at the very outset is that teachers may think it is the notation and the definitions of such a theory that are to be taught. Nothing could be further from the intention of the committee. Indeed, it seems entirely safe to say that the word "function" had best not be used at all in the early courses.

What is desired is that the idea of relationship or dependence between variable quantities be imparted to the pupil by the examination of numerous concrete instances of such relationship. He must be shown the workings of relationships in a large number of cases before the abstract idea of relationship will have any meaning for him. Furthermore, the pupil should be led to form the habit of thinking about the connections that exist between related quantities, not merely because such a habit forms the best foundation for a real appreciation of the theory that may follow later, but chiefly because this habit will enable him to think more clearly about the quantities with which he will have to deal in real life, whether or not he takes any further work in mathematics.

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[1] The first draft of this chapter was prepared for the National Committee by E. R. Hedrick, of the University of Missouri. It was discussed at the meeting of the Committee, Sept. 2-4, 1920; revised by the author, and again discussed Dec. 29-30, 1920, and is now issued as part of the committee's report.

Indeed, the reason for insisting so strongly upon attention to the idea of relationships between quantities is that such relationships do occur in real life in connection with practically all of the quantities with which we are called upon to deal in practice. Whereas there can be little doubt about the small value to the student who does not go on to higher studies of some of the manipulative processes criticized by the National Committee, there can be no doubt at all of the value to all persons of any increase in their ability to see and to foresee the manner in which related quantities affect each other.

To attain what has been suggested, the teacher should have in mind constantly not any definition to be recited by the pupil, not any automatic response to a given cue, not any memory exercise at all, but rather a determination not to pass any instance in which one quantity is related to another, or in which one quantity is determined by one or more others, without calling attention to the fact, and trying to have the student "see how it works." These instances occur in literally thousands of cases in both algebra and geometry. It is the purpose of this chapter to outline in some detail a few typical instances of this character.

I. Relationships in Algebra

In algebra the instance of the function idea which usually occurs to one first is in connection with the study of graphs. While this is natural enough, and while it is true that the graph is fundamentally functional in character, the supposition that it furnishes the first opportunity for observing functional relations between quantities betrays a misconception that ought to be corrected.

1. **Use of letters for numbers.**—The very first illustrations given in algebra to show the use of letters in the place of numbers are essentially functional in character. Thus, such relations as $I = prt$ and $A = \pi r^2$, as well as others that are frequently used, are statements of general relationships. These should be used to accustom the student not only to the use of letters in the place of numbers and to the solution of simple numerical problems, but also to the idea, for example, that changes in r affect the value of A. Such questions as the following should be considered: If r is doubled, what will happen to A? If p is doubled, what will happen to I? Appreciation of the meaning of such relationships will tend to clarify the entire subject under consideration. Without such an appreciation, it may be doubted whether the student has any real grasp of the matter.

2. **Equations.**—Every simple problem leading to an equation in the first part of algebra would be better understood for just such a discussion as that mentioned above. Thus, if two dozen eggs are weighed in a basket which weighs 2 pounds, and if the total weight is found to be 5 pounds, what is the average weight of an egg? If x is the weight in ounces of one egg, the total weight with the 2-pound basket would be $24x + 32$ ounces. If the student will first try the effect of an average weight of 1 ounce, of 1½ ounces, 2 ounces, 2½ ounces, the meaning of the problem will stand out clearly. In every such problem, preliminary trials really amount to a discussion of the properties of a linear function.

3. **Formulas of pure science and of practical affairs.**—The study of formulas as such, aside from their numerical evaluation, is becoming of more and more importance. The actual uses of algebra are not to be found solely nor even principally in the solution of numerical problems for numerical answers. In such formulas as those for falling bodies, levers, etc., the manner in which changes in one quantity cause (or correspond to) changes in another, are of prime importance, and their discussion need cause no difficulty whatever. The formulas under discussion here include those formulas of pure science and of practical affairs which are being introduced more and more into our texts on algebra. Whenever such a formula is encountered, the teacher should be sure that the students have some comprehension of the effects of changes in one of the quantities upon the other quantity or quantities in the formula.

As a specific instance of such scientific formulas, consider, for example, the force F, in pounds, with which a weight W, in pounds, pulls outward on a string (centrifugal force) if the weight is revolved rapidly at a speed v, in feet per second, at the end of a string of length r feet. This force is given by the formula

$$F = \frac{W v^2}{32 r}$$

When such a formula is used the teacher should not be contented with the mere insertion of numerical values for W, v, and r to obtain a numerical value for F.

The advantage obtained from the study of such a formula lies quite as much in the recognition of the behavior of the force when one of the other quantities varies. Thus the student should be able to answer intelligently such questions as the following: If the weight is assumed to be twice as heavy, what is the effect upon the force? If the speed is taken twice as great, what is the effect upon the force? If the radius becomes twice as large, what is the effect upon the force? If the speed is doubled, what change in the weight would result in the same force? Will an increase in the speed cause an increase or decrease in the force? Will an increase in the radius r cause an increase or a decrease in the force?

As another instance (of a more advanced character) consider the formula for the amount of a sum of money P, at compound interest at r per cent, at the end of n years. This amount may be denoted by A_n. Then we shall have $A_n = P(1+r)^n$. Will doubling P result in doubling A_n? Will doubling n result in doubling A_n? Since the compound interest that has accumulated is equal to the difference between P and A_n, will the doubling of r double the interest? Compare the correct answers to these questions with the answers to the similar questions in the case of simple interest, in which the formula reads $A_n = P + Prn$ and in which the accumulated interest is simply Prn.

The difference between such a study of the effect produced upon one quantity by changes in another and the mere substitution of numerical values will be apparent from these examples.

4. Formulas of pure algebra.—Formulas of pure algebra, such as that for $(x+h)^2$, will be better understood and appreciated if accompanied by a discussion of the manner in which changes in h cause changes in the total result. This can be accomplished by discussing such concrete realities as the error made in computing the area of a square field or of a square room when an error has been made in measuring the side of the square. If x is the true length of the side, and if the student assumes various possible values for the error h made in measuring x, he will have a situation that involves some comprehension of the functional workings of the formula mentioned. The same formula relates to such problems as the change from one entry to the next entry in a table of squares.

A similar situation, and a very important one, occurs with the pure algebraic formula for $(x+a)(y+b)$. This formula may be said to govern the question of the keeping of significant figures in finding the product xy. For if a and b represent the uncertainty in x and y, respectively, the uncertainty in the product is given by this formula. The student has much to learn on this score, for the retention of meaningless figures in a product is one of the commonest mistakes of both student and teacher in computational work.

Such formulas occur throughout algebra, and each of them will be illuminated by such a discussion. The formulas for arithmetic and geometric progression, for example, should be studied from a functional standpoint.

5. **Tables.**—The uses of the functional idea in connection with numerical computation have already been mentioned in connection with the formula for a product. Work which appears on the surface to be wholly numerical may be of a distinctly functional character. Thus any table, e. g., a table of squares, corresponds to or is constructed from a functional relation, e. g., for a table of squares, the relation $y = x^2$. The differences in such a table are the differences caused by changes in the values of the independent variable. Thus, the differences in a table of squares are precisely the differences between x^2 and $(x + h)^2$ for various values of x.

6. **Graphs.**—The functional character of graphic representations was mentioned at the beginning of this section. Every graph is obviously a representation of a functional relationship between two or more quantities. What is needed is only to draw attention to this fact and to study each graph from this standpoint. In addition to this, however, it is desirable to point out that functional relations may be studied directly by means of graphs without the intervention of any algebraic formula. Thus, such a graph as a popuation curve, or a curve representing wind pressure, obviously represents a relationship between two quantities, but there is no known formula in either case. The idea that the three concepts, tables, graphs, algebraic formulas, are all representations of the same kind of connection between quantities, and that we may start in some instances with any of the three, is a most valuable addition to the student's mental equipment, and to his control over the quantities with which he will deal in his daily life.

II. Relationships in Geometry

Thus far the instances mentioned have been largely algebraic, though certain mensuration formulas of geometry have been mentioned. While the mensuration formulas may occur to one first as an illustration of functional concepts in geometry, they are by no means the earliest relationships that occur in that study.

1. Congruence.—Among the earliest theorems are those on the congruence of triangles. In any such theorem, the parts necessary to establish congruence evidently determine completely the size of each other part. Thus, two sides and the included angle of a triangle evidently determine the length of the third side. If the student clearly grasps this fact, the meaning of this case of congruence will be more vivid to him, and he will be prepared for its important applications in surveying and in trigonometry. Even if he never studies those subjects, he will nevertheless be able to use his understanding of the situation in any practical cases in which the angle between two fixed rods or beams is to be fixed or is to be determined, in a practical situation such as house building. Other congruence theorems throughout geometry may well be treated in a similar manner.

2. Inequalities.—In the theorems regarding inequalities, the functional quality is even more pronounced. Thus, if two triangles have two sides of one equal respectively to two sides of the other, but if the included angle between these sides in the one triangle is greater than the corresponding angle in the other, then the third sides of the triangles are unequal in the same sense. This theorem shows that as one angle grows, the side opposite it grows, if the other sides remain unchanged. A full realization of the fact here mentioned would involve a real grasp of the functional relation between the angle and the side opposite it. Thus, if the angle is doubled, will the side opposite it be doubled? Such questions arise in connection with all theorems on inequalities.

3. Variations in figures.—A great assistance to the imagination is gained in certain figures by imagining variations of the figure through all intermediate stages from one case to another. Thus, the angle between two lines that cut a circle is measured by a proper combination of the two arcs cut out of the circle by the two lines. As the vertex of the angle passes from the center of the circle to the circumference and thence to the outside of the circle, the rule changes, but these changes may be borne in mind, and the entire scheme may be grasped, by imagining a continuous change from the one position to the other, following all the time the changes in the intercepted arcs. The angle between a secant and a tangent is measured in a manner that can best be grasped by another such continuous motion, watching the changes in the measuring arcs as the motion occurs. Such observations are essentially functional in character, for they consist in careful observations of the relationships between the angle to be measured and the arcs that measure it.

4. **Motion.**—The preceding discussion of variable figures leads naturally to a discussion of actual motion. As figures move, either i whole or in part, the relationships between the quantities involved may change. To note these changes is to study the functional relationship between the parts of the figures. Without the functional idea, geometr would be wholly *static*. The study of fixed figures should not be the sole purpose of a course in geometry, for the uses of geometry are no wholly on static figures. Indeed, in all machinery, the geometri figures formed are in continual motion, and the shapes of the figure formed by the moving parts change. The study of motion and of moving forms, the *dynamic* aspects of geometry, should be given at leas some consideration. Whenever this is done, the functional relation between the parts become of prime importance. Thus a linkage of th form of a parallelogram can be made more nearly rectangular by makin the diagonals more nearly equal, and the linkage becomes a rectangle i the diagonals are made exactly equal. This principle is used in practic in making a rectangular framework precisely true.

5. **Proportionality theorems.**—All theorems which assert tha certain quantities are in proportion to certain others, are obviously functional in character. Thus even the simplest theorems on rectangle assert that the area of a rectangle is directly proportional to its height, i the base is fixed. When more serious theorems are reached, such a the theorems on similar triangles, the functional ideas involved ar worthy of considerable attention. That this is eminently true will b realized by all to whom trigonometry is familiar, for the trigonometri functions are nothing but the ratios of the sides of right triangles. Bu even in the field of elementary geometry a clear understanding of th relation between the areas (and volumes) of similar figures and th corresponding linear dimensions is of prime importance.

III. Relationships in Trigonometry

The existence of functional relationships in trigonometry is evidenced by the common use of the words "trigonometric functions' to describe the trigonometric ratios. Thus the sine of an angle is definite ratio whose value depends upon and is determined by the siz of the angle to which it refers. The student should be made conscious of this relationship and he should be asked such questions as the following: Does the sine of an angle increase or decrease as the angle changes from zero to 90°? If the angle is doubled, does the sine of the angle double? If not, is the sine of double the angle more or less than twice the sine of the original angle? How does the value of the sine behave as the angles increase from 90° to 180°? From 180° to 270°? From 270° to 360°? Similar questions may be asked for the cosine and for the tangent of an angle.

Such questions may be reinforced by the use of figures that illustrate the points in question. Thus an angle twice a given angle should be drawn, and its sine should be estimated from the figure. A central angle and an inscribed angle on the same arc may be drawn in any circle. If they have one side in common, the relations between their sines will be more apparent. Finally, the relationships that exist may be made vivid by actual comparison of the numerical values found from the trigonometric tables.

Not only in these first functional definitions, however, but in a variety of geometric figures throughout trigonometry do functional relations appear. Thus the law of cosines states a definite relationship between the three sides of a triangle and any one of the angles. How will the angle be affected by increase or decrease of the side opposite it, if the other two sides remain fixed? How will the angle be affected by an increase or a decrease of one of the adjacent sides, if the other two sides remain fixed? Are these statements still true if the angle in question is obtuse?

As another example, the height of a tree, or the height of a building, may be determined by measuring the two angles of elevation from two points on the level plain in a straight line with its base. A formula for the height (h) in terms of these two angles (A, B) and the distance (d) between the points of observation, may be easily written down [$h = d \sin A \sin B / \sin(A—B)$]. Then the effect upon the height of changes in one of these angles may be discussed.

In a similar manner, every formula that is given or derived in a course on trigonometry may be discussed with profit from the functional standpoint.

IV. Conclusion

In conclusion, mention should be made of the great rôle which the idea of functions plays in the life of the world about us. Even when

no calculation is to be carried out, the problems of real life frequentl
involve the ability to think correctly about the nature of the relation
ships which exist between related quantities. Specific mention ha
been made already of this type of problem in connection with intere
on money. In everyday affairs, such as the filling out of formula
for fertilizers or for feeds or for spraying mixtures on the farm, th
similar filling out of recipes for cooking (on different scales from tha
of the book of recipes), or the proper balancing of the ration in th
preparation of food, many persons are at a loss on account of the
lack of training in thinking about the relations between quantitie
Another such instance of very common occurrence in real life is i
insurance. Very few men or women attempt intelligently to unde
stand the meaning and the fairness of premiums on life insurance an
on other forms of insurance, chiefly because they cannot readily gras
the relations of interest and of chance that are involved. These relation
are not particularly complicated, and they do not involve any gre
amount of calculation for the comprehension of the meaning and of th
fairness of the rates. Mechanics, farmers, merchants, housewives, a
well as scientists and engineers have to do constantly with quantitie
of things, and the quantities with which they deal are related to othe
quantities in ways that require clear thinking for maximum efficiency.

One element that should not be neglected is the occurrence of suc
problems in public questions which must be decided by the votes o
the whole people. The tariff, rates of postage and express, freigh
rates, regulation of insurance rates, income taxes, inheritance taxe
and many other public questions involve relationships between quan
tities—for example, between the rate of income taxation and the amoun
of the income—that require habits of functional thinking for intelligen
decisions. The training in such habits of thinking is therefore a vita
element toward the creation of good citizenship.

It is believed that transfer of training does operate between suc
topics as those suggested in the body of this paper and those jus
mentioned, because of the existence of such identical or common ele
ments, whereas the transfer of the training given by courses in mathe
matics that do not emphasize functional relationships might be ques
tionable.

While this account of the functional character of certain topics i
geometry and in algebra makes no claim to being exhaustive, the topic
mentioned will suggest others of like character to the thoughtful teache

It is hoped that sufficient variety has been mentioned to demonstrate the existence of functional ideas throughout elementary algebra and geometry. The committee feels that if this is recognized, algebra and geometry can be given new meaning to many children, and indeed to many educators, and that all students will be better able to control the actual relations which they meet in their own lives.

CHAPTER VIII

Terms and Symbols in Elementary Mathematics[1]

A. Limitations imposed by the committee upon its work.—The committee feels that in dealing with this subject it should explicitly recognize certain general limitations, as follows:

1. No attempt should be made to impose the phraseology of any definition, although the committee should state clearly its general views as to the meaning of disputed terms.

2. No effort should be made to change any well-defined current usage unless there is a strong reason for doing so, which reason is supported by the best authority, and, other things being substantially equal, the terms used should be international. This principle excludes the use of all individual efforts at coining new terms except under circumstances of great urgency. The individual opinions of the members, as indeed of any teacher or body of teachers, should have little weight in comparison with general usage if this usage is definite. If an idea has to be expressed so often in elementary mathematics that it becomes necessary to invent a single term or symbol for the purpose, this invention is necessarily the work of an individual; but it is highly desirable, even in this case, that it should receive the sanction of wide use before it is adopted in any system of examinations.

3. On account of the large number of terms and symbols now in use, the recommendations to be made will necessarily be typical rather than exhaustive.

I. Geometry

B. Undefined terms.—The committee recommends that no attempt be made to define, with any approach to precision, terms whose definitions are not needed as parts of a proof.

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[1] The first draft of this chapter was prepared by a subcommittee consisting of David Eugene Smith (chairman), W. W. Hart, H. E. Hawkes, E. R. Hedrick, and H. E. Slaught.

Especially is it recommended that no attempt be made to define precisely such terms as *space, magnitude, point, straight line, surface, plane, direction, distance,* and *solid,* although the significance of such terms should be made clear by informal explanations and discussions.

C. Definite usage recommended.—It is the opinion of the committee that the following general usage is desirable:

1. *Circle* should be considered as the curve; but where no ambiguity arises, the word "circle" may be used to refer either to the curve or to the part of the plane inclosed by it.

2. *Polygon* (including *triangle, square, parallelogram,* and the like) should be considered, by analogy to a circle, as a closed broken line; but where no ambiguity arises, the word polygon may be used to refer either to the broken line or to the part of the plane inclosed by it. Similarly, *segment of a circle* should be defined as the figure formed by a chord and either of its arcs.

3. *Area of a circle* should be defined as the area (numerical measure) of the portion of the plane inclosed by the circle. *Area of a polygon* should be treated in the same way.

4. *Solids.* The usage above recommended with respect to plane figures is also recommended with respect to solids. For example, *sphere* should be regarded as a surface, its volume should be defined in a manner similar to the area of a circle, and the double use of the word should be allowed where no ambiguity arises. A similar usage should obtain with respect to such terms as *polyhedron, cone,* and *cylinder.*

5. *Circumference* should be considered as the length (numerical measure) of the circle (line). Similarly, *perimeter* should be defined as the length of the broken line which forms a polygon; that is, as the sum of the lengths of the sides.

6. *Obtuse angle* should be defined as an angle greater than a right angle and less than a straight angle, and should therefore not be defined merely as an angle greater than a right angle.

7. The term *right triangle* should be preferred to "right-angled triangle," this usage being now so standardized in this country that it may properly be continued in spite of the fact that it is not international. Similarly for *acute triangle, obtuse triangle,* and *oblique triangle.*

8. Such English plurals as *formulas* and *polyhedrons* should used in place of the Latin and Greek plurals. Such unnecessary Lat abbreviations as *Q. E. D.* and *Q. E. F.* should be dropped.

9. The definitions of *axiom* and *postulate* vary so much that tl committee does not undertake to distinguish between them.

D. Terms made general.—It is the recommendation of the con mittee that the modern tendency of having terms made as general possible should be followed. For example:

1. *Isosceles triangle* should be defined as a triangle having tw equal sides. There should be no limitation to two and only two equ sides.

2. *Rectangle* should be considered as including a square as special case.

3. *Parallelogram* should be considered as including a rectangl and hence a square, as a special case.

4. *Segment* should be used to designate the part of a straight lir included between two of its points as well as the figure formed by a arc of a circle and its chord, this being the usage generally recognize by modern writers.

E. Terms to be abandoned.—It is the opinion of the con mittee that the following terms are not of enough consequence in ele mentary mathematics at the present time to make their recognitio desirable in examinations, and that they serve chiefly to increase th technical vocabulary to the point of being burdensome and unnecessary

1. *Antecedent* and *consequent.*

2. *Third proportional* and *fourth proportional.*

3. *Equivalent.* An unnecessary substitute for the more precis expressions "equal in area" and "equal in volume," or (where no con fusion is likely to arise) for the single word "equal."

4. *Trapezium.*

5. *Scholium, lemma, oblong, scalene triangle, sect, perigon, rhom boid* (the term "oblique parallelogram" being sufficient), and *refle angle* (in elementary geometry).

6. Terms like *flat angle, whole angle,* and *conjugate angle* are nc of enough value in an elementary course to make it desirable to rec ommend them.

7. *Subtend,* a word which has no longer any etymological meaning to most students and teachers of geometry. While its use will naturally continue for some time to come, teachers may safely incline to such forms as the following: "In the same circle equal arcs *have* equal chords."

8. *Homologous,* the less technical term "corresponding" being preferable.

9. Guided by principle A2 and its interpretation, the committee advises against the use of such terms as the following: *Angle-bisector, angle-sum, consecutive interior angles, supplementary consecutive exterior angles, quader* (for rectangular solid), *sect, explement, tranverse angles.*

10. It is unfortunate that it still seems to be necessary to use such a term as *parallelepiped,* but we seem to have no satisfactory substitute. For rectangular parallelepiped, however, the use of *rectangular solid* is recommended. If the terms were more generally used in elementary geometry it would be desirable to consider carefully whether better ones could not be found for the purposes than *isoperimetric, apothem, icosahedron,* and *dodecahedron.*

F. Symbols in elementary geometry.—It should be recognized that a symbol like ⊥ is merely a piece of shorthand designed to afford an easy grasp of a written or printed statement. Many teachers and a few writers make an extreme use of symbols, coining new ones to meet their own views as to usefulness, and this practice is naturally open to objection.[2] There are, however, certain symbols that are international and certain others of which the meaning is at once apparent and which are sufficiently useful and generally enough recognized to be recommended.

For example, the symbols for triangle, △, and circle, ⊙, are international, although used more extensively in the United States than in other countries. Their use, with their customary plurals, is recommended.

The symbol ⊥, generally read as representing the single word "perpendicular" but sometimes as standing for the phrase "is perpendicular to," is fairly international and the meaning is apparent. Its use is therefore recommended. On account of such a phrase as "the

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[2] This is not intended to discourage the use of algebraic methods in the solution of such geometric problems as lend themselves readily to algebraic treatment.

$\perp AB$," the first of the above readings is likely to be the more widely used, but in either case there is no chance for confusion.

The symbol $\parallel$ for "parallel" or "is parallel to" is fairly international and is recommended.

The symbol $\sim$ for "similar" or "is similar to" is international and is recommended.

The symbols $\cong$ and $\equiv$ for "congruent" or "is congruent to" both have a considerable use in this country. The committee feels that the former, which is fairly international, is to be preferred because it is the more distinctive and suggestive.

The symbol $\angle$ for "angle" is, because of its simplicity, coming to be generally preferred to any other and is therefore recommended.

Since the following terms are not used frequently enough to render special symbols of any particular value the world has not developed any that have general acceptance and there seems to be no necessity for making the attempt: Square, rectangle, parallelogram, trapezoid, quadrilateral, semicircle.

The symbol $\overset{\frown}{AB}$ for "arc AB" cannot be called international. While the value of the symbol $\frown$ in place of the short word *arc* is doubtful, the committee sees no objection to its use.

The symbol $\therefore$ for "therefore" has a value that is generally recognized, but the symbol $\because$ for "since" is used so seldom that it should be abandoned.

With respect to the lettering of figures, the committee calls attention for purposes of general information to a convenient method, found in certain European and in some American textbooks, of lettering triangles: Capitals represent the vertices, corresponding small letters represent opposite sides, corresponding small Greek letters represent angles, and the primed letters represent the corresponding parts of a congruent or similar triangle. This permits of speaking of α (alpha) instead of "angle A," and of "small a" instead of BC. The plan is by no means international with respect to the Greek letters. The committee is prepared, however, to recommend it with the optional use of the Greek forms.

In general, it is recommended that a single letter be used to designate any geometric magnitude whenever there is no danger of ambiguity. The use of numbers alone to designate magnitudes should be avoided by the use of such forms A_1, A_2, ...

With respect to the symbolism for limits, the committee calls attention to the fact that the symbol ≐ is a local one, and that the symbol → (for "tends to") is both international and expressive and has constantly grown in favor in recent years. Although the subject of limits is not generally treated scientifically in the secondary school, the idea is mentioned in geometry and a symbol may occasionally be needed.

While the teacher should be allowed freedom in the matter, the committee feels that it is desirable to discourage the use of such purely local symbols as the following:

≗ for "equal in degrees,"
ass for "two sides and an angle adjacent to one of them," and
sas for "two sides and the included angle."

G. Terms not standardized.—At the present time there is not sufficient agreement upon which to base recommendations as to the use of the term *ray* and as to the value of terms like *coplanar*, *collinear*, and *concurrent* in elementary work. Many terms, similar to these, will gradually become standardized or else will naturally drop out of use.

II. Algebra and Arithmetic

H. Terms in algebra.—1. With respect to equations the committee calls attention to the fact that the classification according to degree is comparatively recent and that this probably accounts for the fact that the terminology is so unsettled. The Anglo-American custom of designating an equation of the first degree as a *simple equation* has never been satisfactory, because the term has no real significance. The most nearly international terms are *equation of the first degree* (or "first degree equation") and *linear equation*. The latter is so brief and suggestive that it should be generally adopted.

2. The term *quadratic equation* (for which the longer term "second degree equation" is an unnecessary synonym, although occasionally a convenient one) is well established. The terms *pure quadratic* and *affected quadratic* signify nothing to the pupil except as he learns the meaning from a book, and the committee recommends that they be dropped. Terms more nearly in general use are *complete quadratic* and *incomplete quadratic*. The committee feels, however, that the distinction thus denoted is not of much importance and believes that it can well be dispensed with in elementary instruction.

3. As to other special terms, the committee recommends abandoning, so far as possible, the use of the following: *Aggregation* for grouping; *vinculum* for bar; *evolution* for finding roots, as a general topic; *involution* for finding powers; *extract* for find (a root); *absolute term* for constant term; *multiply an equation, clear of fractions, cancel* and *transpose,* at least until the significance of the terms is entirely clear; *aliquot part* (except in commercial work).

4. The committee also advises the use of either *system of equations* or *set of equations* instead of "simultaneous equations," in such an expression as "solve the following set of equations," in view of the fact that at present no well established definite meaning attaches to the term "simultaneous."

5. The term *simplify* should not be used in cases where there is possibility of misunderstanding. For purposes of computation, for example, the form $\sqrt{8}$ may be simpler than the form $2\sqrt{2}$, and in some cases it may be better to express $\sqrt{\frac{3}{4}}$ as $\sqrt{0.75}$ instead of $\frac{1}{2}\sqrt{3}$. In such cases, it is better to give more explicit instructions than to use the misleading term "simplify."

6. The committee regrets the general uncertainty in the use of the word *surd,* but it sees no reasonable chance at present of replacing it by a more definite term. It recognizes the difficulty generally met by young pupils in distinguishing between *coefficient* and *exponent,* but it feels that it is undesirable to attempt to change terms which have come to have a standardized meaning and which are reasonably simple. These considerations will probably lead to the retention of such terms as *rationalize, extraneous root, characteristic,* and *mantissa,* although in the case of the last two terms "integral part" and "fractional part" (of a logarithm) would seem to be desirable substitutes.

7. While recognizing the motive that has prompted a few teachers to speak of "positive x" instead of "plus x", and "negative y" instead of "minus y" the committee feels that attempts to change general usage should not be made when based upon trivial grounds and when not sanctioned by mathematicians generally.

I. Symbols in algebra.—The symbols in elementary algebra are now so well standardized as to require but few comments in a report of this kind. The committee feels that it is desirable. however, to call attention to the following details:

1. Owing to the frequent use of the letter x, it is preferable to use the center dot (a raised period) for multiplication in the few cases in which any symbol is necessary. For example, in a case like $1 \cdot 2 \cdot 3 \cdot \ldots (x-1) \cdot x$, the center dot is preferable to the symbol $\times$; but in cases like $2a(x-a)$ no symbol is necessary. The committee recognizes that the period (as in $a.b$) is more nearly international than the center dot (as in $a \cdot b$); but inasmuch as the period will continue to be used in this country as a decimal point, it is likely to cause confusion, to elementary pupils at least, to attempt to use it as a symbol for multiplication.

2. With respect to division, the symbol $\div$ is purely Anglo-American, the symbol $:$ serving in most countries for division as well as ratio. Since neither symbol plays any part in business life, it seems proper to consider only the needs of algebra, and to make more use of the fractional form and (where the meaning is clear) of the symbol $/$, and to drop the symbol $\div$ in writing algebraic expressions.

3. With respect to the distinction between the use of $+$ and $-$ as symbols of operation and as symbols of direction, the committee sees no reason for attempting to use smaller signs for the latter purpose, such an attempt never having received international recognition, and the need of two sets of symbols not being sufficient to warrant violating international usage and burdening the pupil with this additional symbolism.

4. With respect to the distinction between the symbols $\equiv$ and $=$ as representing respectively identity and equality, the committee calls attention to the fact that, while the distinction is generally recognized, the consistent use of the symbols is rarely seen in practice. The committee recommends that the symbol $\equiv$ be not employed in examinations for the purpose of indicating identity. The teacher, however, should use both symbols if desired.

5. With respect to the root sign, $\sqrt{\ }$, the committee recognizes that convenience of writing assures its continued use in many cases instead of the fractional exponent. It is recommended, however, that in algebraic work involving complicated cases the fractional exponent be preferred. Attention is also called to the fact that the convention is quite generally accepted that the symbol $\sqrt{a}$ (a representing a positive number) means only the positive square root and that the symbol $\sqrt[n]{a}$ means only the principal nth root, and similarly for $a^{\frac{1}{2}}$, $a^{\frac{1}{n}}$.

The reason for this convention is apparent when we come to consider the value of $\sqrt{4}+\sqrt{9}+\sqrt{16}+\sqrt{25}$. This convention being agreed to, it is improper to write $x=\sqrt{4}$, as the complete solution of $x^2-4=0$, but the result should appear as $x=\pm\sqrt{4}$. Similarly, it is not in accord with the convention to write $\sqrt{4}=\pm 2$, the conventional form being $\pm\sqrt{4}=\pm 2$; and for the same reason it is impossible to have $\sqrt{(-1)^2}=\pm 1$ since the symbol refers only to a positive root. These distinctions are not matters to be settled by the individual opinion of a teacher or a local group of teachers; they are purely matters of convention as to notation, and the committee simply sets forth, for the benefit of teachers, this statement as to what the convention seems to be among the leading writers of the world at the present time.

When imaginaries are used, the symbol i should be employed instead of $\sqrt{-1}$ except possibly in the first presentation of the subject.

7. As to the factorial symbols $5!$ and $\underline{|5}$, to represent $5\cdot 4\cdot 3\cdot 2\cdot 1$, the tendency is very general to abandon the second one, probably on account of the difficulty of printing it, and the committee so recommends. This question is not, however, of much importance in the general courses in the high school.

8. With respect to symbols for an unknown quantity there has been a noteworthy change within a few years. While the Cartesian use of x and y will doubtless continue for two general unknowns, the recognition that the formula is, in the broad use of the term, a central feature of algebra has led to the extended use of the initial letter. This is simply illustrated in the direction to solve for r the equation $A=\pi r^2$. This custom is now international and should be fully recognized in the schools.

9. The committee advises abandoning the double colon (: :) in proportion, and the symbol $\propto$ in variation, both of these symbols being now practically obsolete.

J. Terms and symbols in arithmetic.—1. While it is rarely wise to attempt to abandon suddenly the use of words that are well established in our language, the committee feels called upon to express regret that we still require very young pupils, often in the primary grades, to use such terms as *subtrahend*, *addend*, *minuend* and *multi-*

plicand. Teachers themselves rarely understand the real significance of these words, nor do they recognize that they are comparatively modern additions to what used to be a much simpler vocabularly in arithmetic. The committee recommends that such terms be used, if at all, only after the sixth grade.

2. Owing to the uncertainty attached to such expressions as "to three decimal places," "to thousandths," "correct to three decimal places," "correct to the nearest thousandth," the following usage is recommended: When used to specify accuracy in computation, the four expressions should be regarded as identical. The expression "to three decimal places" or "to thousandths" may be used in giving directions as to the extent of a computation. It then refers to a result carried only to thousandths, without considering the figure of ten-thousandths; but it should be avoided as far as possible because it is open to misunderstanding. As to the term, "significant figure," it should be noted that 0 is always significant except when used before a decimal fraction to indicate the absence of integers or, in general, when used merely to locate the decimal point. For example the zeros italicised in the following are "significant," while the others are not: 0.5, 9.5*0*, 1*0*2, 3*0*,200. Further, the number 2396, if expressed correct to three significant figures, would be written 24*0*0.[3] It should be noted that the context or the way in which a number has been obtained is sometimes the determining factor as to the significance of a 0.

3. The pupil in arithmetic needs to see the work in the form in which he will use it in practical life outside the schoolroom. His visualization of the process should therefore not include such symbols as +, —, ×, ÷, which are helpful only in writing out the analysis of a problem or in the printed statement of the operation to be performed. Because of these facts the committee recommends that only slight use be made of these symbols in the written work of the pupil, except in the analysis of problems. It recognizes, however, the value of such symbols in printed directions and in these analyses.

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[3] The italicising of significant zeros is here used merely to make clear the committee's meaning: The device is not recommended for general adoption.

III. General Observations and Recommendations

K. General observations.—The committee desires also to record its belief in two or three general observations.

1. It is very desirable to bring mathematical writing into closer touch with good usage in English writing in general. That we have failed in this particular has been the subject of frequent comment by teachers of mathematics as well as by teachers of English. This is all the more unfortunate because mathematics may be and should be a genuine help toward the acquisition of good habits in the speaking and writing of English. Under present conditions, with a style that is often stilted and in which undue compression is evident, we do not offer to the student the good models of English writing of which mathematics is capable, nor indeed do we always offer good models of thought processes. It is to be feared that many teachers encourage the use of a kind of vulgar mathematical slang when they allow such words as "tan" and "cos," for tangent and cosine, and habitually call their subject by the title "math".

2. In the same general spirit the committee wishes to observe that teachers of mathematics and writers of textbooks seem often to have gone to an extreme in searching for technical terms and for new symbols. The committee expresses the hope that mathematics may retain, as far as possible, a literary flavor. It seems perfectly feasible that a printed discussion should strike the pupil as an expression of reasonable ideas in terms of reasonable English forms. The fewer technical terms we introduce, the less is the subject likely to give the impression of being difficult and a mere juggling of words and symbols.

3. While recognizing the claims of euphony, the fact that a word like "historic" has a different meaning from "historical" and that confusion may occasionally arise if "arithmetic" is used as an adjective with a different pronunciation from the noun, the committee advises that such forms as geometric be preferred to geometrical. This is already done in such terms as analytic geometry and elliptic functions, and it seems proper to extend the custom to include *arithmetic, geometric, graphic,* and the like.

L. General recommendations.—In view of the desirability of a simplification of terms used in elementary instruction, and of establishing international usage so far as is reasonable, the committee recommends that the subject of this report be considered by a committee to be appointed by Section IV of the next International Congress of Mathematicians, such committee to contain representatives of at least the recognized international languages admitted to the meetings.

2. The committee suggests that examining bodies, contributors to mathematical journals, and authors of textbooks endeavor to follow the general principles formulated in this report.

to

PART II

INVESTIGATIONS

PART II. INVESTIGATIONS

CHAPTER IX

The Present Status of "Disciplinary Values" in Education

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1. Introduction

At the present time courses in Mathematics are being reorganized upon the basis of the needs of the citizen and the question of valid aims in the teaching of the subject is a particularly vital one. Among these aims have always been included those known as "disciplinary" and it is necessary to assess the value of such aims.

The question of disciplinary values of school subjects is an old one and, in a somewhat altered form, is still with us. From Plato's day until now the principal reason given for the retention of certain subjects in the curriculum has been that they disciplined the mind. The psychological view upon which this reason was founded was known as faculty psychology. It was thought that if a student exercised his memory upon any material, such as Greek or Latin, the "memory faculty" would be improved in such a way as to make itself evident wherever memory was used; also it was thought that, if a student acquired skill in reasoning in mathematics, this skill would improve his reasoning in every other field. The questions raised by such propositions are now discussed as problems in the "transfer of training," and very definite questions have been formulated for modern psychology to answer: Is it possible for different functions of the mind to be transferred from the field in which they have been trained to other fields? If so, how does this transfer take place? What is it exactly that is transferred? Is it the subject matter itself, or is it habits set up in connection with subject matter? Or is it volitions or feelings, ideas or attitudes? Or is it all of these?

These questions did not all arise at once. They represent the evolution that has taken place in the subject of transfer during the past thirty years. The first question to come up was whether or not any so-called formal discipline was possible under the implications of the new psychology. Very decided opposite views were held in regard to this question, and arguments based upon introspection on the one hand, and what appeared to be logical necessity on the other, proved equally unconvincing. It was then that William James of Harvard University conceived the idea of getting at the truth by means of a test of actual experience. This plan commended itself to psychologists, and gradually the settlement of the question was by general consent left to experiment.

It was not, however, until the Thorndike-Woodworth experiments in 1901 that scientific investigation began to take place. In the twenty years that have elapsed since that time twenty-nine important experiments on transfer have been made. Few of them have made a study of habits at all vital from the teacher's point of view, and many of them are open to severe criticism on account of the way in which they were conducted. No one of them has been checked by being repeated exactly as it was originally given, and very few have dealt with the same mental functions. But in spite of all deficiencies, these experiments have made definite contributions to the subject of transfer. Indeed, they are probably the best basis modern psychology has to offer for prophesying the general value of specific training.

The present investigation has proceeded along the following lines: 1. A study of the experiments and discussions of psychologists on the subject of transfer was made. 2. A set of statements of what are thought to be logical inferences from these experiments and discussions was drawn up. 3. The opinions of forty of our leading educational psychologists with regard to the truth of these statements was solicited. 4. An analysis of the replies and an attempt to formulate conclusions from them has been made.

2. The Experimental Data

Outlines of the experiments which have formed the material for this study will be found in Plates I-IV.[1] In Plates I-III will be found the experiments which deal directly with transfer. In practically all of them, the ability of selected individuals, called "subjects," to do certain things in connection with certain material has been tested at two different times. In the interval between the two tests some or all of the subjects have been trained to do the same things in connection with other material, and in some cases with the same material, as well. In what respects the results of the two tests differ, to what extent and in what way variations are due to practice, have received attention at the hands of the investigators.

The series of experiments outlined in Plate IV are concerned for the most part with the extent to which a tendency on the part of a group of students to do well in one school subject or test exercise is accompanied by a like tendency on their part to do well in other school subjects or test exercises. Such a tendency is commonly known as *correlation*, and its measure, as the *coefficient of correlation*, generally denoted by r. In the cases considered here, r has for the most part been computed from teachers' marks by the use of a formula known as Pearson's "product-moment" formula, and in the more careful studies the "probable error" has been computed. These investigations were carried on for the purpose of finding out if the relation between the abilities of students in different subjects is sufficiently close to support the claim that training in one subject is influencing performance in others. Since there are other factors, notably native ability, which might tend to make performance in one subject follow rather closely that in another, large values of r would merely indicate the necessity of further experimentation in order to find if transfer of training was a contributing factor. Small values of r, on the other hand, would make further experimentation unnecessary.

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[1] Plates I-III are taken from H. O. Rugg's *The Experimental Determination of Mental Discipline in School Studies*, and are used here by the kind permission of Dr. Rugg. These plates will be found in a pocket on the inside of the back cover.

Unfortunately, no definite criteria exist for judging "size" in the case of a correlation coefficient computed from teachers' marks. It is true that $r = 1$ represents perfect correspondence such as exists for the two sets of measures, given bases and computed altitudes in a number of similar triangles. For measured bases and altitudes r no longer equals 1, but a decimal whose size depends upon the accuracy of the measurements. In other words, the greater the inaccuracies in the measures the lower the value of r, while all the time the true correlation is 1. Spearman sought to correct for this "attenuation" in r and succeeded in constructing a formula which does eliminate the effects of errors which are due to pure chance. Certain psychologists consider this formula applicable when the measures are school grades. They would use it and then compare their results with $r = 1$ in determining what values of r are to be designated as high, medium or low. There are mathematicians, on the other hand, who hesitate to use such an instrument of precision as the Spearman formula when the measures are teachers' marks, and who suggest as a criterion correlation coefficients for successive courses in the same subject, as giving the greatest degree of correspondence that can be expected when the data are teachers' marks.[2]

The principal facts, drawn from the results of these experiments, from which later inferences have been made, are the following:

1. A very great majority of the experiments show transfer of training.

2. Under constructive criticism certain defects in the experiments have been pointed out, and consequent improvements made in their technique. For example: It has become the general practice to form upon the basis of a first test two equal groups of practically the same ability, one of which, the practice group, is trained, while the other, the control group, is not. Both groups take a final test, the results of which are compared in a variety of ways, for the purpose of finding to what extent, and in what ways, either group has surpassed the other. The school room as well as the laboratory is now used as

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[2] See the next chapter (Chap. X) for a further discussion of this phase of the subject.

the place of experiment, and the young person as well as the trained adult as the subject of experiment. The material used in the training has changed also. The tendency now is to use class room exercises in place of the more or less artificial materials formerly employed.

3. Many of the experiments have dealt with mental functions which would not be likely to show much transfer under the best of conditions.

4. The more objective the relation between the matter of the training and that of the tests, the more noticeable the transfer, especially in cases where the training was narrowly specific.

5. Negative transfer, or interference due to practice, is sometimes found.

6. Carefully conducted experiments directed toward the conditions of transfer show certain concomitants of specific training to have been operative in bringing about transfer, among which are: *(a)* The development of an ideal. *(b)* Understanding of the method of procedure. *(c)* Conviction of the worthwhileness of the training. *(d)* Maintenance of a high level of attention.

7. The relation of abilities of students in school subjects is sufficiently close to make it not impossible that their training in one subject is influencing their performance in others.

3. A Formulation of Inferences

From the general considerations listed above the following seven statements were formulated and submitted to psychologists for their consideration. The psychologists were asked for any further information they would be willing to give, and it was hoped that a free expression of opinion on their part in regard to any debatable point would tend to obviate some of the objectionable features of a questionnaire.

I. Transfer of training is an established fact, and may be positive, negative, or zero.

II. The true amount of transfer from one field to another has not yet been found by experiment, on account of one or more of the following handicaps:

- *(a) The maturity or previous training of the subjects tested.*
- *(b) The absence from the training of the factors most favorable to transfer.*
- *(c) The inadequacy of the tests to measure the traits sought.*

III. It is a reasonable inference that a substantial amount of transfer to some related field would be found by an adequate test, if in teaching children, emphasis were placed upon the trait which the selected subject was most capable of developing, and if the factors controlling transfer were present in the training.

IV. Even if no great amount of tranfer of training to any one field should ever be found by experiment, it would still be true that if small amounts of transfer of a valuable trait extended to a large number of fields, the sum total of all these small amounts would be a very valuable educational asset.

V. Negative transfer or interference may take place when in the training of a certain trait, auxiliary habits are cultivated, which have to be broken down before the trait can function in a new situation.

VI. Zero transfer may occur when the habits tending to interfere, and those tending to transfer, just offset each other.

VII. There are elements of situations so fundamental in their nature that they occur again and again in connection with almost anything else. Special training with these elements has general value.

4. The Judgment of Psychologists and General Conclusions

Of the forty psychologists to whom the statements were sent twenty-seven replied. Three of these did not give permission that their names should be used. The names of the remaining twenty-four, together with their replies in full, will be found in §5 of this chapter.

In the following table an attempt has been made to show roughly the nature of the responses to each statement. The figures in the body of the table indicate numbers of men; totals are given both in numbers and in percentages. Questions misunderstood are listed as omitted.

Statement	Yes	Yes Qualified	Doubtful	No	Omitted	Total
I	21	2			1	24
II	19	2		1	2	24
II a	13	2	2	2	5	24
II b	17	1	2	2	2	24
II c	15	2	2	2	3	24
III	11	3	2	1	7	24
IV	11	5	1	2	5	24
V	20	1	3			24
VI	17	5			2	24
VII	21	1			2	24
Total	165	24	12	10	29	240
Percents	69	10	5	4	12	100

The general conclusions drawn from this table and from the comments[3] which some of the psychologists added to their answers may be stated as follows:

1. The two extreme views for and against disciplinary values practically no longer exist. As the question now stands, as transfer of training, the psychologists quoted here almost unanimously agree that transfer does exist.

2. A large majority agree that there is a possibility of negative transfer, and of zero transfer, caused by interference effects. Professor Thorndike is of the opinion that negative transfer is comparatively rare and can be avoided by proper methods of training.

3. Very few if any experiments have shown the full amount of transfer between the fields chosen for investigation. The reason for this is to be found in the imperfections of the experimental setting. Of the defects mentioned in Statement II, absence from the training of the factors most favorable to transfer is the one most generally considered important, though strong emphasis is placed in several instances upon the inadequacy of the tests.

4. The amount of transfer in any case where transfer is admitted at all, is very largely dependent upon methods of teaching. This is probably the strongest note struck by the psychologists in their comments. Twelve of them out of twenty-four make some explicit reference to the matter.

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[3] See § 5, below.

5. A majority of the psychologists, judging from their remarks in connection with Statements III, IV, and VII seem to believe that, with certain restrictions, transfer of training is a valid aim in teaching.

6. Transfer is most evident with respect to general elements—ideas, attitudes, and ideals. These act in many instances as the carriers in transfer. Often they form the common element so generally held to be the sine qua non of transfer. Twenty-two of the twenty-four psychologists express the opinion that special training in connection with these elements has general value; and one of them, Professor Sanford, adds: "This I believe to be true, and to be the basis of the generally held belief among practical teachers of the existence and value of 'formal training.'"

5. The Replies of the Psychologists

In the following replies of individual psychologists, the Roman numerals refer to the corresponding statements formulated in §3 above.

J. R. Angell.

I. I regard this as a substantially accurate statement.

II. This also seems to me to be a correct formulation, with particular emphasis on the point (*c*).

III. This formulation seems to me to be a little vague, although I am not disposed to take exception to it. My own formula would be stated in terms of the attitude of attention and the undoubted ability to develop these essential attitudes in ways which are widely useful over great varieties of subject matter.

IV. This, I think, is a defensible statement.

V. This formulation seems to me correct, although it is rather more conservative than I should put it, in that it seems to confine negative transfer to "auxiliary habits." I should regard the negative relations as extending even beyond this region.

VI. This is a theoretically possible condition which presumably rarely occurs.

VII. I subscribe entirely to this statement.

W. C. Bagley.

I, II, IIb, V, VI, VII. Yes.
II a. Perhaps.
II c. To some extent.
IV. Perhaps.

The evidence from the experiments is clearly in favor of a certain amount of transfer.

In my opinion, the possibilities of transfer are increased by the kind of teaching that makes the student conscious of the procedure as such, and keenly appreciative of its value as a *general* procedure.

The theory of "identical elements" I regard as sound—but it is not rich in pedagogical suggestiveness. The theory of transfer through "concepts of method," and "ideals of procedure" furnishes a definite suggestion for teaching. The two theories are not inconsistent with one another: a person who has gained an understanding and an *appreciation* of a procedure will not be limited to the "identical elements" which come by accident; he will *search* for identities—for places at which the procedure may be applied.

H. V. BINGHAM.

I, II, IIb, V, VI, VII. Yes.

IIa. No.

IIc. No.

III. Yes, but the amount of transfer is probably small.

IV. I am in doubt.

What astonishes me is that in preparing this list you did not add one or two additional paragraphs in order to get the consensus of opinion on such propositions as the following:

In view of the probability that the amount of transfer of specific training from one subject matter to another is relatively small, a mathematics curriculum should be almost exclusively constructed with a view to securing a wide range and variety of specific habits and items of knowledge whose value quite apart from any question of the development of general abilities is in each instance readily demonstrable.

EDWIN G. BORING.

I. Yes. Naturally, since it is impossible to think of the educable abilities as anything but complex and the elements must be common to several in many instances.

II. Yes.

IIa. Yes. Undoubtedly in some experiments.

IIb. Yes.

But I should think this deficiency was related to IIc below, since it is the adequate experimental setting that is hard to achieve.

IIc. Yes.

III. Yes.

This proposition is framed so that "yes" is the only possible answer unless one is willing to assert that all transfer is unreasonable. For if one admits transfer, then the transfer would have to occur if the factors controlling it were present in the training; and if the transfer had occurred it would have to be found by an adequate test, since any test that did not find it would not be adequate.

IV. Yes.

V. Yes, emphatically.

VI. Yes, a corollary from I and V.

VII. Yes, extremely probable.

It seems to me that the questionnaire leaves out of account the fundamental difficulty. To measure "the true amount of transfer" requires careful experimental technique, and at the same time *norms* and controlled experiments. You cannot get reliability at a small amount of transfer unless you can tell what the same individual would have done without practice in the initial field; but you have already spoiled it by practice. You have therefore to seek a generalized individual as a standard for measurement, in a group large enough to be constant mentally. It is almost impossible to get unselected human material that will yield comparable average groups, and it is hard enough to get even the large groups. The emphasis on size of group further limits the amount of experimental work that can be done on each individual, so that the tests become less adequate as the groups become larger.

The form of the propositions seems to me to negate any great value that can be achieved from a statistical treatment of the replies to them. Any psychologist knows that the formulation of an instruction which will not be suggestive is an extremely difficult task. Any psychologist knows further that even psychologists are not proof against suggestion. These questions seem to me to be (a) highly suggestive in the direction of affirmative answers, (b) sophistical to the extent of forcing affirmative answers, and (c) misleading, in that three of them (III, IV, and VI), admit at the level at which the discussion is staged of only one answer. This discovery of a bias within the set of questions makes me wonder whether they are conceived in the spirit of scientific inquiry or in an endeavor to establish a propaganda for some purpose already agreed upon.

STEPHEN S. COLVIN.

I. Yes.

II. I find it difficult to answer this question. It does not seem to me that transfer can be said to take place from one field to another in general but only in specific instances. It often happens that tests measure the true amount of transfer in any particular experiment, but that these results cannot be generalized, and consequently the true amount of transfer under various circumstances cannot be said to have been found.

III, IV, V, VI. Yes.

VII. Yes, these are general attitudes, dispositions, prejudices, etc.

GEORGE O. FERGUSON.

I assent to all of the statements in your list except number II, and I assent to number II subject to reservations in particular instances.

Would it not be useful to point out conditions under which transfer is not likely to be of value?

V. A. C. HENMON.

I find myself in accord with all of the propositions submitted concerning transfer of training.

My study of the experimental literature has led me to these conclusions:

1. There is some transfer shown in every carefully controlled study.

2. The amount of transfer varies widely, but in general it is less than the extreme advocates of formal training claim, and on the other hand is greater than the advocates of specific training admit.

3. Care must be taken to secure transfer, or general values, as the transfer may not take place unconsciously.

4. The amount of transfer is not a particularly significant figure, as a gain of 3 or 4 per cent, which expressed quantitatively seems small may be of great significance and value.

5. The doctrine that there is transfer in so far as there are common or identical elements has not helped much in resolving quarrels about disciplinary values of school subjects; for, even if this principle is accepted as a basis for discussion of values of any subject, one may claim that the common or identical elements with other subjects or life situations are numerous or important, and another that they are few or unimportant. Experiment which is still lacking is the court of last resort.

JOSEPH JASTROW.

I am prepared to endorse every one of the proposals submitted. I do this with little hesitation except for II. I believe that the factors indicated are the ones most at work; but I am uncertain as to their relative importance, (*b*) and (*c*) seem to me to be of more importance than (*a*). Yet on general principles (*a*) is the factor which most of all is likely to be operative. It means that transfer has taken place, and that its further operation is naturally limited.

It is my general view, which I have expressed for years, that negative evidence in regard to transfer has been much exaggerated, that much of it relates to details in which transfer is not likely to take place and, moreover, would be disguised if it occurs.

I am entirely convinced that the evidence for the value of disciplinary studies stands. It is to be modified, especially in details, in the light of recent findings. These do not invalidate the principle; perhaps they do not strengthen it as much as was expected; yet they support it. They also show the limitations.

CHARLES H. JUDD.

I, III, V, VI, VII. Yes.

I find it impossible to answer questions II and IV. There are some fields in which the amount of transfer under given conditions has been established, and I do not believe that the particular qualifications which you have enumerated are especially significant. There seems to me also to be in number IV the necessity of referring to methods of teaching, as I have indicated, rather than particular traits.

I find it very difficult to answer these questions because I do not think they are formulated so as to bring out the chief problems involved. I do not think that any subject transfers automatically and in every case. The real problem of transfer is a problem of so organizing training that it will carry over in the minds of students into other fields. There is a method of teaching a subject so that it will transfer, and there are other methods of teaching the subject so that the transfer will be very small. Mathematics as a subject cannot be described in my judgment as sure to transfer. All depends upon the way in which the subject is handled. My answers ought then to be qualified in terms of this general position. I have attempted to discuss the matter fully in a chapter in my "Psychology of High-School Subjects," and I prefer very much to use the phrase which I have there used. Transfer is a form of generalization, and training can be given so as to encourage generalization, or training can be of such a type as to hinder generalization.

GEORGE TRUMBULL LADD.

I. Yes, either positive or negative. I doubt its ever being zero. No influence on mental life can become zero.

II. All three considerations, (*a*), (*b*), and (*c*) will act as handicaps.

III. Experimental control would be almost sure to show a "substantial amount" of transfer, though the amount and the method might be difficult to fix with exactness.

IV. Yes, undoubtedly.

V. Yes. Indeed, it is likely that where any system of rather complicated character is persistently used some of its effects will tend to establish habits that in and of themselves will be of a negative or destructive character. Education requires a sort of constant "making and breaking of the cake of custom."

VI. As I have said I doubt the possibility of a real "zero," though as nearly as we can get at the truth the positive and negative, or favorable and unfavorable, effects of the methods employed may apparently about balance each other.

VII. Undoubtedly, yes.

R. M. OGDEN.

The above seem to me to be fair statements. While some are obviously more significant than others, I believe I should be able to defend them all.

W. B. PILLSBURY.

IIb, IIc, V, and VII. I regard these as established.

I, II, IIa, III, IV, VI. May be true, but are not proved.

There is little general agreement on this whole topic, still less, if one means by transfer the effect of classroom work in any subject upon general ability or effectiveness in any other relation. All agree that habits may be trained and that many habits can be used in different connections; whether a child would reason better in politics because he had had a thorough training in mathematics, for example, is not proved by any actual work or measurements so far as I am familiar with the literature.

EDMUND C. SANFORD.

I. I believe that some degree of transfer is an established fact, speaking broadly of total forms of acquired skill, and that its result may be positive, negative, or zero.

II. I do not believe that the question of the extent, nature and conditions of transfer (or interference) has yet been *exhaustively* studied, though I am inclined to believe that the general results so far attained are valuable and important. I do not feel ready to assess the importance of the factors listed under (*a*), (*b*), and (*c*).

III. This statement seems to me to be equivalent to saying that if transference exists in substantial amount and can be demonstrated, teaching which regarded the necessary conditions could show transference results. *In this hypothetical form* I should judge that every one must answer in the affirmative. If it is equivalent to the simpler question whether the manner of teaching may effect the degree of transfer I should feel again that any one who admitted transfer at all would have to admit a high degree of probability that the manner of teaching was a factor of importance.

IV. Yes, if the amount though small were not too small. Even if many fields were favorably affected, it might still be true that more economical pedagogical means might exist.

V. Without doubt.

VI. Yes, but the case of an exact balance would be rare, of course.

VII. This I believe to be true and to be the basis of the generally held belief among practical teachers of the existence and value of "formal training."

The fundamental elements referred to in VII seem to be "moral" and "general" in the sense of very little specialized methods and habits of working. They are identical or nearly so in many situations and so come within the field in which transfer is generally admitted. But this of course cuts both ways, if they are fundamental in many situations, they are also cultivated by many sorts of activity—or can be if the situations are properly utilized pedagogically.

Speaking still more generally and on the question of the teaching of mathematics rather than on transfer, I am extremely doubtful whether we are not coming to a time in the not very distant future when no subject can make much of an argument for itself on the basis of indirect results. It will have to stand or fall on what it itself can do in its own right. This, of course, does not mean that the test will be merely a vocational one—though that factor will be a large one—but on whether the student when he has completed the course (or if he is very young, those who choose for him) feels that he has gotten something worth while out of it. This value may be something quite non-utilitarian. I can imagine a teacher of mathematics who was so enthusiastic himself for the subject and so skillful in presenting it that his classes would be largely attended even under a system of free election by students who had no future "use" for mathematics in mind. No teacher—even an angel from heaven—could draw all students to a real interest in and perception of the beauty of mathematics, but then perhaps all students ought not to be required to take mathematics. Is it not, like a good many other good things, for the elect?

WALTER DILL SCOTT.

I concur most heartily with statements VII, I, and V. I see no reason to deny VI although such a situation might be difficult to find. The statements under II do not seem to me to be justified. I would be inclined to deny likewise III and IV.

C. E. SEASHORE.

I approve all seven points.

Daniel Starch.

I, II, V, VI, VII. Yes.

IIa. No.

IIb and c. Yes, in part.

III. A larger transfer would probably be found, but whether it would be "substantial" would have to be determined.

IV. Yes, but it would depend upon the amount of transfer and the time and effort involved.

Edward K. Strong.

I, II, IIa, IIb, IIc, V, VI, VII. Yes.

In connection with IIc I should emphasize that our present tests are inadequate not only in testing specific habits which are learned, but also that they do not in any way at present test general attitudes towards one's work or general methods of handling material and of thinking. It is my opinion that the higher the intelligence of the individual the more the possibility of transfer, and the more this transfer will take place in the realm of general attitudes and methods. Stated in another way, the less the intelligence of the child, the more emphasis must be placed on teaching in terms of transfer.

Statement III is not clear to me.

Number IV I would agree with, but I know of no experimental evidence one way or the other in the matter.

Lewis M. Terman.

I, V, VI. Yes.

II, IIa, IIb, IIc. Sometimes one, sometimes another factor has prevented us from getting the real facts; (*a*), (*b*) and (*c*) are only part of them.

III. All depends on *how* related the field was.

IV. Yes, probably.

VII. I agree with what seems to be the intended meaning of this. "Almost anything else"—a little strong.

E. L. Thorndike.

I, IV, VI, VII. Yes.

II. The precise amount of transfer from one field to another has not yet been found by experiment, but the experiments give the best present expectations in the matter.

IIa, IIb, IIc. I don't know about these.

III. "Substantial" too vague.

V. Negative transfer may take place, but this seems to be rare, and may probably usually be avoided by the choice of right methods in training.

VI.—or if two functions have zero or nearly zero community.

Howard C. Warren.

II, IIa, IIc, IV, V, VI, VII. Yes.

I. Yes. I understand the question to mean transfer of training from one *field* to another—not necessarily transfer of *specific training*.

IIb. Doubtful. I would not subscribe at present.
III. Doubtful. I would not subscribe definitely.

M. F. Washburn.

I, V, VI, VII. Yes.

Statements II, III, and IV are so much more complicated than the others that one feels less sure of understanding them.

John B. Watson.

In regard to I, I should say that there is transfer of training in this sense, in learning Russian I learn to use the dictionary, to sit still for an hour and work at a given problem, etc. When I come to study German, exactly the same type of behavior is involved. There is no transfer in any sense which has that meaning, but a part of a learning process may be demanded in some other act of skill. I certainly agree to this.

In regard to III my answer on I above to some extent answers this. The trouble is you have asked the question so that it should be answered in terms of some kind of mental content.

In regard to II, I should answer in the affirmative that no two acts have been scrutinized well enough when *x* has been learned before *y* to show just how much of the activity involved in *y* had already been learned in *x*.

In regard to IIc, I would say that our tests are at present totally inadequate for making the measurements sought for. Furthermore, I am absolutely sure that any such test is totally irrelevant on the practical side. The elementary processes involved in mathematics have to be taught regardless of the question of transfer.

In regard to IV, my feeling again is that an answer is not relevant to the problem. If it refers again to training in mathematics I gladly put myself down as affirming that even if there is no transfer at all such training is a valuable educational asset. V and VI are unanswerable.

In regard to VII if I know what you mean at all I assent most heartily. I think you mean that the mathematical behavior, that is, the ability to add sums, subtract, multiply, determine interest rates, etc., is so universally demanded of all developing and developed human beings that training in such behavior is essential regardless of whether or not that training may influence other training for good or bad. I heartily agree with this.

R. S. Woodworth.

I, II, IIa, IIb, IIc, III, V. Yes.

VI.—or when there is nothing specific to transfer.

(IV, VII no response.)

To guarantee any transfer, the element to be transferred must be brought specifically to the pupil's attention, generalized into a principle, and the application of the principle to other fields made clear. For example, if neatness is trained in written arithmetic work, the value of neatness must be discussed and the ways in which neatness is secured made plain and more than a mere rule of thumb; it should be reduced to general principles, and conditions of neatness in other fields should be made plain. In short, the matter to be transferred must be made the specific subject of study.

Robert M. Yerkes.

I and V. Yes.

II. This cannot be answered categorically and simply without serious risk of injustice to the facts.

IIa, b, c. All are serious sources of error.

III. This depends upon the "traits" or forms of reaction in question. Yes, in some cases; no in others.

IV. Yes, decidedly.

VI. Presumably.

VII. I believe so.

These questions demand a confession of faith in addition to statements of fact. It is my opinion that we need to study the whole matter experimentally with extreme thoroughness and critical care before we commit ourselves to important pedagogical decisions.

CHAPTER X

The Theory of Correlation Applied to School Grades

BY

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In the past few years, the words "correlation" and "correlation coefficient" have appeared frequently in educational journals in connection with the discussion of the influence of one school subject upon another.[1] It is the object of this study to investigate the reliance that can be placed upon the numerical results obtained in the application of the correlation method to data based upon school marks and to find out whether the data themselves offer any explanation of the numerical results. A summary of the results of the study will be found at the end of the discussion of the data, page 122.

The meaning of correlation.—Without going into technicalities the meaning of correlation is easily explained. If we have two series of paired numbers, for example heights and arm lengths of each individual of a group, or prices of flour and of cotton on certain dates, or marks in two school subjects of individual pupils of a school, we are interested in the connection, if any, between the two sets of figures. There may be little or no connection between two such sets of numbers, as in the case of ages and heights of a group of adults. Again there may be a very close relation as would be shown in the case where the pairs of numbers are the radii and corresponding circumferences of a set of circles. It is clear that the set of pairs of numbers representing height and arm length is different from these two extreme cases. We express the connection between arm length and height roughly by saying that in general a tall man has a long arm, but given the height of a man we cannot compute the length of his arm. We need something besides the words "in general" to tell us how the two sets of numbers are related or *correlated*. We seek then some number which will measure for us the degree of this relationship or of this *correlation* between paired numbers. Many measures have been devised but the one in widest use is the measure often called Pearson's *coefficient of correlation,* or the product-moment coefficient, or simply the *correlation coefficient*. It is universally represented by the letter r.

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[1] See Bibliography, pp. 123-124.

The correlation coefficient varies from —1 to 1. For the extremes of the range the correlation is *perfect*. In perfectly correlated series, given one of any pair of numbers we can find the other by solving a linear algebraic equation. An example of this perfect correlation is that of the radii and circumferences of circles, in which the correlation coefficient is 1. If a decrease in one of the pairs of a perfectly correlated series is accompanied by an increase in the other the coefficient is —1. If $r = 0$ there is no tendency to this linear relationship, though there may be a close connection of another kind. The correlation coefficient then may be described as a measure of the approach to linear relationship.

The data for this study consist of the complete records of graduates of six high schools: Urbana, Champaign, Evanston, Quincy, all in Illinois, Topeka, Kansas, and Spokane, Washington. From 200 to 700 complete records were secured from each school. The records were separated as to sex and thus twelve sets of records were studied. As indicated in the tables in the appendix many pairs of correlation coefficients were computed together with their probable errors. The grades were given on the usual percentage system and nearly all grades used in computing the coefficients were the averages of two or more semester grades. The grades were grouped in blocks of five. For example, all grades between 82.5 and 87.5 were taken as 85. The numerical work was carefully checked but no corrections were made for the error introduced by thus grouping the grades.

The first step in the calculation of a correlation coefficient is the construction of a so-called *correlation table*. For example, the table for algebra and the average in the last three years of English from the girls of one school is shown on the next page.

The grades for algebra are at the left hand side and the grades for English are at the top. The total of all the numbers in a row is the number of pupils who received the grade in algebra corresponding to the row. Thus, 94 pupils received a grade of 90 in algebra. Of these 94 pupils, 2 received a grade of 75 in English, 7 a grade of 80, 29 of 85, 44 of 90, and 12 of 95.

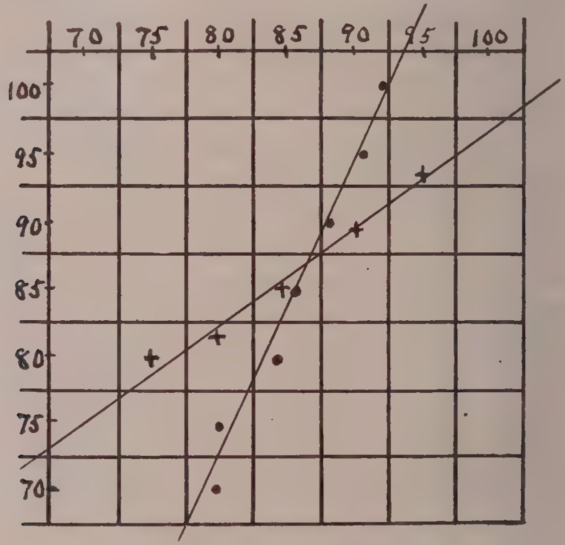
The average grade in English of the 94 pupils who received a grade of 90 in algebra is 88.03. In the same way the averages of the grades in the different rows can be calculated. If dots are put in the rows to represent these averages we will have, after erasing the numerals of the table, the diagram on the next page. If these dots lie on or near a straight line the correlation coefficient is a good measure of the relationship. If there were no correlation the dots would lie on a vertical line. The departure from the vertical of the straight line on which the dots lie is also a measure of relationship. For a relationship to exist, it is not necessary that these dots lie on a straight line; they may be on a curve. The measure of the condensation of all the points of the diagram about such a curve is called the *correlation ratio* and is usually represented by the Greek letter η. If this curve is a straight line, η and r are equal. If we take the means of the columns

Average of Last Three Years English

Freshman Algebra	70	75	80	85	90	95	100	Totals of Rows
100				1	5	6		12
95			1	10	36	24		71
90		2	7	29	44	12		94
85		2	16	27	23	4		72
80		4	18	22	19			63
75		6	12	13	4	1		36
70			1					1
Totals of Columns		14	55	102	131	47		349

A Correlation Table

we have another set of points (represented by crosses in the diagram). Thus for every correlation table there are two sets of points which should lie near straight lines if the correlation coefficient is to be a good measure of relationship. In the diagram, if we take the intersection of the two lines as the origin, the equation of the line through the crosses is $y = .72x$, meaning that for a unit change in x (grade in English) the change in y (mean grade in algebra) is .72. For the line with the dots we have $x = .42\,y$, meaning that for a unit change in the grade in algebra the change in the mean grade in English is .42.



Regression Curves

In the literature, the curves about which the mean points cluster are called *regression curves.* One of the questions then to settle is that of the linearity of the regression curves. If the mean points lie on straight lines the correlation coefficient and the two correlation ratios are all equal in absolute value. There is a test of linearity known as Blakeman's test which tells us how much the correlation coefficient may differ from the correlation ratio and still be a good measure of relationship. In the form we have used here this test is

$$B = N\ (\eta^2 - r^2) < 11.4,$$

where N is the total number of entries in the table.

The correlation ratios were calculated for each coefficient, and are tabulated in the columns (see tables in the appendix) headed η_1 and η_2. In the columns headed B are given the values of $N(\eta^2 - r^2)$ for the larger of the two ratios. All the calculations were worked out to four significant figures, but two only were retained. In about 97 percent of the cases, Blakeman's test was satisfied. In general, then, we may say that if the problem is considered as an abstract numerical exercise the correlation coefficient is a good measure of the relationship between two sets of school grades paired through individual pupils belonging to the same school. The fact that Blakeman's test is not satisfied is no proof that there is no relationship, but shows only that the correlation coefficient is not a good measure of the relationship, if one exists.

A correlation scale.—In order to find out just what should be considered as a high or a low correlation many coefficients were computed for subjects whose relationships were known to be close or otherwise. Some of these coefficients are as follows:

College algebra—plane trigonometry,		
Taken at the same time with the same instructor	.737	
Taken at the same time with different instructors	.620	
Two consecutive courses in Latin (boys)	.680	(av. of 5 coeffs.)
" " " " " (girls)	.710	" " 14 "
" " " " English (boys)	.579	" " 16 "
" " " " " (girls)	.575	" " 16 "
Two courses in Latin, one year intervening (boys)	.580	" " 3 "
" " " " " " " (girls)	.613	" " 5 "
" " " English, one year intervening (boys)	.521	" " 7 "
" " " " " " " (girls)	.510	" " 10 "
" " " Latin, two years intervening (girls)	.430	" " 5 "
" " " English, two years intervening	.410	" " 10 "

Correlations between drawing or manual training and English or other language courses were less than .25.

These coefficients helped to make an otherwise more or less arbitrary scale as shown

Extremely high . High Medium Low Ex. Low
1.00 .70 .55 .40 .25 0

Condensed tables of correlation coefficients.[2]—In the following tables the correlation coefficients are listed in order of size. The numbers given are averages for the six schools of those coefficients which passed Blakeman's test and for which the number of pupils was over

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[2] The complete tables will be found in the appendix to this Chapter, pp. 124-128.

30. Girls and boys are listed separately except for certain elective subjects for which there were not enough data to warrant separation. These average coefficients are made up from the tables in the appendix.

GIRLS. AVERAGE OF COEFFICIENTS OF CORRELATION FOR THE SIX SCHOOLS

Extremely High

1.	2nd yr. Latin—3rd yr. Latin	.74 ± .02
2.	1st yr. Latin—2nd yr. Latin	.71 ± .02
3.	3rd yr. Latin—4th yr. Latin	.68 ± .03

High

4.	2nd yr. Latin—4th yr. Latin	.64 ± .03
5.	1st yr. History—Av. last 3 yrs. English	.62 ± .03
6.	1st yr. English—2nd yr. English	.60 ± .02
7.	2nd yr. English—3rd yr. English	.60 ± .02
8.	1st yr. English—Av. last 3 yrs. English	.59 ± .02
9.	1st yr. Latin—3rd yr. Latin	.58 ± .03
10.	3rd yr. English—4th yr. English	.55 ± .02

Medium

11.	2nd yr. English—4th yr. English	.52 ± .02
12.	Algebra—Geometry	.52 ± .02
13.	1st yr. English—Civics	.52 ± .04
14.	1st yr. English—3rd yr. English	.51 ± .02
15.	Algebra—Av. German	.50 ± .03
16.	1st yr. English—Av. History	.48 ± .03
17.	Algebra—Av. Latin	.47 ± .03
18.	Algebra—Av. History	.47 ± .02
19.	1st yr. Latin—Av. English	.46 ± .04
20.	1st yr. English—Av. Latin	.46 ± .03
21.	Algebra—Civics	.44 ± .03
22.	Algebra—Av. English	.44 ± .02
23.	1st yr. English—4th yr. English	.44 ± .03
24.	1st yr. Latin—4th yr. Latin	.43 ± .04
25.	1st yr. English—Av. German	.41 ± .03

Low

26.	1st yr. English—Geometry	.39 ± .03
27.	Domestic Science—Av. English	.38 ± .03
28.	Domestic Science—Av. German	.38 ± .04
29.	Domestic Science—Geometry	.34 ± .03
30.	Domestic Science—Av. History	.34 ± .03
31.	Domestic Science—Av. Latin	.33 ± .04
32.	Domestic Science—Civics	.26 ± .04

BOYS. AVERAGE OF COEFFICIENTS OF CORRELATION FOR THE SIX SCHOOLS

Extremely High

1.	2nd yr. Latin—3rd yr. Latin	.75 ± .06
2.	1st yr. Latin—4th yr. Latin (one school only)	.69 ± .06

High

3.	1st yr. Latin—2nd yr. Latin	.65 ± .03
4.	2nd. yr. English—3rd yr. English	.60 ± .02
5.	1st yr. Latin—3rd yr. Latin	.58 ± .05
6.	1st yr. History—Av. last 3 yrs. English	.57 ± .03
7.	1st yr. English—2nd yr. English	.57 ± .02
8.	1st yr. Latin—Av. last 3 yrs. English	.55 ± .03

Medium

9.	Algebra—Av. Latin	.53 ± .03
10.	Algebra—Geometry	.52 ± .02
11.	1st yr. English—Av. last 3 yrs. English	.52 ± .02
12.	1st yr. English—Av. German	.51 ± .02
13.	1st yr. English—Av. History	.51 ± .02
14.	1st yr. English—3rd yr. English	.51 ± .03
15.	3rd yr. English—4th yr. English	.51 ± .04
16.	Algebra—Av. German	.50 ± .03
17.	1st yr. English—Av. Latin	.49 ± .03
18.	Algebra—Av. History	.47 ± .02
19.	Algebra—Av. last 3 yrs. English	.47 ± .02
20.	2nd yr. English—4th yr. English	.46 ± .03
21.	1st yr. English—Geometry	.43 ± .02

Low

22.	1st yr. English—Civics	.39 ± .04
23.	Algebra—Civics	.36 ± .03
24.	Manual training—geometry	.34 ± .04
25.	1st yr. English—4th yr. English	.30 ± .03
26.	Manual training—Civics	.25 ± .04

Extremely Low

27.	Manual training—Av. last 3 yrs. English	.23 ± .04
28.	Manual training—Av. German	.21 ± .05
29.	Manual training—Av. History	.18 ± .04
30.	Manual training—Av. Latin	.07 ± .06

BOYS AND GIRLS. AVERAGE OF COEFFICIENTS
FOR SOME CASES WHERE THERE WERE NOT ENOUGH RECORDS TO WARRANT
SEPARATION OF SEXES

Medium

1.	Latin—stenography	.49 ± .05
2.	German—stenography	.45 ± .05
3.	Latin—bookkeeping	.44 ± .05

Low

4.	Geometry—stenography	.39 ± .05
5.	History—bookkeeping	.38 ± .05
6.	English—stenography	.36 ± .04
7.	Geometry—bookkeeping	.35 ± .06
8.	History—stenography	.34 ± .05
9.	German—bookkeeping	.33 ± .06
10.	English—bookkeeping	.29 ± .05
11.	Civics—bookkeeping	.26 ± .08
12.	Civics—stenography	.25 ± .06

Significant differences.—In the tables there seems to be very little difference between boys and girls. The most notable one is the correlation between 1st year English and German, .41 ± .03 for girls and .51 ± .02 for boys. This difference is significant in the following way. Assuming a normal distribution and translating into terms of the theory of probabilities we should expect a deviation as great as .10 once in about 16 trials, meaning by a trial the taking of an average from six schools chosen at random. For consecutive courses in English we found the correlations, girls .575 ± .012, boys .579 ± .011. We should expect the correlation for the girls to reach or exceed .579 in three out of seven trials; that is, we may consider the difference .004 as due to random sampling, not to sex. The correlations: algebra — German, algebra — geometry, are the same for boys or for girls, i. e., for algebra — German $r = .50 \pm .03$, for algebra — geometry, $r = .52 \pm .03$. Investigation shows that the odds are 3 to 1 in favor of the two correlations being identical in a statistical sense. That is, there is not a marked difference between geometry and German in the relationship to algebra measured by the correlation coefficient. A somewhat similar result is shown if we substitute Latin for German. In a very general way we may say that in the above tables a difference of .10 or more in two correlations is significant.

Freshman subjects and last three years of English.—If we center our attention upon the correlations between the average of the last three years of English and various freshman subjects and collect them together we have the following table.

GIRLS	
1st yr. History—average last 3 yrs. English	.62 high
1st yr. English—average last 3 yrs. English	.59 high
1st yr. Latin—average last 3 yrs. English	.46 medium
1st yr. Algebra—average last 3 yrs. English	.44 low medium
1st yr. Domestic Science—average last 3 years English	.38 low
BOYS	
1st yr. History—average last 3 yrs. English	.57 high
1st yr. Latin—average last 3 yrs. English	.55 high
1st yr. English—average last 3 yrs. English	.52 high medium
1st yr. Algebra—average last 3 yrs. English	.47 medium
1st yr. Manual training—average last 3 yrs. English	.23 extremely low
BOYS AND GIRLS	
Stenography—average last 3 yrs. English	.36 low
Bookkeeping—average last 3 yrs. English	.29 low

From this table we see that as far as the relationship with English is concerned algebra is below history, English, and Latin and above domestic science, stenography, bookkeeping, and manual training. The coefficient for algebra — English lies between the coefficient for English — English with one year intervening between the courses, and that for English — English with two years intervening. It is also comparable with the coefficient for freshman Latin — senior Latin. In general, we may say that we should expect a pupil who has received a good grade in algebra in his freshman year to receive good grades in the English courses following, in much the same way that we expect a pupil who is good in freshman English or Latin to receive high marks in his junior or senior English or Latin. But consecutive courses in English or Latin give much higher correlations than algebra — English.

Comparison of algebra with other freshman subjects.—In order to compare algebra with other freshman subjects in their relations to the work of the following three years, six standard subjects were chosen and average grades for years following the first found for each school and sex. These subjects were English, geometry, history, Latin, German, and civics. With each of these was paired freshman English, freshman algebra, domestic science, manual training, freshman history, bookkeeping and stenography. The last two were not separated as to sex. The tables below give the average correlations for the six schools arranged in order of size of the average of the numbers in the two columns.

CORRELATIONS OF FRESHMAN SUBJECTS WITH AVERAGES IN CERTAIN STANDARD SUBJECTS

		Boys	Girls
1st yr. English—Average last 3 yrs. English		.52	.59
" " " " " " history		.51	.48
" " " " " " Latin		.49	.46
" " " " German		.50	.41
" " " " Civics		.39	.52
" " " " Geometry		.43	.39
	Average	.460	.475
1st yr. Algebra—Average Geometry		.52	.52
" " " " last 3 yrs. Latin		.53	.47
" " " " German		.50	.50
" " " " History		.47	.47
" " " " last 3 yrs. English		.47	.44
" " " " Civics		.36	.44
	Average	.475	.473
Domestic Science—Average last 3 yrs. English			.38+
" " " German			.38—
" " " Geometry			.34
" " " History			.34
" " " Latin			.33
" " " Civics			.28
	Average		.342

Manual training—Average geometry		.34
“ “ “ Civics		.25
“ “ “ Last 3 yrs. English		.23
“ “ “ German		.21
“ “ “ History		.18
“ “ “ Latin		.07
	Average	.213

		Boys and Girls
Bookkeeping—Average last 3 yrs. Latin		.44
“ “ “ “ History		.38
“ “ Geometry		.36
“ “ German		.35
“ “ last 3 yrs. English		.33
“ “ Civics		.29
	Average	.358

Stenography—Average last 3 yrs. Latin		.44
“ “ German		.42
“ “ Civics		.40
“ “ last 3 yrs. English		.36
“ “ Geometry		.36
“ “ History		.27
	Average	.375

We may sum up this table in the following short table by grouping the boys and girls together and listing in order of the averages.

	Average of all correlations
1st yr. Algebra	.474
1st yr. English	.468
Stenography	.375
Bookkeeping	.358
Domestic science	.342
Manual training	.213

When we consider the fact that the numbers in the table are the averages of from 30 to 72 correlation coefficients we see that algebra and English stand out from the others with little to choose between the two. When further we notice that the courses in stenography and bookkeeping were taken simultaneously with the subjects with which they were correlated while the English and algebra were taken in the freshman year the difference is emphasized.

Centering our attention in turn upon each of the standard courses, English, geometry, history, Latin, German, civics, we have the following table:

	Boys		Girls
1st year English—average last 3 yrs. English	.52		.59
1st year Algebra “ “ “ “	.47		.44
Domestic science “ “ “ “			.38
Stenography “ “ “ “		.36	
Bookkeeping “ “ “ “		.33	
Manual training “ “ “ “	.23		
1st yr. Algebra—Geometry	.52		.52
1st yr. English “	.43		.39
Stenography “		.36	
Bookkeeping “		.36	
Domestic science “			.34
Manual training “	.34		
1st yr. English — History	.51		.48
1st yr. Algebra “	.47		.47
Bookkeeping “		.38	
Domestic science “			.34
Stenography “		.27	
Manual training “	.18		
1st yr. Algebra — Latin	.53		.47
1st yr. English “	.49		.46
Stenography “		.44	
Bookkeeping “		.44	
Domestic science “			.33
Manual training “	.07		
1st yr. Algebra—German	.50		.50
1st yr. English “	.50		.41
Stenography “			.42
Domestic science “		.38	
Bookkeeping “		.35	
Manual training “	.21		
1st yr. English — civics	.39		.52
1st yr. Algebra “	.36		.44
Stenography “		.40	
Bookkeeping “		.29	
Domestic science “			.28
Manual training “	.25		

In each group of the above table algebra and English come first. Indications are that history would stand higher than either algebra or English, but only a few history correlations were worked out. For the four schools having sufficient data in freshman history the average of the correlations history — average English was, girls .62, boys .57

or higher than the correlation between freshman English and average English.

All the above tables may be summed up as showing that algebra as taught in our schools has little claim to preeminence in its influence upon other subjects coming later in the high school course. Its influence as measured by a correlation coefficient is about the same as that for English but higher than those for vocational courses and probably lower than for history.

The influence of general intelligence. Partial correlation.—In studying the underlying reason for a correlation between two school subjects, one of the first questions is as to the effect of general intelligence. To investigate this it is necessary to have some measure of general intelligence. The data at hand would not give this but an approximation which we may call "school intelligence" was found by taking the average of all school marks in all subjects except the two subjects for which the correlation was under investigation. Each measure of this school intelligence was the average of from 25 to 40 grades given during the four years' course. The averaging was done by two independent computers for the three schools from which we had the largest number of records. Correlations between this school intelligence and various subjects were then found and the so-called *partial correlations* determined.

Partial correlation can be easily illustrated by an example. Suppose the heights of many sons, fathers, and grandfathers be measured. There will be a correlation between the sons' heights and the grandfathers' heights. The question arises as to how much of this correlation is due to the correlations sons — fathers, fathers — grandfathers. The obvious thing to do would be to find many fathers of the same height and then measure the correlation, if any, between the sons and grandfathers. This correlation coefficient is called the partial coefficient of correlation for heights of sons—heights of grandfathers, the heights of fathers being constant.

To see the effect of school intelligence upon the correlation coefficient between two subjects, say algebra and English, we must find the partial correlation coefficient for algebra — English with school intelligence constant. In the case of ordinary correlation coefficients we safeguarded our results by considering the two lines of regression. For three variables the analagous theoretical discussion leads to three equations in three variables (the regression planes). In the simple case of two variables, if we have given one of the two grades we can find the associated mean grade of the other from the equation of the appropriate regression line. In the case of three variables, if we have given two of the variables we can find the associated mean of the third variable from the appropriate regression equation in three variables. The theory is easily extended to n variables.

Considering the three subjects, freshman Algebra, average last 3 yrs. English and average in all other studies for three schools we find a great decrease in the correlation between algebra and English when our so-called school intelligence is held constant. This is shown in the following table.

School	Correlation Algebra-Av. English	Correlation Algebra-School Average	Correlation English-School Average	Partial Correlation Algebra-English, School Average Constant
A, girls	.54	.61	.69	.22
A, boys	.41	.58	.71	.00
D, girls	.53	.64	.66	.25
D, boys	.59	.64	.69	.27
F, girls	.38	.50	.63	.10
F, boys	.51	.62	.55	.25

From this table we see that the correlation between algebra and English is influenced greatly by the general school intelligence. The higher the correlations algebra-school average, English-school average, the greater the difference. This is corroborated by the regression equations. Letting x represent the grade in algebra, y in English, z in school intelligence, the regression equations which give the mean grade in English in terms of the grades in algebra and school intelligence are

School A, girls: $y = .15x + .64z$
" A, boys: $y = .00x + .96z$
" D, girls: $y = .22x + .62z$
" D, boys: $y = .25x + .69z$
" F, girls: $y = .06x + .44z$
" F, boys: $y = .16x + .33z$

The magnitude of the coefficients of x (or z) gives the average change in y associated with the unit change in x (or z). In these equations we see that in each case the effect of z upon y is much greater than the effect of x.

Turning now to the regression equations which show the mean grade in school intelligence associated with grades in algebra and English we have

School A, girls: $z = .22x + .45y$
" A, boys: $z = .23x + .43y$
" D, girls: $z = .33x + .38y$
" D, boys: $z = .29x + .34y$
" F, girls: $z = .57x + .44y$
" F, boys: $z = .31x + .38y$

These show that in general the grade in school intelligence z is associated more closely with the grade in average English y than with the grade in freshman algebra x. This is also shown in the table giving the coefficients for algebra-school average, English-school average. The probable errors in the constants in these equations vary from .03 to .06.

In the case of school A the algebra was replaced by freshman English and for comparison purposes the partial coefficients and regression equations worked out.

School	Correlation 1st yr. English-average English	Correlation 1st yr. English-School average	Correlation Av. English-School average	Partial Correlation 1st yr. English-average English
A, girls	.67	.65	.69	.40
boys	.60	.58	.71	.33

The regression equations giving the mean grade, y, in the English of the last three years in terms of x, freshman English, and z, school intelligence are,

girls, $y = .33x + .52z$
boys, $y = .29x + .73z$

showing that here the general school intelligence affects the grade in the English of the last three years more than does the freshman English. The regression equations giving the mean grade, z, in terms of x and y are

$$z = .26x + .39y$$
$$z = .18x + .43y$$

which is in accord with the correlations with the school average in the table for this case.

It is interesting at this point to see the effect of school intelligence on courses having extremely close relation like consecutive courses in Latin.

These were worked out for school A.

School	Correlation 1st yr. Latin-2nd yr. Latin	Correlation 1st yr. Latin-School average	Correlation 2nd yr. Latin-School average	Partial Correlation 1st yr. Latin-2nd yr. Latin
A, girls	.73	.62	.71	.52
boys	.73	.66	.79	.45

Letting x represent freshman Latin, y Sophomore Latin, and z school average, the regression equations are

$$y = .51x + .70z$$
$$y = .39x + .92z$$

where freshman Latin has much less influence on sophomore Latin than has school intelligence. In the equation for z,

$$z = .14x + .33y$$
$$z = .11x + .41y$$

the sophomore Latin outweighs the freshman Latin in its effect on the general average. We may emphasize this in the following way of looking at the effect of general school intelligence. If we are to select from the freshmen who studied Latin a group of students to form an honor class in sophomore Latin, we would get a better class by selecting not those who excelled in Latin, but those who excelled in their school work generally. This seems to be not at all confined to Latin.

It is reasonable to expect that the correlation between algebra, or English, and what we have called school intelligence is much higher than the correlations with some real measure of general intelligence not associated with school grades. For the cases which have come up in this study, the effect of a lowering of the correlations with intelligence would be to increase the partial correlations. For example, in one of the above tables we have the correlations algebra-English .54, algebra-school intelligence .61, English-school intelligence .69, with a resulting partial correlation coefficient .22 for algebra-English. Using .55 and .65 instead of .61 and .69 we have the partial coefficient .29. Using .50 and .60 we get .35.

The Bureau of Educational Research at the University of Illinois kindly turned over to us some data from their records bearing on this point. The pupils of the Harrison Technical High School had been given two general intelligence tests and the grades in algebra and English secured for 270 pupils. Using a combination of the two intelligence tests as the grade in intelligence the following results were found

Correlation coefficient	algebra—English	.37
“ “	algebra—intelligence	.32
“ “	English—intelligence	.36
Partial correlation coefficient	algebra—English	.28

Here, due to the relatively low correlations with intelligence the difference between the correlation coefficient and the partial correlation coefficient for algebra-English is not as great as is found when school average is a measure of intelligence.

Letting x represent algebra, y English, and z intelligence we find for the regression equation for y in terms of x and z,

$$y = .26x + .27z;$$

that is, the effects of algebra and of general intelligence on the grade in English are practically equal. The regression equation for z in terms of x and y is

$$z = .20x + .28y;$$

that is, the grade in general intelligence is more closely associated with the grade in English than it is with the grade in algebra, as was also the case when the measure of intelligence was the average of school grades.

Taking manual training, average English, and school intelligence for the school having the most data, school A, we find the correlations manual training-English, .22; manual training—general average .29, English—general average .58. The partial correlation for manual training—English is .06. For the same school the partial correlation for algebra-English was .00. Since the probable errors were about .05 these results do not differ much. The reason that in this case the result for manual training seems higher than for algebra is due to the low correlation between manual training and English. A number of partial correlations for English with manual training, bookkeeping, stenography and domestic science were worked out. In each case the results were very small numbers, all lying between $-.10$ and $.10$. In these subjects the correlations with English and school intelligence were lower than for algebra.

It is of course not necessary to stop with three variables in discussing partial correlation. We may have any number. For one school, school A, the following four subjects were taken, freshman algebra, x, freshman English, y, average of the last three years English, z, and the average of all other studies, u. The regression equations for z in terms of x, y, and u are

$$\text{girls, } z = .06x + .25y + .47u$$
$$\text{boys, } z = -.07x + .31y + .77u$$

Since the probable errors of these coefficients are between .03 and .05, the coefficients of x do not mean much. In this case any effect of freshman algebra upon the English comes by way of the school intelligence. This was the school yielding the lowest partial correlations for algebra-English. The regression equations for school intelligence in terms of the other variables are

$$\text{girls, } u = .17x + .17y + .35z$$
$$\text{boys, } u = .21x + .08y + .39z.$$

For this same school the freshman English was replaced by manual training. The regression equation for average English in terms of the others is

$$z = .03x + .02y + .80u.$$

where the coefficient for school intelligence is the predominant coefficient. The equation for u in terms of the others is

$$u = .22x + .12y + 31z.$$

The noticeable thing in all of these regression equations is the relative importance of the coefficient of the variable representing school intelligence. With the exception of subjects having many elements in common such as consecutive courses in the same subject the relation expressed by means of the correlation coefficient is due to the association of each subject with school intelligence. In the case of algebra—English there is in some schools a small residue left after accounting for the school intelligence which may be called the intrinsic relation between algebra and English. This probably would have been larger if a measure of general intelligence had been used instead of school averages. In the case of vocational courses this residue is in general evanescent.

Summary.—The correlation coefficient is in general a good measure of the relation between the two sets of numbers given by grades in two school subjects. However, the variation from school to school in the coefficients for any particular pair of subjects is so great that the coefficient from one school is not very significant. Even in a single school the coefficient will depend upon whether our data come from the whole body of pupils or are restricted to graduates. If it were important to find for example *the* correlation coefficient for English—algebra (if it exists) very many correlation coefficients from all sorts of schools would have to be calculated and various corrections applied. The problem would be very similar to the problem of finding the mortality of a country from observations taken in the separate districts.

In this investigation, in which we are comparing coefficients of correlation in various pairs of subjects, the data are restricted to those taken from the records of high school graduates, a rather restricted body of pupils, and we should expect the coefficients to be somewhat less than those for the general body of pupils. We find that algebra occupies in general an intermediate position both in its relation to future English courses and to other courses of the curriculum. It seems to stand with freshman English significantly above vocational courses.

The relation between two school subjects may be accounted for by the relations of each with general intelligence and by a certain intrinsic relation peculiar to the two subjects. This intrinsic relation is most noticeable in consecutive courses in the same subject, but even here it is not large. In the case of vocational subjects like manual training, stenography, bookkeeping and domestic science this intrinsic relation seems to be lacking. In the case of algebra there seems to be a small residue left unexplained by general intelligence due to the relatively high correlations between algebra and intelligence as measured by general school averages. If a good measure of general intelligence had been used the intrinsic correlations would probably be higher than those found. There seems to be little use of separating the sexes in such an investigation.

BIBLIOGRAPHY

1901. **Wissler, C.** The Correlation of Mental and Physical Tests. Psychological Review, Monograph Supplements, No. 16.

1903. **Smith, A. G.** Heredity. Correlation and Sex Difference in School Abilities. Edited by E. L. Thorndike. Columbia University Contributions to Education, Vol. XI, No. 2.

Burris, W. P. Correlations of the Abilities Involved in Secondary School Work. Edited by E. L. Thorndike, Columbia University Contributions to Education, Vol. XI, No. 2.

Brinkerhoff, E. C., and Morris, G. The Relationships between the Abilities Involved in Secondary School Subjects. Edited by E. L. Thorndike. Columbia University Contributions to Education. Vol. XI, No. 2.

1904. **Spearman, C.** The Proof and Measurement of Association between Two Things. American Journal of Psychology, Vol. 15.

Spearman, C. "General Intelligence," objectively determined and measured. American Journal of Psychology, Vol. 15.

1905. **Lewis, L. C.** A Study in Formal Discipline. School Review, Vol. 13.

Wissler, C. The Spearman Correlation Formula. Science. Vol. XXII.

1906. **Spearman, C.** Footrule for Measuring Correlation. The British Journal of Psychology, Vol. 2.

1907. **Spearman, C.** Demonstration of Formulae for True Measurement of Correlation. American Journal of Psychology, Vol. XVIII.

1908. **Rietz, H. L., and Shade, I.** Correlation of Efficiency in Mathematics and Efficiency in other Subjects. University of Illinois Studies, Vol. VI. No. 10.

1909. **Spearman, C.** Correlation Calculated from Faulty Data. British Journal of Psychology, Vol. 3.

1915. **Moore, C. N.** On Correlation and Disciplinary Values. School and Society, Vol. II.

Moore, C. N. On the Coefficient of Correlation as a Measure of Relationship. Science, Vol. XLII.

1916. **Rietz, H. L.** On the Correlation of the Marks of Students in Mathematics and in the Law. Journal of Educational Psychology, Vol. VII.

1916. **Rugg, H. O.** The Experimental Determination of Mental Discipline in School Studies. Educational Psychology Monographs, No. 17.

1917 **Crathorne, A. R.** Required Mathematics. School and Society, Vol. VI.

1919. **Ingels, Nelle L.** A Statistical Study in Correlation of Efficiency in Secondary Mathematics and Efficiency in other High School Branches. The Mathematics Teacher, Vol. XI, No. 4.

APPENDIX

The Complete Tables of Correlation Coefficients

HIGH SCHOOL A

		GIRLS					BOYS				
		N	r	η_1	η_2	B	N	r	η_1	η_2	B
1	2nd yr. Latin — 3rd yr. Latin	65	.72±.04	.73	.74	2	—	—	—	—	—
2	1st yr. Latin — 2nd yr. Latin	133	.73±.03	.73	.78	10	74	.73±.04	.77	.77	5
3	3rd yr. Latin — 4th yr. Latin	35	.57±.08	.68	70	6	—	—	—	—	—
4	2nd yr. Latin — 4th yr. Latin	35	.57±.08	.70	.66	6	—	—	—	—	—
5	1st yr. History — Av. English	298	.60±.03	.61	.61	6	195	.58±.03	.61	.65	—
6	1st yr. English — 2nd yr. English	351	.67±.02	.69	.68	8	234	.54±.03	.55	.55	4
7	2nd yr. English — 3rd yr. English	335	.68±.02	.69	.70	9	22	.63±.09	.66	.89	9
8	1st yr. English — Av. English	348	.67±.02	.68	.68	3	234	.60±.03	.63	.63	7
9	1st yr. Latin — 3rd yr. Latin	74	.70±.04	.73	.79	11	—	—	—	—	—
10	3rd yr. English — 4th yr. English	347	.65±.02	.66	.66	5	282	.64±.03	.66	.65	—
11	2nd yr. English — 4th yr. English	346	.51±.03	.54	.52	11	232	.57±.03	.58	.59	5
12	Algebra — Geometry	349	.57±.03	.64	.62	25	255	.43±.04	.47	.48	—
13	1st yr. English — 3rd yr. English	338	.56±.03	.57	.57	2	—	—	—	—	—
14	Algebra — German	284	.44±.03	.47	.47	9	285	.45±.04	.48	.50	9
15	1st yr. English — Av. History	347	.50±.03	.51	.50	4	60	.53±.03	.53	.55	5
16	Algebra — Av. Latin	152	.52±.04	.54	.58	11	155	.49±.06	.58	.68	9
17	Algebra — Av. History	347	.54±.03	.55	.54	4	255	.43±.04	.50	.43	—
18	1st yr. Latin — Av. English	249	.51±.03	.55	.53	10	157	.56±.04	.59	.61	8
19	1st yr, English — Av. Latin	152	.46±.04	.51	.51	7	190	.54±.05	.59	.64	10
20	Algebra — Civics	326	.40±.03	.44	.44	11	310	.32±.04	.34	.34	3
21	1st yr. English — 4th yr. English	347	.52±.03	.53	.52	6	233	.45±.04	.58	.48	30
22	1st yr. Latin — 4th yr. Latin	35	.30±.10	.54	.46	6	—	—	—	—	—
23	Algebra — Av. English	349	.54±.03	.68	.55	6	255	.41±.04	.44	.47	2
24	1st yr. English — Civics	326	.47±.03	.48	.47	1	95	.38±.04	.38	.39	4
25	1st yr. English — Av. German	284	.43±.03	.46	.44	8	40	.47±.04	.49	.56	6
26	1st yr. English — Geometry	349	.40±.03	.46	.47	9	60	.44±.04	.45	.47	6
27	Domestic Science* — Av. English	134	.40±.05	.41	.41	1	100	.22±.07	.34	.27	7
28	Domestic Science — Av. German	110	.42±.05	.44	.44	2	71	.27±.07	.35	.33	4
29	Domestic Science — Geometry	134	.30±.05	.31	.35	4	100	.27±.06	.27	.38	8
30	Domestic Science — History	133	.28±.05	.29	.34	5	100	.23±.07	.39	.27	10
31	Domestic Science — Latin	52	.44±.08	.58	.49	7	34	.07±.14	.39	.27	10
32	Domestic Science — Civics	121	.24±.06	.37	.38	11	97	.13±.07	.30	.14	7

*"Domestic Science" applies to the Girls' columns only; for the Boys' columns it should be replaced by "Manual Training".

HIGH SCHOOL B

		GIRLS					BOYS				
		N	r	η_1	η_2	B	N	r	η_1	η_2	B
1	2nd yr. Latin — 3rd yr. Latin	46	.79± .04	.82	.84	4	—	—	—	—	—
2	1st yr. Latin — 2nd yr. Latin	74	.64± 05	.70	.70	6	31	53±.09	.49	.65	4
3	3rd yr. Latin — 4th yr. Latin	30	.75± .05	.80	.84	4	—	—	—	—	—
4	2nd yr. Latin — 4th yr. Latin	31	.74± .05	.75	.76	1	—	—	—	—	—
5	1st yr. history — Av. English	30	67± 07	.69	.78	4	—	—	—	—	—
6	1st yr. English — 2nd yr. English	164	.62± 03	.63	64	5	99	.67±.04	.68	.71	5
7	2nd yr. English — 3rd yr. English	163	.60± 03	.62	.60	3	98	.60±.04	.64	61	5
8	1st yr. English — Av. English	164	.66 ±.03	.67	.67	3	101	.61±.04	.62	.64	4
9	1st yr. Latin — 3rd yr. Latin	48	55± 07	.62	58	4	—	—	—	—	—
10	3rd yr. English — 4th yr. English	145	.52±.04	.62	.56	18	75	.57±.05	.57	58	2
11	2nd yr. English — 4th yr. English	146	.64± 03	.64	.65	8	94	.56±.05	61	56	5
12	Algebra — Geometry	144	53 ± .04	56	.58	9	95	.54±.05	62	55	8
13	1st yr. English — 3rd yr. English	163	.43±.04	.49	.49	8	99	.56±.05	.59	57	4
14	Algebra — German	137	.60± 04	.62	.61	5	79	.43±.07	56	.51	10
15	1st yr. English — Av. History	162	.66 ± 03	.67	78	28	99	.56±.05	57	58	2
16	Algebra — Av. Latin	94	.44±.06	.53	.51	8	50	.15± 10	21	27	2
17	Algebra — Av. History	160	.49±.04	.54	.51	8	95	.45± 06	49	47	3
18	1st yr. Latin — Av. English						32	.54± 09	81	58	2
19	1st yr. English — Av. Latin	94	.55± 05	.55	.60	6	44	.46± 08	49	49	2
20	Algebra — Civics	68	.41± 07	.46	.49	5	60	.44± 07	57	52	8
21	1st yr. English — 4th yr. English	144	.45± 04	.52	.48	9	96	.30± 06	32	33	2
22	1st yr. Latin — 4th yr .Latin	32	.60 ± 08	.70	.63	4	—	—	—	—	—
23	Algebra — Av. English	163	52± 04	.54	53	3	97	.40± 06	.48	43	7
24	1st yr. English — Civics	68	.54± 06	.61	59	8	63	44± 07	50	48	4
25	1st yr. English — Av. German	137	.44± 05	.48	47	6	83	37± 06	40	44	5
26	1st yr. English — Geometry	145	.35±.05	37	.38	3	99	37± 06	.41	41	3
27	Domestic Science* — Av. English	116	.35 ± 06	.36	36	2	56	37± 08	43	37	3
28	Domestic Science — Av. German	98	41± 06	.43	.47	5	48	22± 09	38	46	8
29	Domestic Science — Geometry	101	.40± 06	.41	44	3	54	36± 08	40	51	7
30	Domestic Science — History	113	.28± 06	.30	.28	11	54	.29±.12	29	.38	3
31	Domestic Science — Latin	65	.36±.07	.43	.43	4	—	—	—	—	—
32	Domestic Science — Civics	43	.42±.09	.50	.50	3	34	11± 11	.11	.51	8

HIGH SCHOOL C

		GIRLS					BOYS				
		N	r	η_1	η_2	B	N	r	η_1	η_2	B
1	2nd yr. Latin — 3rd yr. Latin	—	—	—	—	—	—	—	—	—	—
2	1st yr. Latin — 2nd yr. Latin	30	.65±.07	.69	.67	2	30	.60± 08	79	.65	7
3	3rd yr. Latin —4th yr. Latin	—	—	—	—	—	—	—	—	—	—
4	2nd yr. Latin — 4th yr. Latin	—	—	—	—	—	—	—	—	—	—
5	1st yr. History — Av. English	48	.67±.05	.69	.68	1	43	.39± 09	51	45	4
6	1st yr. English — 2nd yr. English	80	.62±.05	.63	.63	1	80	.50± 06	.56	.52	5
7	2nd yr. English — 3rd yr. English	93	.52±.05	.54	.54	2	83	.55± 05	57	.60	6
8	1st yr. English — Av. English	79	.62±.05	.64	.67	5	76	.43±.06	51	.47	5
9	1st yr. Latin — 3rd yr. Latin	—	—	—	—	—	—	—	—		—
10	3rd yr. English — 4th yr. English	63	.32±.08	.34	44	6	53	.28± 09	.30	49	9
11	2nd yr. English — 4th yr. English	63	.45±.07	.46	.50	3	50	.23± 09	.37	.31	4
12	Algebra — Geometry	55	.38±.08	.47	.50	6	79	.55± 05	.59	.61	5
13	1st yr. English — 3rd yr. English	81	.42±.06	.47	.44	4	82	.38± 06	40	.43	3
14	Algebra — German	41	.42±.09	.43	.57	6	44	.54± 07	62	.60	4
15	1st yr. English — Av. History	75	.40±.07	.43	.43	2	77	.49 ± 06	50	.51	1
16	Algebra — Av. Latin	30	.47±.10	.50	.54	2	33	.60± 07	.61	.63	2
17	Algebra — Av. History	77	.25±.07	.36	.35	5	81	.45± 06	47	48	2
18	1st yr. Latin — Av. English	33	25±.11	.32	.40	3	34	.62± 07	69	.62	3
19	1st. yr. English — Av. Latin	30	.29±.12	.30	.46	4	30	.44±.10	.46	.51	2
20	Algebra — Civics	—	—	—	—	—	32	.28± 11	.41	37	3
21	1st yr. English — 4th yr. English	52	.32±.08	.38	.46	6	50	.02±.09	.22	.20	2
22	1st yr. Latin — 4th yr. Latin	—	—	—	—	—	—	—	—	—	—
23	Algebra — Av. English	79	.19±.07	.24	.34	6	81	.54± 05	.59	.60	6
24	1st yr. English — Civics	—	—	—	—	—	31	.00±.12	.35	.44	5
25	1st yr. English — Av. German	40	.41±.09	.46	.44	2	40	.55± 07	57	.58	2
26	1st yr. English — Geometry	54	.45±.07	.47	.51	4	73	.42±.06	.46	.44	3
27	Domestic Science* — Av. English	54	.47±.07	.50	.50	2	32	.03±.12	.16	.18	—
28	Domestic Science — Av. German	31	.40±.11	.43	.44	2	—	—	—	—	—
29	Domestic Science — Geometry	30	.49±.10	.54	.58	3	30	.49±.09	.56	.50	—
30	Domestic Science — History	53	.37±.08	.37	.39	1	32	.10±.11	.28	.29	—
31	Domestic Science — Latin	—	—	—	—	—	—	—	—	—	—
32	Domestic Science — Civics	—	—	—	—	—	—	—	—	—	—

*"Domestic Science" applies to the Girls' columns only; for the Boys' columns it should be replaced by "Manual Training".

HIGH SCHOOL D

		GIRLS					BOYS				
		N	r	η_1	η_2	B	N	r	η_1	η_2	B
1	2nd yr. Latin — 3rd yr. Latin	—	—	—	—	—	47	.70±.05	.73	.71	2
2	1st yr. Latin — 2nd yr. Latin	170	.67±.03	.77	.70	24	104	.62±.04	.70	.64	11
3	3rd yr. Latin — 4th yr. Latin	—	—	—	—	—	—	—	—	—	—
4	2nd yr. Latin — 4th yr. Latin	—	—	—	—	—	—	—	—	—	—
5	1st yr. History — Av. English	—	—	—	—	—	47	.56±.07	.59	.57	2
6	1st yr. English — 2nd yr. English	298	.60±.02	.60	.61	2	196	.57±.03	.62	.62	11
7	2nd yr. English — 3rd yr. English	291	.61±.03	.61	.61	0	192	.66±.03	.67	.68	4
8	1st yr. English — Av. English	298	.64±.02	.64	.66	9	196	.61±.03	.66	.62	2
9	1st yr. Latin — 3rd yr. Latin	87	.46±.06	.49	.48	3	54	.47±.07	.63	.73	10
10	3rd yr. English — 4th yr. English	112	.60±.04	.68	.62	11	97	.68±.04	.71	.77	4
11	2nd yr. English — 4th yr. English	111	.51±.05	.52	.56	6	98	.49±.05	.52	.53	4
12	Algebra — Geometry	288	.55±.03	.56	.56	1	191	.68±.03	.71	.69	7
13	1st yr. English — 3rd yr. English	282	.57±.03	.58	.58	3	192	.54±.03	.55	.61	16
14	Algebra — German	211	.62±.03	.65	.63	8	107	.63±.04	.65	.64	3
15	1st yr. English — Av. History	288	.54±.03	.55	.55	3	184	.46±.04	.48	.47	3
16	Algebra — Av. Latin	251	.55±.03	.56	.56	3	152	.70±.03	.56	.56	3
17	Algebra — Av. History	283	.49±.03	.50	.51	5	184	.48±.04	.49	.51	6
18	1st yr. Latin — Av. English	—	—	—	—	—	147	.58±.04	.59	.59	1
19	1st yr. English — Av. Latin	251	.57±.03	.58	.57	2	152	.64±.03	.68	.64	7
20	Algebra — Civics	—	—	—	—	—	—	—	—	—	—
21	1st yr. English — 4th yr. English	114	.51±.05	.52	.58	6	100	.37±.06	.47	.38	9
22	1st yr. Latin — 4th yr. Latin	36	.22±.11	.34	.32	3	—	—	—	—	—
23	Algebra — Av. English	297	.52±.03	.53	.54	8	193	.64±.03	.64	.63	3
24	1st yr. English — Civics	—	—	—	—	—	—	—	—	—	—
25	1st yr. English — Av. German	212	.42±.04	.43	.44	3	107	.61±.04	.64	.64	5
26	1st yr. English — Geometry	278	.43±.03	.45	.43	4	191	.49±.04	.53	.51	8
27	Domestic Science* — Av. English	255	.36±.04	.37	.39	5	128	.15±.06	.17	.23	4
28	Domestic Science — German	184	.33±.05	.33	.36	4	76	.15±.08	.17	.27	4
29	Domestic Science — Geometry	246	.38±.04	.38	.39	2	124	.21±.06	.26	.30	5
30	Domestic Science — History	248	.36±.04	.36	.37	2	120	.11±.06	.14	.22	4
31	Domestic Science — Latin	217	.30±.04	.32	.31	2	87	.04±.07	.10	.21	3
32	Domestic Science — Civics	125	.11±.06	.20	.30	9	68	.22±.08	.32	.41	2

HIGH SCHOOL E

		GIRLS					BOYS				
		N	r	η_1	η_2	B	N	r	η_1	η_2	B
1	2nd yr. Latin — 3rd yr. Latin	47	.71±.05	.73	.80	6	37	.80±.04	.86	.81	4
2	1st yr. Latin — 2nd yr. Latin	53	.74±.04	.81	.75	6	44	.73±.04	.78	.75	3
3	3rd yr. Latin — 4th yr. Latin	43	.72±.05	.74	.74	2	—	—	—	—	—
4	2nd yr Latin — 4th yr. Latin	43	.67±.06	.69	.71	2	—	—	—	—	—
5	1st yr. History — Av. English	76	.53±.06	.57	.56	3	78	.57±.05	.63	.60	5
6	1st yr. English — 2nd yr. English	—	—	—	—	—	—	—	—	—	—
7	2nd yr. English — 3rd yr. English	—	—	—	—	—	—	—	—	—	—
8	1st yr. English — Av. English	78	.44±.06	.45	.46	1	82	.35±.06	.45	.45	7
9	1st yr. Latin — 3rd yr. Latin	47	.65±.06	.66	.75	7	37	.64±.07	.79	.65	8
10	3rd yr. English — 4th yr. English	77	.49±.06	.59	.57	8	84	.67±.04	.68	.70	4
11	2nd yr. English — 4th yr English	—	—	—	—	—	—	—	—	—	—
12	Algebra —Geometry	76	.47±.06	.49	.48	2	82	.40±.06	.42	.41	1
13	1st yr. English — 3rd yr. English	78	.57±.05	.57	.65	8	—	—	—	—	—
14	Algebra — German	47	.49±.08	.51	.65	9	55	.46±.07	.50	.51	3
15	1st yr. English — Av. History	35	.34±.10	.58	.43	8	—	—	—	—	—
16	Algebra — Av. Latin	53	.35±.08	.39	.38	2	44	.61±.06	.67	.65	3
17	Algebra — Av. History	35	.41±.10	.50	.54	4	—	—	—	—	—
18	1st yr. Latin — Av. English	75	.53±.06	.56	.65	11	74	.42±.07	.56	.47	3
19	1st yr. English — Av. Latin	52	.43±.08	.53	.45	5	44	.33±.06	.37	.37	1
20	Algebra — Civics	30	.59±.08	.74	.63	6	43	.36±.09	.45	.44	3
21	1st yr. English — 4th yr. English	77	.38±.07	.44	.42	3	82	.33±.07	.39	.39	4
22	1st yr. Latin — 4th yr. Latin	43	.56±.07	.59	.59	1	31	.69±.06	.89	.75	9
23	Algebra — Av. English	78	.42±.06	.49	.43	5	82	.33±.07	.35	.36	2
24	1st yr. English — Civics	30	.54±.09	.58	.58	1	43	.35±.09	.39	.42	2
25	1st yr. English — German	47	.40±.08	.43	.47	3	55	.45±.07	.47	.47	1
26	1st yr. English — Geometry	77	.47±.06	.69	.50	2	82	.43±.06	.45	.55	10
27	Domestic Science* — Av. English	—	—	—	—	—	—	—	—	—	—
28	Domestic Science — German	—	—	—	—	—	—	—	—	—	—
29	Domestic Science — Geometry	—	—	—	—	—	—	—	—	—	—
30	Domestic Science — History	—	—	—	—	—	—	—	—	—	—
31	Domestic Science — Latin	—	—	—	—	—	—	—	—	—	—
32	Domestic Science — Civics	—	—	—	—	—	—	—	—	—	—

*"Domestic Science" applies to the Girls' columns only; for the Boys' columns it should be replaced by "Manual Training".

HIGH SCHOOL F

		GIRLS					BOYS				
		N	r	η_1	η_2	B	N	r	η_1	η_2	B
1	2nd yr. Latin — 3rd yr. Latin	50	.76±.04	.78	.82	5	—	—	—	—	—
2	1st yr. Latin — 2nd yr. Latin	67	.79±.03	.80	.80	2	—	—	—	—	—
3	3rd. yr. Latin — 4th yr. Latin	34	.69±.06	.70	.71	1	—	—	—	—	—
4	2nd yr. Latin —4th yr. Latin	34	.58±.08	.66	.80	10	—	—	—	—	—
5	1st yr.History — Av. English	—	—	—	—	—	123	.76±.03	.77	.77	2
6	1st yr. English — 2nd yr. English	186	.51±.04	.52	.51	1	125	.55±.04	.57	.56	3
7	2nd yr. English — 3rd yr. English	186	.54±.04	—	—	—	125	.57±.04	.58	.57	1
8	1st yr. English — Av English	186	.51±.04	.52	.55	8	125	.63±.04	.64	.64	2
9	1st yr. Latin — 3rd yr. Latin	51	.56±.07	.63	.59	5	—	—	—	—	—
10	3rd yr. English — 4th yr English	186	.67±.03	.69	.68	5	125	.58±.04	.65	.69	18
11	2nd yr. English — 4th yr. English	186	.48±.04	.52	.52	7	—	—	—	—	—
12	Algebra — Geometry	121	.66±.03	.67	.67	2	116	.53±.04	.53	.55	8
13	1st yr. English—3rd yr. English	—	—	—	—	—	125	.59±.04	.63	.60	5
14	Algebra — German	67	.42±.07	.52	.45	6	39	.47±.08	.50	.68	9
15	1st yr. English — Av. History	124	.43±.05	.46	.45	3	68	.52±.05	.54	.56	4
16	Algebra — Av. Latin	111	.50±.05	.52	.51	3	63	.62±.05	.65	.63	2
17	Algebra — Av. History	124	.61±.04	.65	.62	7	68	.56±.06	.62	.59	5
18	1st yr. Latin — Av. English	86	.56±.05	.59	.56	3	—	—	—	—	—
19	1st yr. English — Av. Latin	111	.45±.05	.48	.52	7	63	.55±.06	.57	.56	2
20	Algebra — Civics	185	.37±.04	.38	.41	6	123	.38±.05	.40	.41	2
21	1st yr. English — 4th yr. English	—	—	—	—	—	125	.49±.05	.50	.49	1
22	1st yr. Latin — 4th yr. Latin	35	.49±.09	.60	.54	4	—	—	—	—	—
23	Algebra — Av. English	186	.41±.04	.49	.42	13	125	.50±.05	.53	.53	4
24	1st yr. English — Civics	185	.16±.05	.74	.20	97	123	.38±.05	.41	.38	4
25	1st yr English — German	67	.35±.07	.41	.43	4	39	.63±.06	.69	.73	6
26	1st yr. English — Geometry	121	.25±.06	.30	.31	4	116	.40±.05	.45	.42	5
27	Domestic Science* — Av. English	124	.34±.05	.36	.37	2	33	.40±.10	.55	.41	5
28	Domestic Science — German	42	.32±.09	.49	.40	6	—	—	—	—	—
29	Domestic Science — Geometry	69	.14±.08	.40	.43	11	31	.36±.10	.54	.54	5
30	Domestic Science — History	77	.40±.07	.51	.49	8	—	—	—	—	—
31	Domestic Science — Latin	60	.23±.08	.48	.40	—	—	—	—	—	—
32	Domestic Science — Civics	123	.24±.06	.40	.30	14	33	.60±.08	.62	.63	3

HIGH SCHOOL A

BOYS AND GIRLS

		N	r	η_1	η_2	B
1	Bookkeeping — Latin	97	.42±.06	.49	.45	6
2	" — History	181	.41±.04	.43	.43	4
3	" — Av. English	180	.40±.04	.43	.44	5
4	" — German	129	.38±.05	.41	.40	6
5	" — Geometry	180	.39±.04	.44	.43	7
6	" — Civics	175	.40±.04	.42	.48	12
7	Stenography — Av. Eng.	228	.41±.04	.42	.45	2
8	" — Geometry	229	.44±.04	.45	.47	7
9	" — German	180	.37±.04	.40	.44	9
10	" — Civics	226	.22±.04	.31	.26	11
11	" — Latin	132	.28±.05	.41	.33	12
12	" — History	—	—	—	—	—

*"Domestic Science" applies to the Girls' columns only; for the Boys' columns it should be replaced by "Manual Training".

High School B

BOYS AND GIRLS

		N	r	η_1	η_2	B
1	Bookkeeping — Latin	55	.49±.07	.58	.53	6
2	“ — History	97	.40±.06	.44	.42	3
3	“ — English	102	.29±.06	.32	.32	2
4	“ — German	84	.37±.06	.50	.43	8
5	“ — Geometry	86	.39±.06	.59	.47	17
6	“ — Civics	53	.36±.08	.40	.44	3
7	Stenography — Av. English	78	.32±.07	.35	.43	8
8	“ — Geometry	75	.09±.08	.22	.27	5
9	“ — German	81	.32±.07	.33	.39	4
10	“ — Civics	49	.15±.09	.37	.22	6
11	“ — Latin	49	.32±.09	.44	.37	5
12	“ — History	89	.24±.07	.31	.29	4

High School C

BOYS AND GIRLS

		N	r	η_1	η_2	B
1	Bookkeeping — Latin	—	—	—	—	—
2	“ — History	30	.35±.11	.37	.30	1
3	“ — Av. English	31	.31±.11	.36	.39	2
4	“ — German	—	—	—	—	—
5	“ — Geometry	30	.35±.11	.56	.49	6
6	“ — Civics	—	—	—	—	—
7	Stenography — Av. English	38	.15±.11	.18	.34	4
8	“ — Geometry	30	.35±.11	.38	.73	12
9	“ — German	—	—	—	—	—
10	“ — Civics	—	—	—	—	—
11	“ — Latin	—	—	—	—	—
12	“ — History	36	.01± .11	.47	.44	8

High School D

BOYS AND GIRLS

		N	r	η_1	η_2	B
1	Bookkeeping — Latin	35	.44±.09	.59	.51	6
2	“ — History	45	.55±.08	.58	.55	1
3	“ — English	54	.32±.08	.36	.33	2
4	“ — German	30	.32±.11	.46	.34	3
5	“ — Geometry	44	.17±.10	.24	.34	4
6	“ — Civics	29	.36±.12	.40	.45	2
7	Stenography — Av.English	69	.43±.07	.52	.56	9
8	“ — Geometry	59	.47±.07	.49	.55	5
9	“ — German	43	.51±.08	.56	.60	4
10	“ — Civics	32	.38±.10	.39	.45	2
11	“ — Latin	48	.53±.07	.63	.58	6
12	“ — History	63	.43±.07	.48	.44	3

High School F

BOYS AND GIRLS

		N	r	η_1	η_2	B
1	Bookkeeping — Latin	30	.42±.10	.54	.66	7
2	“ — History	76	.18±.07	.25	.29	4
3	“ — Av. English	75	.13±.08	.24	.27	4
4	“ — German	30	.26±.12	.63	.67	4
5	“ — Geometry	44	.50±.08	.54	.51	2
6	“ — Civics	76	.05±.08	.16	.14	2
7	Stenography — Av. English	95	.51±.05	.51	.55	4
8	“ — Geometry	50	.46±.07	.48	.55	5
9	“ — German	34	.62±.07	.70	.71	4
10	“ — Civics	94	.46±.05	.61	.52	15
11	“ — Latin	46	.62±.06	.64	.68	4
12	“ — History	95	.67±.04	.68	.68	1

CHAPTER XI

Mathematical Curricula in Foreign Countries[1]

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No one believes in the bodily transplantation of a course of study or of methods of work from one nation to another; yet every nation can profit from a careful comparative study of the work of other nations.

Administrators and teachers are now forced by events to consider eliminations, additions, and modifications of the courses of study and the methods of instruction. A study of conditions in other countries should be very profitable as a basis for readjustment of our own courses.

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[1] For a more detailed discussion of courses in mathematics given in foreign countries, the reader is referred to *Curricula in Mathematics*, or comparison of courses in the countries represented by the International Commission on the Teaching of Mathematics, by J. C. Brown (U. S. Bureau of Education, Bulletin, 1914, No. 45, Washington, 1915.) It should be noted that the present chapter describes conditions as they were before the war. It appears that changes since then have not as yet been extensive or far reaching.

The curricula in mathematics of the leading nations of Europe have much more in common with each other than with those of the United States. European nations will therefore sometimes find in the practice of their sister nations encouraging confirmation of their own customs, whereas the United States will be confronted by the question as to whether she alone has found the path that is best, at least for herself. Some lines of divergence between the general practice in Europe and that of the United States are mentioned in the concluding remarks of this chapter (p. 172). Historical and other reasons for the existence of the divergencies are easily found. Reasons, sometimes of a theoretical and sometimes of a practical nature, may doubtless also be found sufficient to warrant the present continuance of some of these divergencies. At the same time the possession of authoritative statements of the practice of the world's leading nations, as contained in the reports from which the material that follows is taken, can but prove helpful and stimulating to the educators of the United States.

All of the statements of fact in the chapter are based upon reports of the International Commission to the congress at Cambridge. A complete bibliography of the reports used is given on page 173-176. The data for the various nations have perforce been given with varying degrees of completeness, depending upon the information available in the reports. The reports from all of the countries indicate conditions at the time that the reports were submitted (1912-1914). As the age at which pupils enter school varies somewhat in different countries, the age of the pupil, rather than the school year, was chosen as the basis of comparison.

Unless otherwise stated, it is to be understood that the European schools are for boys only. Relatively little is as yet done in most of the European countries for the mathematical education of girls beyond the fundamentals of arithmetic. The scanty information that is available is given in appropriate connections.

I. General Arrangement of the Courses in Typical Schools of the Various Countries.

AUSTRIA.

Arrangement of school years.

Age of pupil.	School year.	Volksschule.	Bürgerschule.	Gymnasium.	Realschule.	Realgymnasium.
6–7	First	1				
7–8	Second	2				
8–9	Third	3				
9–10	Fourth	4				
10–11	Fifth	5		1	1	1
11–12	Sixth		1	2	2	2
12–13	Seventh		2	3	3	3
13–14	Eighth		3	4	4	4
14–15	Ninth			5	5	5
15–16	Tenth			6	6	6
16–17	Eleventh			7	7	7
17–18	Twelfth			8		8

Compulsory education extends from the sixth to the fourteenth year. Coeducation is the rule in the rural districts and the exception in the cities.

All schools, whether public or private, are subject to the same regulations as the State schools. Most of the secondary schools are maintained by the State.

There are two kinds of elementary schools: (*a*) The common primary, usually called the Volksschule, and (*b*) the superior elementary school, called the Bürgerschule. Many of the Bürgerschulen in the rural communities are ungraded. In the cities the course in these schools comprises from two to eight grades. The Bürgerschulen usually consist of three classes, which follow the fifth year of the primary school. Pupils may enter directly from the primary school. Different courses are offered for boys and girls, and as far as possible the sexes are instructed in different schools.

Sometimes an intermediate course of one year follows the Bürgerschule. The nature of the course which a pupil follows during this year depends largely upon the type of school for which he is preparing. It is now proposed to make the intermediate course an additional class of the Bürgerschule.

There are three general types of secondary schools: (*a*) The Gymnasium, (*b*) the Realschule, and (*c*) the Realgymnasium. The first has a course of eight, the second of seven, and the third of eight years, thus differing from the custom in Germany. The minimum age for admission to the Gymnasium is 10 years, so that the first year in the Austrian Gymnasium corresponds to the second year in the German Gymnasium, and four years of preliminary work are required for entrance. The average age at entrance is 10½ to 11 years.

The Gymnasium offers the traditional classical course and prepares the pupils for entering any university. In the Gymnasium more than 50 per cent of the school time is devoted to the study of Latin, Greek, history, and the mother tongue. About 25 per cent of the time is devoted to mathematics, the history of Austria, geography, physics, and chemistry. Latin and Greek are emphasized.

The Realschule attempts to furnish a thorough knowledge of the modern subjects, with special attention to the sciences. In these schools the pupil is prepared for the study of more advanced natural science and mathematics; no attempt is made to prepare him for any particular vocation.

The course in the Realgymnasium is intended for those who wish some secondary education but do not expect to enter a university or a higher technical school. A graduate of the Realgymnasium may enter certain classes of the Gymnasium or of the Realschule.

A type of secondary school called the Reform Gymnasium, standing intermediate in courses offered between the classical Gymnasium and the Realschule, was organized in 1908. It has an eight-year course. The four lower classes are identical with the corresponding classes of the Realschule. In the four upper classes Latin, a modern language, and free-hand drawing are required; in other respects the course is like the upper grades of the Gymnasium. Greek is not taught in the Reform Gymnasium.

BELGIUM.

Arrangement of school years.

Age of pupil.	School year.	Primary school.	Middle school.	Athénée royal.
6–7	First	1		
7–8	Second	2		
8–9	Third	3		
9–10	Fourth	4		
10–11	Fifth	5		
11–12	Sixth	6		
12–13	Seventh		1	1
13–14	Eighth		2	2
14–15	Ninth		3	3
15–16	Tenth			4
16–17	Eleventh			5
17–18	Twelfth			6
18–19	Thirteenth			7

The primary school in Belgium comprises the first six school years. About one-third of the number are coeducational. The primary schools are under the control of the commune, but the State reserves the right of inspection. After completing the prescribed work of the primary school—that is, at about the age of 12—a pupil may enter either the middle school or the Athénée Royal. The lowest class in each of these schools corresponds, therefore, to our seventh grade.

The course in the middle schools is three years in length, corresponding to our seventh, eighth, and ninth grades. These schools were created to meet the needs of the higher artisan and the commercial classes, and emphasis is placed upon immediate utility. The diploma is valuable in many business pursuits. Attendance at the middle school is compulsory for all who expect to take a Government position. Pupils leaving these schools are qualified for the lower public positions and for positions in commercial, industrial, and mechanic arts. The pupil must pass an examination before he is admitted.

The middle schools, like the primary schools, are under the control of the commune, but the State exercises the right of inspection. A pupil who has completed the course in the middle schools, and who desires to prepare for the university, may enter the fourth from the upper class in the Athénée.

The Athénées Royaux are the official schools of higher grade, and they have a course seven years in length. They are usually for boys only. To be admitted to the lowest class of these schools a pupil must be at least 11 years of age and must pass an entrance examination. In general, however, the lowest class corresponds to our seventh grade. (The boy leaves at about the age of 19, after 13 years in school.) These institutions are supported by the State and are independent of the commune. They correspond, in some respects, to the secondary schools of the United States.

There are three kinds of Athénées Royaux: (*a*) The humanistic, with seven years of Latin and five years of Greek; (*b*) the Latin humanities, with seven years of Latin and no Greek, but with an extensive course in mathematics; (*c*) the modern humanities with seven years devoted to a modern language. The course in the modern humanities, is frequently divided into two sections in the three upper classes. These sections are (*a*) scientific and (*b*) commercial.

Pupils who complete any of the three courses in the Athénée

Royal and pass a final examination receive a diploma which admits them to the university. Graduates of the classical course are admitted to any faculty in any of the universities. Graduates of the Latin-scientific course are admitted to the higher schools of mining, engineering, and manufacturing. Graduates of the modern humanities are admitted to the commercial and consular sections in the Universities of Ghent and Liège.

The organization of parallel courses equivalent to the established classical course follows French rather than German precedent. In Germany the various types of courses are found in different schools: The Gymnasium, Realgymnasium, and Realschule.

The secondary schools of Belgium are a very important factor of the national life. The opportunity to secure a good education is offered to all, and no social or class distinctions determine the kind of education that a boy or girl shall receive. The great intellectual leaders of the country are usually products of the Athénée. In Belgium the classical course is still regarded as a dignified and scholarly course, but the utilitarian subjects are regarded as of equal importance and dignity with the classics.

DENMARK.

Arrangement of school years.

Age of pupil.	School year.	Forskole.	Mellemskole or intermediate.	Realclasse.	Gymnasium.
6–7	First	1			
7–8	Second	2			
8–9	Third	3			
9–10	Fourth	4			
10–11	Fifth	5			
11–12	Sixth	6	1		
12–13	Seventh	7	2		
13–14	Eighth	(8)	3		
14–15	Ninth		4		
15–16	Tenth			1	1
16–17	Eleventh				2
17–18	Twelfth				3

Subdivisions of the Gymnasium—Periods per week devoted to mathematics in each.

	First year.	Second year.	Third year.
Classical course	2	2	2
Modern language course	2	2	2
Mathematical-scientific course	6	6	6

The schools of Denmark have always been closely associated with the church, but in recent years the church control has been largely nominal. The bishop and the clergy aid in the selection of teachers and in the general administration of the schools. The schools are under the control of civil authorities.

Elementary schools are called Forskoler. Compulsory school attendance begins at the age of 7 and ends at 14. Most of these schools are free, and books are usually furnished without cost. In most of the rural districts the schools are coeducational, but this is not true of the schools in the larger cities.

The minimum length of the school year is 240 days of six periods each.

There are three divisions of the primary school. The first division is of three years; the second, two years; and the third, two years. The municipal school of Copenhagen is regarded as the best in the country, and the methods and courses of study used there are freely copied elsewhere. Coeducation has been introduced into some of the elementary schools of Copenhagen with marked success. The course at Copenhagen is seven years in length; in some parts of the country it is eight.

The middle and secondary schools are in process of development, and it is difficult to draw sharp distinctions between the various kinds of schools. In some of the municipal elementary schools the highest classes overlap some of the classes of the intermediate schools. In Copenhagen there are continuation schools where pupils who have finished the first seven school years may secure further instruction in certain subjects. A short course in mathematics and its practical applications is given. These schools are growing rapidly.

The four lowest classes of the higher schools comprise the intermediate schools. A pupil enters the intermediate school at the age of 11 or 12 and completes the course at the age of 15 or 16. The intermediate schools are called "Mellemskoler." They were established in 1903 in the attempt to eliminate the abrupt break between the elementary and the secondary schools. Most of the schools of this type outside of Copenhagen are coeducational.

After completing the four years of the Mellemskole, a pupil may enter the Realclasse, which continues for one year, or he may enter the three-year Gymnasium, which prepares for the university.

The Gymnasium course is composed of three subdivisions: (*a*) The classical course, (*b*) the modern language course, (*c*) the mathematical-scientific course.

FINLAND.

Arrangement of school years.

Age of pupil.	School year	Primary school.	Lycée.
7-8	First	1	
8-9	Second	2	
9-10	Third	3	
10-11	Fourth	4	
11-12	Fifth	5	1
12-13	Sixth	6	2
13-14	Seventh	7	3
14-15	Eighth		4
15-16	Ninth		5
16-17	Tenth		6
17-18	Eleventh		7
18-19	Twelfth		(8)

The elementary primary schools of Finland are coeducational; the higher primary schools are not. A small fee is charged at all primary schools. Education is not compulsory beyond the elementary school.

The primary schools in cities are usually divided into three periods:

(*a*) The elementary primary school of two years.

(*b*) The ordinary primary school of four years.

(*c*) The complementary school of one year. A pupil must be at least 9 years of age before he can enter the ordinary primary. The age for entering the complementary school varies from 14 to 17 years. As the population of Finland is largely rural and exceptionally homogeneous, the courses in all primary schools are practically the same.

There are four types of secondary schools:

(*a*) The classical lycée.

(*b*) The real lycée.

(*c*) The preparatory schools.

(*d*) Finishing schools (for girls only).

There are 26 State lycées for boys and 16 for girls. The pupils who enter the lycées are from 9 to 12 years of age. In the real lycée one more hour a week is given to mathematics than in the classical lycée, and much more emphasis is placed upon the study of physics.

FRANCE.

Arrangement of school years.

Age of pupil.	School year.	Primary school.	Higher primary.	Lycée.
6–7	First	1		
7–8	Second	2		
8–9	Third	3		
9–10	Fourth	4		
10–11	Fifth		1	1
11–12	Sixth		2	2
12–13	Seventh		(3)	3
13–14	Eighth			4
14–15	Ninth			5
15–16	Tenth			6
16–17	Eleventh			7
17–18	Twelfth			(8)

Primary education in France is free, compulsory, and secular. The law requires that every commune must have at least one primary school for boys, and every commune of more than 5,000 population must have a primary school for girls. Education is compulsory from the sixth to the thirteenth year.

Primary instruction is given up to about the age of 10 or 11; then more or less differentiation in training occurs. There is an elementary course of two years; the pupils in this course range in age from 6 to 9 years. This is followed by an intermediate course of two years; the pupils are from 9 to 11 years of age. Finally, there is a higher primary school of two or three years. Most of the pupils take only the first two of these courses and then enter a secondary school. The higher primary course is not offered in all schools.

Higher primary instruction is of two kinds: (*a*) The higher primary course, and (*b*) the complementary course. The first differs from the second in that it is entirely separate from the elementary school and is under a different director. The complementary course is connected with the elementary school and is under the same direction. The higher primary school usually has a course of two, sometimes of three, years. The complementary course is one year. Pupils who complete the course in the higher primary school are prepared for agriculture, industry, and commerce. However, the aim in these schools is not primarily to prepare the pupil for a particular vocation, but to furnish a general practical knowledge of several vocations. Those who wish to prepare for examinations for entrance to more advanced schools are put into a special section. To be admitted to either of the above divisions, a pupil must have a certificate from the elementary school and must have taken at least one year of additional work.

Secondary education is neither free nor compulsory. The pupil usually enters the secondary school at the age of 10 or 11 and graduates at the age of 17 or 18. The secondary school is divided into two cycles, the first of four and the second of three years. Pupils who enter the first section are required to study Latin and they may elect Greek; in the second section particular emphasis is put upon the study of French and the sciences; Latin and Greek are not offered.

Electives are offered in the second and third years of the second cycle. In one course Greek and Latin are continued. In a second course Latin and either English or German are studied. In a third course the sciences and modern languages predominate, and but little Latin is offered. In the fourth no Latin is offered; emphasis is put upon the sciences and upon modern languages.

GERMANY.

Arrangement of school years.

Age of pupil.	School year.	Volksschule.	Bürgerschule.	Gymnasium.	Realgymnasium.	Oberrealschule.
6–7	First	1				
7–8	Second	2				
8–9	Third	3				
9–10	Fourth	4	1	1	1	1
10–11	Fifth	5	2	2	2	2
11–12	Sixth	6	3	3	3	3
12–13	Seventh	7	4	4	4	4
13–14	Eighth	8	5	5	5	5
14–15	Ninth		6	6	6	6
15–16	Tenth		7	7	7	7
16–17	Eleventh		8	8	8	8
17–18	Twelfth		9	9	9	9

Classes in a nine-year secondary school.

	Classes.	Usual age of pupils.
		Years.
Lower stage	Sexta—VI	9
	Quinta—V	10
	Quarta—IV	11
Intermediate stage	Untertertia—UIII	12
	Obertertia—OIII	13
	Untersekunda—UII	14
Upper stage	Obersekunda—OII	15
	Unterprima—UI	16
	Oberprima—OI	17

The following table indicates the average number of periods per week allotted to the study of mathematics in each type of secondary school:

Periods per week in mathematics.

	Class.									Total.
	VI	V	IV	UIII	OIII	UII	OII	UI	OI	
Gymnasium	4	4	4	3	3	4	4	4	4	34
Realschule	4	4	4	4	5	5	5	5	5	41
Oberrealschule	5	5	6	6	5	5	5	5	5	47

In Germany centralization of educational authority is not carried to so high a degree as in France. Each of the German States is independent in local affairs, and the educationl system is characterized by flexibility rather than rigidity.

In the United States a pupil may enter the elementary school and continue his course through the university. The "educational ladder" is unbroken. In Germany there are two distinct systems, the lower or elementary school and the higher school system. After a pupil has passed his fourth school year it is almost impossible to transfer from one to the other. Elementary education is usually based on State laws; secondary education is usually regulated by ordinances of a local character.

Education is compulsory from the sixth to the fourteenth year (sixth to the thirteenth year in Württemberg), and there are practically no illiterates in Germany.

ELEMENTARY SCHOOLS.

The elementary schools are called Volksschulen, and are all free. The majority of the teachers are men, but the percentage of women is steadily increasing. Most of the Volksschulen have an eight-year course.

Most of the German States make special provision in the elementary schools for the education of backward and of exceptionally gifted children. The brighter children have a richer curriculum, usually including a foreign language. Normal pupils require eight years to complete the course in the Volksschule. Very able pupils, after two years' attendance, are put into special classes which prepare them for the Gymnasium in one and one-half instead of two years. For backward pupils, courses of from 7 to 10 years are offered.

The middle schools of Germany are called Mittelschulen, higher elementary schools, or Bürgerschulen. They are intermediate between the lower elementary and the secondary school from the point of view of courses offered. Attendance at these schools is not a prerequisite for admission to the secondary schools. Mittelschulen are especially common in the states of southern Germany. Some of these are for boys, some for girls, and some are attended by both boys and girls. The course is usually nine years in length, and the lower grades overlap the primary school. The curricula in these schools are adapted as far as possible to local and individual needs. The schools are largely utilitarian in aim.

SECONDARY SCHOOLS.

The public secondary schools are almost all undenominational or interdenominational. The elementary schools are usually denominational, except in Baden and Hesse, and the Volkesschulen in some of the cities.

Separate schools are usually maintained for boys and girls, except in Württemburg, Baden, and Hesse, where mixed classes are common. In all of the States, mixed classes are found in the small schools in rural communities. Most of the secondary schools for boys are public, and many of those for girls are private. The maintenance of the elementary school usually falls upon the community. Cities and towns maintain their own secondary schools. There is a tendency to provide more liberally for public secondary education for girls.

There are three kinds of higher schools with nine-year courses: (*a*) Gymnasium, (*b*) Realgymnasium, (*c*) Oberrealschule.

The Gymnasium is the classical secondary school. Both Latin and Greek are taught.

In the Realgymnasium no Greek is taught. Latin, the modern languages, and mathematics are emphasized.

In the Oberrealschule Latin and Greek are not taught, but emphasis is placed upon the modern languages and science.

In all three types of schools German, history, religion, and mathematics are taught.

The fundamental idea in organizing the Realschule was to prepare for the commercial professions, as the Gymnasium prepares for the learned professions. The curricula in most of the Realschulen have been enlarged to meet the demands for broader culture.

After the first three years in a secondary school it is almost impossible for a pupil to transfer to another type of school. This means that parents must decide by the time their children are 9 or 10 years of age which type of school they wish them to enter. In order that this decision, which is almost irrevocable, may be postponed until the child is a few years older, institutions called Reformgymnasia and Realgymnasia have been established. In these schools a common foundation for all three types of secondary schools is laid in the first three years. At the end of this period one section begins the study of English, and emphasis is later put upon the natural sciences (Realschule and Oberrealschule). Another section begins the study of Latin, and two years later this section is subdivided, one division (Gymnasium) beginning the study of Greek and the other section (Realgymnasium) beginning the study of English. This general plan is sometimes called the Frankfort system.

Most of the states now maintain separate high schools for girls. The curricula of these schools are not unlike those of the Realschule, except that less emphasis is placed upon science and mathematics.

For a description of the conducting of classes in the various types of schools the reader may consult *Mathematics in the Schools of Prussia,* by J. W. A. Young, published by Longmans, Green & Co., New York. See also *The present Teaching of Mathematics in Germany* (Bureau of Publications, Teachers College, New York City) for details in regard to courses in mathematics in the secondary schools of the more important German States.

HOLLAND.

Arrangement of school years.

Age of pupil.	School year.	Primary school.	Bürgerschule.	Middle school.	Gymnasium.
6–7	First	1			
7–8	Second	2			
8–9	Third	3			
9–10	Fourth	4			
10–11	Fifth	5			
11–12	Sixth	6			
12–13	Seventh		1	1	1
13–14	Eighth		2	2	2
14–15	Ninth			3	3
15–16	Tenth			4	4
16–17	Eleventh			5	5
17–18	Twelfth				6

There are many large cities in Holland, and to a considerable extent they determine the general character of the primary and secondary education. Although complete religious liberty prevails throughout the kingdom, the various religious denominations exercise great influence in educational matters.

The primary school comprises the first six school years. The pupils usually enter at the age of 6. Boys and girls attend the same school and are instructed together. Primary instruction is given in the day schools, the evening schools, and in the continuation schools. Attendance at the day school is obligatory for all children from 7 to 13 years of age.

There are between five and six thousand public day schools in Holland, and almost two thousand private primary schools that have been subsidized by the state.

The secondary schools may be public or private. There are four general subdivisions:

1. Bürgher schools.
2. Higher Bürgher schools.
3. Industrial, trade, and technical schools.
4. Agricultural schools.

These schools are sometimes called the middle schools to distinguish them from the Gymnasia.

The Bürgher schools are especially for the children of the tradesman, mechanic, and the agriculturist. The law requires that such a school must be maintained in every community having a population of more than 10,000.

The Higher Bürgher schools train those who expect to become engineers, architects, and technologists. Those who expect to enter the service of the state also attend these schools. The diploma admits the pupil without examination to certain of the special higher schools.

The Higher Bürgher schools have two courses, one of five years and the other of three. The tendency is to extend the course to six years. These schools offer especially thorough instruction in the sciences and modern languages. Entrance is by examination. A candidate for admission must be at least 12 years of age. A pupil who successfully completes a course is given a certificate which is of value to him if he seeks a civil or commercial appointment. There are about 100 Higher Bürgher schools in Holland. Both boys and girls may be enrolled in these schools. Only male teachers are employed.

The Higher Bürgher schools offer the majority of the pupils the best opportunity for a good, general secondary education. A graduate of these schools cannot enter a university until he has studied one year of Latin and of Greek. In Belgium the sciences and the modern languages are considered as equivalent to the ancient languages in dignity and in importance, but the classical idea still prevails quite largely in Holland.

There are numerous types of industrial and technical schools.

In addition to the schools mentioned above every community of 20,000 inhabitants must have a Gymnasium. These prepare especially for the universities. The course is six years in length. The age of admission is 12 to 13 years.

HUNGARY.

Arrangement of school years.

Age of pupil.	School year.	Volksschule.	Bürgerschule.	Gymnasium.	Realschule.
6–7	First	1			
7–8	Second	2			
8–9	Third	3			
9–10	Fourth	4			
10–11	Fifth	5	1	1	1
11–12	Sixth	6	2	2	2
12–13	Seventh		3	3	3
13–14	Eighth		4	4	4
14–15	Ninth		(5)	5	5
15–16	Tenth		(6)	6	6
16–17	Eleventh		(7)	7	7
17–18	Twelfth		(8)	8	8

The number of hours devoted each week to the study of mathematics and of geometric drawing in the Gymnasium and Realschule is indicated in the following:

Periods per week in mathematics and geometric drawing.

Classes.	Gymnasium.			Realschule.		
	Mathematics.	Geometric drawing.	Total.	Mathematics.	Geometric drawing.	Total.
First	4	3	7	4	4	8
Second	4	3	7	4	4	8
Third	3	2	5	3	2	5
Fourth	3	2	5	4	2	6
Fifth	3		3	5		5
Sixth	4		4	4		4
Seventh	3		3	4		4
Eighth	2		2	3		3
Total	26	10	36	31	12	43

Boys and girls are usually taught in separate institutions, but coeducational elementary schools are not uncommon in the less densely populated districts. The course in the elementary school is six years in length. The law provides for a minimum school year of eight months in the country and nine in the city. Education is compulsory from the sixth to the twelfth year.

The Bürgerschulen are much the same as the corresponding schools in Austria. The course is usually four years in length, but some schools offer six or eight year courses. Some of the Bürgerschulen are supported by the state, some by religious organizations, and others by the communities. The girl or boy may enter these schools after completing the fourth class of the Volkschule. The entering age is usually 10. Both male and female teachers are employed.

In the Bürgerschule for boys the same subjects are usually taught in the first four years as in the first four years of the Gymnasium and Realschule, with the exception of Latin and French. The Bürgerschule does not attempt to prepare the pupil for higher education, but seeks to emphasize the practical subjects and to lead to the higher vocations. Pupils may be transferred from certain classes of the Bürgerschule to the Gymnasium and Realschule and vice versa by passing examinations. The change that is gradually taking place in the Bürgerschule is evidenced by the fact that Latin is now taught in some of the boys' schools, and some of these schools are not unlike the Gymnasium and Realschule.

In the Gymnasium Latin is studied in each of the eight classes and Greek in the last four. The study of Latin is obligatory. In many Gymnasia a pupil is permitted to elect some subject instead of Greek. German is taught after the second class.

Latin is not a required subject in the Realschule, but it is sometimes offered in the last four classes. In the Realschule German is taught in all classes and French in the last six. Mathematics is treated more extensively in the Realschule than in the Gymnasium.

Separate secondary schools of a distinctive type have recently been established for girls. These schools offer a six-year course and may be entered by examination after the completion of the sixth year of the elementary school. There are now two kinds of high schools for girls. In one of these especial emphasis is placed upon the modern languages and domestic science; in the other, the course closely resembles that of the Gymnasium for boys. Both men and women are employed as teachers in the high schools for girls.

ITALY.

Arrangement of school years.

Age of pupil.	School year.	Elementary school.	Secondary school.		
			Classical school.		Modern school.
			Ginnasio.	Liceo.	
6-7	First	1			
7-8	Second	2			
8-9	Third	3			
9-10	Fourth	4			
10-11	Fifth	5	1		1
11-12	Sixth	6	2		2
12-13	Seventh		3		3
13-14	Eighth		4		4
14-15	Ninth		5		5
15-16	Tenth			1	6
16-17	Eleventh			2	7
17-18	Twelfth			3	

The elementary school usually consists of six grades. The first three grades comprise the inferior and the next three the superior course. A pupil may enter school at the age of 6, and education is compulsory from the sixth to the twelfth year. After completing the fourth year of the elementary school, the pupil who is going to a higher school may take an examination, and if he passes this he may enter the secondary school. No pupil is allowed to remain in the elementary school after he is 15 years of age or in the inferior course after he is 12 years of age.

Secondary schools are usually erected and equipped by the local authorities, and the other expenses are shared by the State. There are two types of secondary schools besides the normal schools:

I. The classical school and the modern school with Latin—
   (*a*) Ginnasio—5-year course.
   (*b*) Liceo—3-year course.

II. Modern school without Latin—Technical school—
   (*a*) Scuola tecnica and scuola complementare—3-year course.
   (*b*) Instituto tecnico—4-year course.
   Instituto nautico—3-year course.

The complete classical course covers eight years, and the modern course covers seven years.

Girls are admitted to secondary schools upon the same conditions as boys.

JAPAN.

Arrangement of school years.

Age of pupil.	School year.	Ordinary primary.	Higher primary.	Middle school.
6–7	First	1		
7–8	Second	2		
8–9	Third	3		
9–10	Fourth	4		
10–11	Fifth	5		
11–12	Sixth	6		
12–13	Seventh		1	1
13–14	Eighth		2	2
14–15	Ninth		3	3
15–16	Tenth			4
16–17	Eleventh			5
17–18	Twelfth			(6)

Periods per week in mathematics.

	First year.	Second year.	Third year.	Fourth year.	Fifth year.	Sixth year.
Ordinary primary	5	6	6	6	4	4
Higher primary	4	4	4			
Middle school	4	4	4	4	2	

The state has entire control of education, and school attendance is compulsory from the sixth to the fourteenth year. The minister of education is a member of the imperial cabinet.

The elementary school consists of two subdivisions. The first extends over the first six grades and is called the ordinary primary. The second, called the higher elementary, extends over the next two or three years. Education is compulsory in the ordinary primary, but not in the higher elementary school. In some localities supplementary education is now provided for those who can pursue their regular education beyond the compulsory stages.

Boys and girls are usually taught in the same school and in the same class during the elementary school period, but the middle and higher schools are not coeducational.

After completing the course in the ordinary primary school, the pupil who expects to enter a higher school goes at once to the middle school instead of to the higher primary.

The course in the middle school is usually five years in length; sometimes a supplementary year is added. The graduates of middle schools are qualified to enter higher special schools of various kinds and special industrial schools. They may enter military, naval, or navigation schools, or the higher normal school.

The middle schools were established to give boys a higher common education, but many of these schools have become virtually preparatory schools. The authorities are going to close the high schools preparing for the imperial universities and to establish new higher middle schools, for the purpose of imparting to those who have finished the middle school course a higher common education more thorough than before.

After graduating from a middle school, a boy who expects to enter the university may enter a higher school, having a course of three years, which prepares for the university. A boy who enters the university after preparing in this manner is 20 or 21 years of age. Admission to the higher middle schools is usually on the basis of competitive examinations. During the first year five hours per week are devoted to mathematics; during the second and third years, four hours per week. The course in mathematics in these schools includes trigonometry, algebra, analytic geometry, and the calculus.

After a girl finishes the ordinary primary school, she may enter a girls' high school or she may enter the higher primary school. The course in a girl's high school is four or five years. This is sometimes supplemented by two additional years. This supplementary course is the only provision for the higher education of women except the normal school and certain technical schools. No girl is allowed to enter the imperial university.

ROUMANIA.

Arrangement of school years.

Age of pupil.	School year.	Primary school.	Secondary school.	
			Gymnasium.	Lycée.
6-7	First	1		
7-8	Second	2		
8-9	Third	3		
9-10	Fourth	4		
10-11	Fifth	(5)	1	
11-12	Sixth		2	
12-13	Seventh		3	
13-14	Eighth		4	
14-15	Ninth		(5)	1
15-16	Tenth			2
16-17	Eleventh			3
17-18	Twelfth			4

The schools of Roumania are divided into three groups: (*a*) Primary; (*b*) commercial, technical, and private; (*c*) secondary and higher.

The primary and the secondary schools are free, and education is compulsory between the ages of 7 and 14 years.

In the cities the primary school course is 4 years in length, and the school year is 10 months. In rural communities the course is 5 years in length, and the school year is 9 months.

The secondary schools are of two kinds: The gymnasium and the lycée. In some cities both types are found in the same school. Pupils may be admitted to the gymnasium by examination or upon presentation of a certificate from the primary school. Admission to the lycée is by certificate from the gymnasium.

There are three parallel courses in the lycée, somewhat as in the French schools. A pupil may elect the mathematics-science course, the Latin-science course, or the classical course. Only a few of the public secondary schools are for girls.

A pupil who does not expect to go to a university spends five years in the gymnasium instead of entering the lycée.

RUSSIA.

Arrangement of school years.

Age of pupil.	School year.	Primary school.	Real-schule.	Gym-nasium.
7–8	First	1		
8–9	Second	2		
9–10	Third	3		
10–11	Fourth	4	1	1
11–12	Fifth	(5)	2	2
12–13	Sixth	(6)	3	3
13–14	Seventh		4	4
14–15	Eighth		5	5
15–16	Ninth		6	6
16–17	Tenth		7	7
17–18	Eleventh		(8)	8

The primary schools usually have courses of three or four years, but in some cities the course is five or six years in length. Five periods a week are devoted to arithmetic.

The secondary schools are of three types, the classical Gymnasium, the modern Gymnasium, and the Realschule.

There is usually a one year preparatory course in the Gymnasia, making the length of the entire Gymnasia courses nine years.

Gymnasia for girls are quite numerous, and the course of study indicates that quite a high standard is sought. Most gymnasia for women have courses seven years in length. There are some gymnasia with three and four year courses.

SWEDEN.

Arrangement of school years.

Age of pupil.	School year.	Primary school.	Real-schule.	Gymnasium.
6–7	First	1		
7–8	Second	2		
8–9	Third	3		
9–10	Fourth	4	1	
10–11	Fifth	5	2	
11–12	Sixth	6	3	
12–13	Seventh		4	
13–14	Eighth		5	
14–15	Ninth		6	1
15–16	Tenth			2
16–17	Eleventh			3
17–18	Twelfth			4

Periods per week in mathematics.

Gymnasia.	Classes.			
	First.	Second.	Third.	Fourth.
Realgymnasium	7	6	6	6
Latin gymnasium	5	4	4	5

Pupils usually enter school at the age of 6 or 7, and attendance is compulsory up to the age of 14. The school year is 34½ weeks in length. Both the church and the school are under the supervision of the same department of the government.

The primary school has a six year course. The parish is usually the school district, and instruction is carried on under the supervision of local inspection and of inspectors appointed by the government. Coeducation is common in the primary schools of the rural districts. From five to seven periods per week are devoted to mathematics in the primary school.

The higher school is divided into the Realschule and the Gymnasium. The aim of the Realschule is to provide a general practical education beyond the primary school. The course of six years is concluded without an examination. In all Realschulen four or five periods per week are devoted to mathematics.

After completing the first five years of the Realschule, a pupil may enter the first year of the Realgymnasium or of the Latin Gymnasium, instead of taking the sixth year of the Realschule. In the Gymnasium, pupils are urged to take subjects for which they have special talent, and considerable freedom to discontinue a subject is permitted.

SWITZERLAND.

Arrangement of school years.

Age of pupil.	School year.	Primary school.	Lower middle school (Sekundarschule).	Higher middle school.	
				Gymnasium.	Realschule.
6–7	First	1			
7–8	Second	2			
8–9	Third	3			
9–10	Fourth	4			
10–11	Fifth	5	1		
11–12	Sixth	6	2		
12–13	Seventh		3	1	
13–14	Eighth		4	2	
14–15	Ninth		(5)	3	1
15–16	Tenth			4	2
16–17	Eleventh			5	3
17–18	Twelfth			6	4
18–19	Thirteenth			7	5

The Cantons differ greatly in the number of periods per week devoted to mathematics. In the primary school one period daily is the general rule.

The following table shows the number of periods per week devoted to mathematics, including bookkeeping, in the literary Gymnasium, Realgymnasium, and Realschule of Zürich.

Periods per week in mathematics.

	Years.						
	First.	Second.	Third.	Fourth.	Fifth.	Sixth.	Seventh.
Literary Gymnasium	5	5½	3	3	3½	4	1½
Realgymnasium	5	5½	6	5½	5½	4½	2½
Realschule	9½	8	9	9	5½		

The chief characteristic of education might be said to be its variability. This is caused by the great geographic, economic, and religious differences prevailing in the various Cantons. There is considerable dissatisfaction in Switzerland because of the great variety in educational systems, and there is a tendency toward centralization of authority in education. The school year varies from 38 to 48 weeks.

The primary schools are said to be unusually efficient in the training they give. The entering age varies from 6 to 7 years in the different Cantons, and the course is six years in length. All primary schools are free, and coeducation predominates except in the large cities.

In many Cantons a pupil who has completed the fourth year of the primary school may enter an advanced school called the lower middle school of the Sekundarschule. In this school one or more foreign languages and algebra are taught. In some Cantons a pupil enters this type of school after completing the sixth year of the primary school. The course is then from three to five years in length. These schools correspond to the Bürgerschulen in Germany and Austria and to the upper primary schools in France.

A pupil who has completed the six-year course of the primary school may enter the higher middle school. There are many names used in the various Cantons for the different types of schools of secondary grade, but most of these schools may be called either Gymnasia or Realschulen. The Realschulen of Switzerland correspond to the Oberrealschulen of Germany. In French Switzerland there is even more differentiation of courses than in German Switzerland, and many of the secondary schools resemble the French lycées.

Table 1 is a graphic representation of the arrangement of the school years in different countries. A heavily dotted line indicates that in *some* cases the type of school under consideration extends over the period represented.

The table portrays general conditions and not exceptional cases.

TABLE 1.—*Arrangement of school years.*

Age	6–7	7–8	8–9	9–10	10–11	11–12	12–13	13–14	14–15	15–16	16–17	17–18	18–19
School year	1	2	3	4	5	6	7	8	9	10	11	12	13
Austria:													
Volksschule, 5-year	——	——	——	——	——	……	……	……	……	……	……	……	……
Bürgerschule, 3-year	……	……	……	……	……	——	——	——	……	……	……	……	……
Gymnasium, 8-year	……	……	……	……	——	——	——	——	——	——	——	——	……
Realschule, 7-year	……	……	……	……	——	——	——	——	——	——	——	……	……
Realgymnasium, 8-year	……	……	……	……	——	——	——	——	——	——	——	——	……
Belgium:													
Primary, 6-year	——	——	——	——	——	——	……	……	……	……	……	……	……
Middle, 3-year	……	……	……	……	……	……	——	——	——	……	……	……	……
Athénée, 7-7½-year	……	……	……	……	……	‥—	——	——	——	——	——	——	——
Denmark:													
Folkeskole, 8-year	——	——	——	——	——	——	——	——	……	……	……	……	……
Intermediate, 4-year	……	……	……	……	……	——	——	——	——	……	……	……	……
Realklasse, 1-year	……	……	……	……	……	……	……	……	……	——	……	……	……
Gymnasium, 3-year	……	……	……	……	……	……	……	……	……	——	——	——	……
Finland:													
Primary, 7-year	——	——	——	——	——	——	——	……	……	……	……	……	……
Lycée, 8-year	……	……	……	……	——	——	——	——	——	——	——	——	……
France:													
Primary, 4-year	——	——	——	——	……	……	……	……	……	……	……	……	……
Higher Primary, 3-year	……	……	……	……	——	——	——	……	……	……	……	……	……
Lycée, 8-year	……	……	……	……	——	——	——	——	——	——	——	——	……
Germany:													
Volksschule, 8-year	——	——	——	——	——	——	——	——	……	……	……	……	……
Bürgerschule, 9-year	……	……	……	——	——	——	——	——	——	——	——	——	……
Gymnasium, 9-year	……	……	……	——	——	——	——	——	——	——	——	——	……
Realgymnasium, 9-year	……	……	……	——	——	——	——	——	——	——	——	——	……
Oberrealschule, 9-year	……	……	……	——	——	——	——	——	——	——	——	——	……

TABLE 1.—*Arrangement of school years*—Continued.

Age	6–7	7–8	8–9	9–10	10–11	11–12	12–13	13–14	14–15	15–16	16–17	17–18	18–19
School year	1	2	3	4	5	6	7	8	9	10	11	12	13
Holland:													
Primary, 6-year	———	———	———	———	———	———	.....	.....	.....	.....	.....	.....	.....
Bürgerschule, 2-year	.....	.....	.....	.....	.....	.....	———	———	.....	.....	.....	.....	.....
Middle, 5-year	.....	.....	.....	.....	.....	.....	———	———	———	———	———	.....	.....
Gymnasium, 6-year	.....	.....	.....	.....	.....	.....	———	———	———	———	———	———	.....
Hungary:													
Volksschule, 6-year	———	———	———	———	———	———	.....	.....	.....	.....	.....	.....	.....
Bürgerschule, 4-6-year	.....	.....	.....	.....	———	———	———	———	•••••	•••••	•••••	.....	.....
Gymnasium, 8-year	.....	.....	.....	.....	———	———	———	———	———	———	———	———	.....
Realschule, 8-year	.....	.....	.....	.....	———	———	———	———	———	———	———	———	.....
Italy:													
Elementary, 6-year	———	———	———	———	———	———	.....	.....	.....	.....	.....	.....	.....
Ginnasio, 5-year	.....	.....	.....	.....	———	———	———	———	———	.....	.....	.....	.....
Liceo, 3-year	.....	.....	.....	.....	.....	.....	.....	.....	.....	———	———	———	.....
Modern, 7-year	.....	.....	.....	.....	———	———	———	———	———	———	———	.....	.....
Japan:													
Ordinary Primary, 6-year	———	———	———	———	———	———	.....	.....	.....	.....	.....	.....	.....
Higher Primary, 3-year	.....	.....	.....	.....	.....	.....	———	———	———	.....	.....	.....	.....
Middle, 6-year	.....	.....	.....	.....	.....	.....	———	———	———	———	———	———	.....
Roumania:													
Primary, 5-year	———	———	———	———	———	.....	.....	.....	.....	.....	.....	.....	.....
Gymnasium, 5-year	.....	.....	.....	.....	———	———	———	———	———	.....	.....	.....	.....
Lycée, 4-year	.....	.....	.....	.....	.....	.....	.....	.....	———	———	———	———	.....
Russia.													
Primary, 3–5 year	———	———	———	•••••	•••••	.....	.....	.....	.....	.....	.....	.....	.....
Realschule, 7–8 year	.....	.....	.....	———	———	———	———	———	———	———	•••••	.....	.....
Gymnasium, 8 year	.....	.....	.....	———	———	———	———	———	———	———	———	.....	.....
Sweden:													
Primary, 6-year	———	———	———	———	———	———	.....	.....	.....	.....	.....	.....	.....
Realschule, 6-year	.....	.....	.....	———	———	———	———	———	———	•••••	.....	.....	.....
Gymnasium, 4-year	- - -	- - -	- - -	- - -	- - -	- - -	- - -	- - -	———	———	———	———	- - -
Switzerland:													
Primary, 6-year	———	———	———	———	———	———	- - -	- - -	- - -	- - -	- - -	- - -	- - -
Lower Middle, 5-year	- - -	- - -	- - -	- - -	———	———	———	———	•••••	- - -	- - -	- - -	- - -
Gymnasium, 7-year	- - -	- - -	- - -	- - -	- - -	- - -	———	———	———	———	———	———	———
Realschule, 5-year	- - -	- - -	- - -	- - -	- - -	- - -	- - -	- - -	———	———	———	———	———
United States of America:													
Elementary, 8-year	———	———	———	———	———	———	———	———	- - -	- - -	- - -	- - -	- - -
Secondary, 4-year	- - -	- - -	- - -	- - -	- - -	- - -	- - -	- - -	———	———	———	———	- - -

II. Summaries of the Work in Mathematics by Years.

THE WORK OF THE FIRST SCHOOL YEAR.

There is not much divergence between the courses in arithmetic in the most progressive schools of the various countries. In general it may be said that the aim is to teach the children to count and to read and write the numbers to 100; to perform easy additions and subtractions; to know the fractions, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, and to make a few easy multiplications and divisions involving numbers less than 20. Practically all the work is oral, and objects are freely used. In many of the schools of the United States, on the other hand, only incidental number work is given during the first school year. In some schools regular number work is begun the latter half of the first year. In a few schools incidental number work is given during the first two school years.

In general it may be said that a pupil in most of the more progressive European schools makes somewhat more progress in arithmetic during his first school year than the pupil in the corresponding grade in the United States.

THE WORK OF THE SECOND SCHOOL YEAR.

The course of the second school year varies more than that of the first year. In general, the aim of the work in the better schools may be said to be to teach the children to count, read, and write numbers to 1,000; to perform the fundamental operations with numbers less than 100; and to learn the simple units of measure. In several countries, multipliers and divisors are limited to one figure. The pupils are taught to count to 100 by twos, fives, and tens. The 45 addition combinations are learned in this year and the multiplication tables involving products up to 10 times 10 are usually studied. These are usually so taught as to prepare for division at the same time.

Subtraction is usually taught by the addition method and is studied at the same time as addition. The fractions $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ are taught, and objects are very extensively used. The simple denominate numbers are studied and much attention is devoted to measures and estimates. Oral work predominates. Numerous concrete problems involving the experiences of the pupils are given.

The course in the most progressive schools of the United States compares favorably with the most advanced courses in Europe. In the great majority of the schools of the United States, however, not so much is attempted in arithmetic as in the best schools of Europe during the second school year. The longer school year and the longer school day enable the European teachers to devote more time to drill in fixing the number facts, and the pupils leaving the second grade there are probably more thoroughly grounded in the fundamentals than is the case in this country.

THE WORK OF THE THIRD SCHOOL YEAR.

There is greater variety in the third-year courses than in those of the first and second school years. In a few of the countries—for example, Belgium and Italy—the notation of decimal fractions is introduced. This is usually not done in the United States until the latter part of the fourth or the early part of the fifth year. It is a common practice abroad to introduce fractions with denominate numbers. In all of the European countries and in Japan oral arithmetic greatly predominates. In Japan a special part of the recitation is set apart for this oral drill. The textbooks in several of the countries, notably Germany, Austria, and Italy, are collections of problems rather than expositions of number.

A great deal of emphasis is put upon accuracy and speed in the four fundamental operations. Both oral and written drills are extensively used. The study of denominate numbers is usually continued and extended. In some of the countries a good deal of attention is devoted to the drawing of the simplest geometric figures.

In many of the schools subtraction is taught by the addition method.

At the end of his third school year the German boy may either (1) continue in the Volksschule, (2) pass to the Bürgerschule, or (3) enter a secondary school (Gymnasium, Realgymnasium, or Oberrealschule). The Russian pupil may enter either the Realschule or the Gymnasium. If he prefers he may continue in the primary school a year or two more. In Sweden the primary school lasts six years, but a pupil may enter the Realschule after he has completed the third school year.

THE WORK OF THE FOURTH SCHOOL YEAR.

At the close of the fourth school year (in Germany, and Russia the third school year) the pupil may enter a "higher" school of some sort; in Austria the Gymnasium, the Realgymnasium, or the Realschule; in France, the higher primary or lycée; in Germany, the Gymnasium, the Realgymnasium, or the Oberrealschule; in Hungary, the Bürgerschule, the Gymnasium or the Realschule; in Italy the Ginnasio, or the modern school; in Roumania, the Gymnasium; and in Switzerland, the middle school. Conditions are such that pupils cannot well enter a "higher" school except at the beginning of the course; so that the fourth (or the third) school year is the last year of the primary school for many European pupils.

The attempt is made to fix the four operations for abstract and denominate numbers firmly in mind by the time the pupil has completed his fourth school year. In all of the countries a large amount of both oral and written drill is provided. Speed and accuracy in the fundamentals are watchwords everywhere. In addition, the concept of common fractions is much extended, and the decimal notation is introduced. In several of the countries, only addition and subtraction of common and decimal fractions are taught, but in others multiplication and division are also included in the course. When this is done, only the easy cases are usually considered.

The general use of the metric system in the countries of continental Europe makes the introduction of decimals practicable at an earlier date than in the United States. Usually the subject of decimal fractions is closely correlated with the metric system. In several of the European countries a good deal of emphasis is put on the study of the simplest geometric figures. In general, it may be said that the courses in European countries include all that is offered in the United States during the fourth school year and a good deal of the work of the fifth school year. The formal study of common or decimal fractions is seldom begun in the United States before the fifth school year.

In most of the European countries emphasis is put upon computation rather than upon reasoning during the first four school years.

THE WORK OF THE FIFTH SCHOOL YEAR.

The work of the fifth school year in most European countries is decidedly more extensive than in most of the schools of the United States. It is unusual in this country for the general ideas of percentage to be introduced before the sixth school year. In several of the European countries this work is introduced during the fifth school year. Proportion is rarely introduced in this country in the fifth school year, but it is not uncommonly introduced in that year abroad. Tests of divisibility and prime factors are given much more attention in the European countries than here. Continual emphasis is put on accuracy and speed in the fundamental operations.

The general use of the metric system gives the European teacher an excellent field of application for decimal fractions, and the two topics are closely related.

Common fractions and denominate numbers are more closely related than is usually the case in the United States.

Probably the most marked difference between fifth-grade work here and abroad is the large amount of time and attention put upon the propædeutic study of geometry in the European countries. Very frequently this work is given under the subject of drawing and, when this is the case, the arithmetic and drawing are usually taught by the same teacher. In all cases arithmetic is very closely related to drawing and in several of the countries the relation between arithmetic and manual training is very close.

It may be said in general, that the work of the fifth year in the European schools is considerably in advance of the work in the schools of the United States. The courses abroad include all that is included in the fifth-year courses in this country and a great deal that is not included here.

THE WORK OF THE SIXTH SCHOOL YEAR.

The course in mathematics in practically all of the European countries is decidedly more advanced than in the United States.

The sixth school year is the last year of primary instruction in many of the countries.

The courses abroad include all that is given during the corresponding school year in the United States and also many subjects that are not included in the course in this country.

One marked contrast to the work in this country is found in the emphasis that is put upon the function concept. This feature of the work is especially emphasized in the schools of Germany, Austria, and Hungary, but it is also given some attention in several other countries. In some countries the function concept is introduced in connection with the work of mensuration or graphs; in others, it is first presented with direct and inverse proportion. The tendency abroad is to increase still further the emphasis already placed upon the idea of function.

Another marked contrast to the work in the United States is found in the emphasis that is put upon the propædeutic study of geometry. The courses abroad, almost without exception, provide for the study of intuitive or observational geometry. The amount of time given to this work varies, but the general prevalence of such work is indicative of the importance attached to it. The work is closely related to the arithmetic and the drawing. In several of the countries provision is made for the study of geometric drawing. The pupils learn to use the ruler, protractor, compasses, and triangle, and to make the simple geometric constructions. This work is closely correlated with the work in intuitive geometry and the classes are usually taught by the same teacher. In Germany the systematic and serious study of geometry begins with the sixth school year. In Germany and England easy loci problems are introduced.

Short methods and abbreviated processes receive more emphasis abroad than in this country. This is especially true in Austria, Belgium, Germany, and Hungary.

Alligation is taught in several of the countries; for example, in Germany and Russia. The subject is seldom taught in the United States

The elementary ideas of algebra and of algebraic computation are introduced during the sixth school year in a few of the European countries; for example, in France, Russia, and Sweden.

The time devoted to the study of mathematics abroad is about the same, on the average, as in this country. In some of the countries the time devoted to mathematics is somewhat in excess of that in the United States, if we consider the course in drawing as a part of the course in mathematics.

THE WORK OF THE SEVENTH SCHOOL YEAR.

In all of the European countries the course in mathematics is decidedly more advanced than in the United States.

In practically all of the countries abroad the subject of intuitive or observational geometry is emphasized during the seventh school year. This is the case in a few schools in the United States, but it is the exception rather than the rule. In some schools abroad the most elementary ideas of trigonometry are introduced in connection with the study of similar figures. The pupils abroad receive much training in constructive work, drawing to scale, reducing and enlarging figures.

In some of the schools the pantograph and the planimeter are used. Squared paper is extensively used in England and to some extent in a few other countries. In Austria, France, Germany, Hungary, Italy, and Sweden especial attention is devoted to instruction in the elements of geometric drawing.

Geometry and arithmetic are very closely correlated in the instruction. In England more emphasis is put upon the geometric locus than in any other country during this school year.

In the United States more emphasis is probably put upon a few of the applications of percentage, but several topics in arithmetic not given much attention here are introduced abroad. Alligation is taught in few American schools, but many pupils in the schools of Germany, Hungary, and Switzerland study the subject. Recurring decimals, cube root, the check for multiplication and division by casting out the nines, and the function concept are more extensively taught abroad than in the United States. The function concept receives marked and growing emphasis in Austria, Germany, France, Hungary, and Switzerland.

It is the exception in the United States for any algebra beyond the simple equation and the evaluation of easy formulas to be introduced in the seventh school year. In practically all of the European countries, on the other hand, the pupils learn to solve simple linear equations containing one unknown, to evaluate the formulas of mensuration and percentage, and to perform the four fundamental operations with simple algebraic expressions. In a few countries some easy factoring and graph work are introduced.

THE WORK OF THE EIGHTH SCHOOL YEAR.

In the United States the eighth school year is the last year of the elementary school. In many of the schools but little work in mathematics is attempted beyond the study of mensuration, proportion, and some of the applications of percentage. In an increasing number of schools some work in algebra and in geometry is being introduced during the eighth school year. The nature and extent of this work varies, but in general it may be said to consist chiefly of the simple equation and the evaluation of formulas.

In all European countries the course in mathematics is more extensive than in the United States. In no European country is less geometry or algebra offered during the eighth school year than in the United States, and in practically all the countries the work in both of these subjects is much more extensive than in this country. In Austria, Germany, and Switzerland the most emphasis in algebra is put upon the four operations, the solution of simple equations, and proportion. In a few of the countries the subject of factoring is also taught. This is especially true in certain schools of Austria, Roumania, and Holland.

In practically all of the countries, except the United States, a good deal of emphasis is put upon the study of intuitive geometry during the eighth school year. Especial attention is devoted to the study of congruent and similar figures and to simple constructions. In a few of the countries the pantograph is used in connection with the study of similar figures, and the terms sine, cosine, and tangent are introduced. This is notably true in France. In a few of the modern lycées of France logarithms are introduced. The subject of geometric drawing is given a good deal of attention in several of the countries, especially in Austria, Hungary, and in some of the cantons of Switzerland. In a few of the schools of Austria some instruction is given in the subject of descriptive geometry. In practically all of the schools abroad the subjects of geometry and arithmetic are more closely related than in this country. Continued emphasis is put upon the function concept, especially in Austria, Germany, and Switzerland.

In most of the foreign countries less time is devoted to the study of arithmetic during the eighth school year than in the United States. The study of alligation is continued in some of the schools of Austria, Belgium, Switzerland, and Japan; and a good deal of emphasis is put upon the study of tests for divisibility.

It is customary in all of the European countries to teach algebra and geometry simultaneously. During the eighth school year the time is about evenly divided between these two subjects, from two to three hours a week being devoted to each. An attempt is not made to fuse the subjects, but the interrelations between them are kept constantly in mind and the pupil is not permitted to forget his geometry while studying algebra, or vice versa. Each subject is considered an instruction unit, but it is used whenever possible as a tool in the study of the other. By the time a European boy has completed his eighth school year, he is at least a full year in advance of the American boy in his knowledge of mathematics.

THE WORK OF THE NINTH SCHOOL YEAR.

The ninth school year is usually the first year of the secondary school in the United States. In the majority of secondary schools the subject of algebra is completed up to quadratic equations during the ninth school year. In a few of the schools less work is attempted in algebra and more attention is devoted to the subject of intuitive geometry. In such schools, emphasis is put upon the use of the simple geometric instruments, the making of easy constructions, measurements, and simple designs.

In all European schools both algebra and geometry are taught during the ninth school year. In most of the countries the time is divided about evenly between these two subjects, from two to three hours a week being devoted to each. Here, as in the preceding school year, the subjects supplement each other, and no attempt is made to fuse them. The relations between two subjects are emphasized much more extensively abroad than in the United States. In most of the schools of Europe the distinction between plane and solid geometry is less marked than in the schools of the United States. This is, in part at least, due to the fact that models and drawings are very extensively used abroad. During the ninth school year the work in algebra in most of the European schools is but slightly more advanced than in the best schools of the United States. In some of the schools of Austria, France, Denmark, Holland, and Hungary the course is somewhat more advanced than in the United States. In these schools logarithms, proportion, and quadratics are studied.

In practically all of the European countries geometry is studied during the ninth school year. Especial emphasis is put upon the study of proportional lines, congruent and similar figures, areas, and volumes. In all of the schools models are extensively used. In Belgium, France, Germany, and Holland the study of similar figures is supplemented by elementary exercises in surveying, and in some of the French schools the terms sine, cosine, tangent, and cotangent are introduced and used.

Descriptive geometry is introduced in some of the Realschulen and Realgymnasia of Austria. In France, Sweden, and in the Realgymnasia of Switzerland some attention is devoted to the subject of geometric drawing. In most of the countries continued emphasis is put upon the idea of function. This is especially true in Austria, France, Germany, Sweden, and Switzerland. The subject of logarithms is introduced in most of the countries and in some of the schools of Germany the pupils become familiar with the slide-rule.

In a few of the countries the study of arithmetic is continued during the ninth school year. Especial emphasis is put upon the use of tables for finding interest, and upon abridged processes.

Arithmetic, algebra, and geometry are more closely related in the schools abroad than in the United States. Each subject is considered as an instruction unit, but it is closely correlated with the others. When a European boy has completed his ninth school year he has a mastery of all the topics in algebra that the American boy has studied, and in addition to these he is familiar with practically all of the more important theorems of plane and solid geometry. In some of the schools of Finland, Germany, and Russia the elements of trigonometry are introduced.

THE WORK OF THE TENTH SCHOOL YEAR.

The tenth school year is the second year of most of the secondary schools in the United States. In most of the schools the entire year is devoted to the study of plane geometry. In some of the schools half of the year is devoted to the study of algebra and the other half to the study of plane geometry.

In the schools of Europe the mathematics of the tenth school year is distinctly in advance of that in the United States. The difference in the mathematical courses in the two countries is most marked in the subjects of geometry, trigonometry, and drawing.

In most of the European schools the course in algebra is much the same as in the first half of the third year in the secondary schools in the United States, that is, the course begins with the study of quadratic equations and continues through the progressions and the binomial theorem. In many of the European countries the subjects of logarithms, compound interest, and annuities are given a good deal of attention.

The courses in geometry abroad usually include the study of congruent and similar figures, cyclometry, regular polyhedra, and the formulas for the surfaces and volumes of the common solids.

The subject of trigonometry is taught during the tenth school year in almost all of the European countries. The extent of the course varies, but usually the entire subject of plane trigonometry is completed during this school year. In a few of the countries spherical trigonometry is also studied, and the knowledge thus gained is applied in the study of solid geometry and of mathematical geography.

Descriptive geometry is introduced in the French scientific lycée, in a few Realschulen and Oberrealschulen of Germany, and in some of the schools of Holland and Hungary. The study of geometric drawing is given more emphasis in France than in other countries during the tenth school year. The elements of analytic geometry are introduced in a few of the Realgymnasia and Oberrealschulen of Germany and in some of the Realschulen of Russia. In practically all of the European countries, by the time a boy has completed the tenth school year, if he has pursued the scientific course, he has studied all the mathematics that is offered during the entire twelve school years in most of the schools of the United States.

THE WORK OF THE ELEVENTH SCHOOL YEAR.

The eleventh school year is the third year of the secondary school of the United States. In most of the schools the course in mathematics includes a half year of algebra and a half year of solid geometry. In a few of the schools, the first half year is devoted to the study of solid geometry and the last half to trigonometry.

In most of the schools of Europe the work in algebra during the eleventh school year includes the study of arithmetic and geometric progressions, permutations and combinations, the theory of probability, the binomial theorem, and determinants. The preceding topics are included in the advanced courses in Austria, France, Germany, Holland, Hungary, Sweden, Switzerland, and Roumania. The subject of trigonometry, which was completed in most schools during the tenth school year, is applied to surveying. Much more emphasis is put upon the subjects of cosmography and mechanics than in the United States. Descriptive geometry is studied in Austria, Germany, France, Holland, Hungary, and Switzerland, and the study of geometric drawing is continued in Austria. Analytic geometry is introduced in certain of the schools of Austria, France, Germany, Hungary, and Sweden, and the elements of differential calculus are introduced in a few of the Realschulen of Austria and in the Oberrealschulen of Germany. The calculus is applied in the study of physics. Spherical trigonometry is taught in several of the countries and it is usually applied in the study of mathematical geography.

When a European boy has completed the eleventh school year, if he has elected the scientific course, he has studied more mathematics than is offered in any except a few of the most progressive secondary schools in the United States.

THE WORK OF THE TWELFTH SCHOOL YEAR.

The twelfth school year is the last year of the secondary school in the United States. In many of the schools the first half of the year is devoted to the study of solid geometry and the last half to trigonometry or to business arithmetic. In some of the schools the first half of the year is devoted to the study of plane trigonometry and the last half to the study of college algebra. It is rare that courses in spherical trigonometry, beyond the study of the right spherical triangle, are offered in the United States.

In practically all of the European countries the twelfth school year begins or ends with a comprehensive review of the mathematics of the preceding years. Special emphasis is put upon such reviews in Austria, Germany, Denmark, Holland, and Hungary. The practical applications of mathematics are emphasized in most of the countries, and in Austria and Germany some attention is devoted to the history of mathematics. Spherical trigonometry is offered in the schools of Holland, Germany, Hungary, and Switzerland; and analytic geometry is studied in Denmark, Germany, Austria, Hungary, Sweden, Switzerland, and Roumania. In a few of the cantons of Switzerland solid analytic geometry is studied. Descriptive geometry is offered in Germany, Austria, Hungary, and Switzerland, and advanced algebra is studied in most of the countries. Differential and integral calculus are offered in the schools of Austria, Belgium, Denmark, France, Sweden, Switzerland, Russia, Germany, and Roumania, and in some English schools.

The relations between algebra and geometry are especially emphasized in France and the relations between mathematics and physics receive special emphasis in Germany, Holland, and Switzerland.

When a European boy has completed his twelfth school year he has had the opportunity to study more mathematics than is offered in the secondary schools of the United States. He has had more practice in applying his mathematics to physics, cosmography, and mathematical geography than is the case with the American boy. He sees the unity of mathematics in a way that is seldom true in the case of the American boy. He can use his arithmetic and algebra in the solution of geometric problems and his arithmetic and geometry in the solution of algebraic problems much better than the average American boy. He has some knowledge of analytic geometry and of the calculcus. The frequent drills and reviews so common in European schools have caused him to master a large number of mathematical facts and formulas and he is taught to use these in practical ways. Mathematics to him is an interesting and a fruitful subject, because he has learned to appreciate something of its deeper significance.

III. Graphic Representation of Work in Mathematics

Tables 2 to 6 indicate the time at which each subject is introduced into the various types of schools and the school years during which the subject is taught.

The nomenclature of some of the mathematical subjects varies to such an extent in the various countries that it is not possible to be exact in all details in such tables. These tables are intended to indicate usual practices, and not exceptional cases.

The term geometry, as used in Table 4, includes all that is usually thought of in this country under the headings of observational, demonstrative (plane and solid), descriptive geometry, and geometric drawing. In many cases the reports of the countries do not indicate the divisions between two types of geometry sufficiently to justify the construction of a separate graph for each.

TABLE 2.—*Years of study of arithmetic.*

Age	6–7	7–8	8–9	9–10	10–11	11–12	12–13	13–14	14–15	15–16	16–17	17–18
School year	1	2	3	4	5	6	7	8	9	10	11	12
Austria:												
Volksschule, 5-year course	——	——	——	——	——	……	……	……	……	……	……	……
Bürgerschule, 3-year	……	……	……	……	……	——	——	——	……	……	……	……
Gymnasium, 5-year	……	……	……	……	——	——	——	——	——	……	……	……
Realschule, 5-year	……	……	……	……	——	——	——	——	——	……	……	……
Realgymnasium, 4-year	……	……	……	……	——	——	——	——	……	……	……	……
Belgium:												
Primary, 6-year	——	——	——	——	——	——	……	……	……	……	……	……
Middle, 3-year	……	……	……	……	……	……	——	——	——	……	……	……
Athénée, 5-year	……	……	……	……	……	……	——	——	——	——	——	……
Denmark:												
Folkeskole, 8-year	——	——	——	——	——	——	——	……	……	……	……	……
Intermediate, 4-year	……	……	……	……	……	——	——	——	——	……	……	……
Realklasse, 1-year	……	……	……	……	……	……	……	……	……	——	……	……
England:												
Elementary, 8-year	——	——	——	——	——	——	——	——	……	……	……	……
Secondary, 1-year	……	……	……	……	……	……	……	……	……	——	……	……

TABLE 2.—*Years of study of arithmetic*—Continued.

Age	6–7	7–8	8–9	9–10	10–11	11–12	12–13	13–14	14–15	15–16	16–17	17–18
School year	1	2	3	4	5	6	7	8	9	10	11	12
Finland:												
Primary, 7-year	——	——	——	——	——	——	——	......	......	......	......	......
Lycée, 4-year	......	......	......	......	——	——	——	——	......	......	......	......
France:												
Primary, 4-year	——	——	——	——	......	......	......	......	......	......	......	......
Higher primary, 4-year	......	......	......	......	——	——	——	......	......	......	——	......
Lycée, 4-year	......	......	......	......	——	——	——	——	......	......	......	......
Germany:												
Volksschule, 8-year	——	——	——	——	——	——	——	——	......	......	......	......
Bürgerschule, 6-year	......	......	......	——	——	——	——	——	——	......	......	......
Gymnasium, 6-year	......	......	......	——	——	——	——	——	——	......	......	......
Realgymnasium, 5-year	......	......	......	——	——	——	——	——	......	......	......	......
Oberrealschule, 5-year	......	......	......	——	——	——	——	——	......	......	......	......
Holland:												
Primary, 6-year	——	——	——	——	——	——	......	......	......	......	......	......
Bürgerschule, 2-year	......	......	......	......	......	......	——	——	......	......	......	......
Middle, 2-year	......	......	......	......	......	......	——	——	......	......	......	......
Gymnasium, 2-year	......	......	......	......	......	......	——	——	......	......	......	......
Hungary:												
Volksschule, 6-year	——	——	——	——	——	——	......	......	......	......	......	......
Bürgerschule, 4-year	......	......	......	......	——	——	——	——	......	......	......	......
Gymnasium, 3-year	......	......	......	......	——	——	——	......	......	......	......	......
Realschule, 3-year	......	......	......	......	——	——	——	......	......	......	......	......
Italy:												
Elementary, 6-year	——	——	——	——	——	——	......	......	......	......	......	......
Ginnasio, 5-year	......	......	......	......	——	——	——	——	——	......	......	......
Modern, 4-year	......	......	......	......	——	——	——	——	......	......	......	......
Liceo, 1-year	......	......	......	......	......	......	......	......	......	——	......	......
Japan:												
Ordinary primary, 6-year	——	——	——	——	——	——	......	......	......	......	......	......
Higher primary, 3-year	......	......	......	......	......	......	——	——	——	......	......	......
Middle, 3-year	......	......	......	......	......	......	——	——	——	......	......	......
Roumania:												
Primary, 5-year	——	——	——	——	——	......	......	......	......	......	......	......
Gymnasium, 3-year	......	......	......	......	——	——	——	......	......	......	......	......
Lycée, 1-year	......	......	......	......	......	......	......	......	——	......	......	......
Russia:												
Primary, 3–5-year	——	——	——	......	......	......	......	......	......	......	......	......
Gymnasium, 4-year	......	......	......	——	——	——	......	......	......	......	——	......
Realschule, 3-year	......	......	......	——	——	——	......	......	......	......	......	......
Sweden:												
Primary, 6-year	——	——	——	——	——	——	......	......	......	......	......	......
Realschule, 6-year	......	......	......	——	——	——	——	——	——	......	......	......
Switzerland:												
Primary, 6-year	——	——	——	——	——	——	......	......	......	......	......	......
Lower middle, 3-year	......	......	......	......	——	——	——	......	......	......	......	......
Gymnasium, 2-year	......	......	......	......	......	......	——	——	......	......	......	......
Realschule, 1-year	......	......	......	......	......	......	......	......	——	......	......	......
United States of America:												
Elementary, 8-year	——	——	——	——	——	——	——	——	......	......	......	......
Secondary, 1-year	......	......	......	......	......	......	......	......	......	......	......	......

TABLE 3.—*Years of study of algebra.*

Age	6-7	7-8	8-9	9-10	10-11	11-12	12-13	13-14	14-15	15-16	16-17	17-18	18-19
School year	1	2	3	4	5	6	7	8	9	10	11	12	13
Austria:													
Bürgerschule, 2-year							——	——					
Gymnasium, 5-year							——	——	——	——	——		
Realschule, 5-year							——	——	——	——	——		
Realgymnasium, 5-year							——	——	——	——	——		
Belgium:													
Middle, 2-year							••••	——	——				
Athénée, 5-year									——	——	——	——	——
Denmark:													
Intermediate, 2-year							••••	——	——				
Realklasse, 1-year										——			
Gymnasium, 3-year										——	——	——	
England:													
Elementary, 2-year							——	——					
Secondary, 3-year									——	——	——		
Private preparatory, 5-year								——	——	——	——	——	
Finland:													
Lycée, 5-year							——	——	——	——	——		
France:													
Higher primary, 2-year						——	——						
Lycée, 6-year						——	——	——	——	——	——		
Germany:													
Bürgerschule, 5-year							——	——	——	——	——		
Gymnasium, 5-year							——	——	——	——	——		
Realgymnasium, 5-year							——	——	——	——	——		
Oberrealschule, 5-year							——	——	——	——	——		

Age	6-7	7-8	8-9	9-10	10-11	11-12	12-13	13-14	14-15	15-16	16-17	17-18
School year	1	2	3	4	5	6	7	8	9	10	11	12
Holland:												
Bürgerschule, 2-year							——	——				
Middle, 5-year							——	——	——	——	——	
Gymnasium, 6-year							——	——	——	——	——	——
Hungary:												
Bürgerschule, 5-year							••••	——	••••	••••	——	——
Gymnasium, 5-year							••••	——	——	——	——	——
Realschule, 5-year							••••	——	——	——	——	——
Italy:												
Ginnasio, 3-year							——	——	——			
Liceo, 2-year										——	——	
Modern, 5-year							——	——	——	——	——	
Japan:												
Middle, 3-4-year								——	——	——	••••	
Roumania:												
Gymnasium, 2-year							••••	——	——			
Lycée, 4-year								——	——	——	——	——
Russia:												
Gymnasium, 6-year						——	——	——	——	——	——	
Realschule, 5-year						——	——	——	——	——		

TABLE 3.—*Years of study of algebra*—Continued.

Age	6–7	7–8	8–9	9–10	10–11	11–12	12–13	13–14	14–15	15–16	16–17	17–18
School year	1	2	3	4	5	6	7	8	9	10	11	12
Sweden:												
Realschule, 4-year	......	......	......	......	......	——	——	——	——	......	......	......
Gymnasium, 3-year	......	......	......	......	......	......	......	......	——	——	——	......
Switzerland:												
Gymnasium, 6-year	......	......	......	......	......	......	••••••	——	——	——	——	——
Realschule, 4-year	......	......	......	......	......	......	......	......	——	——	——	——
United States of America:												
Elementary, 1-year	......	......	......	......	......	......	......	••••••	......	......	......	......
Secondary, 2-year	......	......	......	......	......	......	......	......	——	......	——	••••••

TABLE 4.—*Years of study of geometry.*

Age	6–7	7–8	8–9	9–10	10–11	11–12	12–13	13–14	14–15	15–16	16–17	17–18	18–19
School year	1	2	3	4	5	6	7	8	9	10	11	12	13
Austria:													
Volksschule, 1-year	......	......	......	......	——	......	......	......	......	......	......	......	......
Bürgerschule, 3 years	......	......	......	......	......	——	——	——	......	......	......	......	......
Gymnasium, 6-year	......	......	......	......	——	——	——	——	——	——	......	......	......
Realschule, 6-year	......	......	......	......	——	——	——	——	——	——	......	......	......
Realgymnasium, 6-year	......	......	......	......	——	——	——	——	——	——	......	......	......
Belgium:													
Primary, 1-year	......	......	......	......	......	——	......	......	......	......	......	......	......
Middle, 3-year	......	......	......	......	......	......	——	——	——	......	......	......	......
Athénée, 7-year	......	......	......	......	......	......	——	——	——	——	——	——	——
Denmark:													
Folkeskole, 2-year	......	......	......	......	......	——	——	......	......	......	......	......	......
Intermediate, 4-year	......	......	......	......	......	——	——	——	——	......	......	......	......
Realklasse, 1-year	......	......	......	......	......	......	......	......	......	——	......	......	......
Gymnasium, 2-year	......	......	......	......	......	......	......	......	......	——	——	......	......
England:													
Elementary, 5-year	......	......	......	——	——	——	——	——	......	......	......	......	......
Secondary, 3-year	......	......	......	......	......	......	......	......	——	——	——	......	......
Private Preparatory, 3-year	......	......	......	......	......	......	......	——	——	——	......	......	......
Finland:													
Primary, 1-year	......	......	......	......	......	......	——	......	......	......	......	......	......
Lycée, 4-year	......	......	......	......	......	......	......	——	——	——	——	......	......
France:													
High Primary, 3-year	......	......	......	......	——	——	——	......	......	......	......	......	......
Lycée, 8-year	......	......	......	......	——	——	——	——	——	——	——	——	......

TABLE 4.— *Years of study of geometry*—Continued.

Age	6-7	7-8	8-9	9-10	10-11	11-12	12-13	13-14	14-15	15-16	16-17	17-18	18-19
School year	1	2	3	4	5	6	7	8	9	10	11	12	13
Germany:													
Volksschule, 7-year	.....	.....	.....	.....	——	——	——	——	——	——	——	.....	.....
Bürgerschule, 7-year	.....	.....	.....	.....	——	——	——	——	——	——	——	.....	.....
Gymnasium, 7-year	.....	.....	.....	.....	——	——	——	——	——	——	——	.....	.....
Realgymnasium, 7-year	.....	.....	.....	.....	——	——	——	——	——	——	——	.....	.....
Oberrealschule, 7-year	.....	.....	.....	.....	——	——	——	——	——	——	——	.....	.....

Age	6-7	7-8	8-9	9-10	10-11	11-12	12-13	13-14	14-15	15-16	16-17	17-18
School year	1	2	3	4	5	6	7	8	9	10	11	12
Holland:												
Bürgerschule, 2-year	.....	.....	.....	.....	.....	.....	——	········	.....	.....	.....	.....
Middle, 5-year	.....	.....	.....	.....	.....	.....	——	——	——	——	——	.....
Gymnasium, 6-year	.....	.....	.....	.....	.....	.....	——	——	——	——	——	——
Hungary:												
Volksschule, 2-year	.....	.....	.....	.....	——	——	.....	.....	.....	.....	.....	.....
Bürgerschule, 8-year	.....	.....	.....	.....	——	——	——	——	········	········	········	········
Gymnasium, 8-year	.....	.....	.....	.....	——	——	——	——	——	——	——	——
Realschule, 8-year	.....	.....	.....	.....	——	——	——	——	——	——	——	——
Italy:												
Elementary, 1-year	.....	.....	.....	.....	.....	——	.....	.....	.....	.....	.....	.....
Gymnasium, 5-year	.....	.....	.....	.....	——	——	——	——	——	.....	.....	.....
Liceo, 2-year	.....	.....	.....	.....	.....	.....	.....	.....	.....	——	——	.....
Modern, 6-year	.....	.....	.....	.....	——	——	——	——	——	——	.....	.....
Japan:												
Middle, 3–4 year	.....	.....	.....	.....	.....	.....	.....	.....	——	——	——	········
Roumania:												
Gymnasium, 4-year	.....	.....	.....	.....	.....	——	——	——	——	.....	.....	.....
Lycée, 3-year	.....	.....	.....	.....	.....	.....	.....	.....	——	——	——	.....
Russia:												
Gymnasium, 3-year	.....	.....	.....	.....	.....	.....	——	——	——	.....	.....	.....
Realschule, 3-year	.....	.....	.....	.....	.....	.....	——	——	——	.....	.....	.....
Sweden:												
Realschule, 4-year	.....	.....	.....	.....	.....	——	——	——	——	.....	.....	.....
Gymnasium, 3-year	.....	.....	.....	.....	.....	.....	.....	.....	——	——	——	.....
Switzerland:												
Primary, 2-year	.....	.....	.....	.....	——	——	.....	.....	.....	.....	.....	.....
Lower Middle, 3-year	.....	.....	.....	.....	——	——	——	.....	.....	.....	.....	.....
Gymnasium, 6-year	.....	.....	.....	.....	.....	.....	——	——	——	——	——	——
Realschule, 3-year	.....	.....	.....	.....	.....	.....	.....	.....	——	——	——	.....
United States of America:												
Elementary, 1-year	.....	.....	.....	.....	.....	.....	.....	········	.....	.....	.....	.....
Secondary, 1½-year	.....	.....	.....	.....	.....	.....	.....	.....	.....	——	...——	.....

TABLE 5.—*Years of study of trigonometry.*

Age	6–7	7–8	8–9	9–10	10–11	11–12	12–13	13–14	14–15	15–16	16–17	17–18	18–19
School year	1	2	3	4	5	6	7	8	9	10	11	12	13
Austria:													
Gymnasium, 2-year	.....	.....	.....	.....	.....	.....	.....	.....	.....	——	——	——	.....
Realschule, 2-year	.....	.....	.....	.....	.....	.....	.....	.....	.....	——	——	.....	.....
Realgymnasium, 2-year	.....	.....	.....	.....	.....	.....	.....	.....	.....	——	——	.....	.....
Belgium:													
Athénée, 2-year	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	——	——	——
Denmark:													
Gymnasium, 1-year	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	——	.....	.....
England:													
Secondary, 2-year	.....	.....	.....	.....	.....	.....	.....	.....	.....	——	——	.....	.....
Private preparatory, 2-year	.....	.....	.....	.....	.....	.....	.....	.....	.....	——	——	.....	.....
Finland:													
Lycée, 3-year	.....	.....	.....	.....	.....	.....	.....	.....	——	——	——	.....	.....
France:													
Lycée, 2-year	.....	.....	.....	.....	.....	.....	.....	.....	.....	——	——	.....	.....
Germany:													
Bürgerschule, 2-year	.....	.....	.....	.....	.....	.....	.....	.....	.....	——	——	.....	.....
Gymnasium, 3-year	.....	.....	.....	.....	.....	.....	.....	.....	.....	——	——	——	.....
Realgymnasium, 3-year	.....	.....	.....	.....	.....	.....	.....	.....	——	——	——	.....	.....
Oberrealschule, 3-year	.....	.....	.....	.....	.....	.....	.....	.....	——	——	——		
Holland:													
Middle, 2-year	.....	.....	.....	.....	.....	.....	.....	.....	.....	——	——	.....	.....
Gymnasium, 2-year	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	——	——	.....
Hungary:													
Gymnasium, 2-year	.....	.....	.....	.....	.....	.....	.....	.....	.....	——	——	.....	.....
Realschule, 2-year	.....	.....	.....	.....	.....	.....	.....	.....	.....	——	——	.....	.....

Age	6–7	7–8	8–9	9–10	10–11	11–12	12–13	13–14	14–15	15–16	16–17	17–18
School year	1	2	3	4	5	6	7	8	9	10	11	12
Italy:												
Liceo, 1-year	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	——	.....
Japan:												
Middle, 2-year	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	——	——
Roumania:												
Lycée, 2-year	.....	.....	.....	.....	.....	.....	.....	.....	.....	——	——	.....
Russia:												
Gymnasium, 3-year	.....	.....	.....	.....	.....	.....	.....	.....	.....	——	——	.....
Realschule, 2-year	.....	.....	.....	.....	.....	.....	.....	.....	——	——	.....	.....
Sweden:												
Gymnasium, 2-year	.....	.....	.....	.....	.....	.....	.....	.....	.....	——	——	.....
Realgymnasium, 2-year	.....	.....	.....	.....	.....	.....	.....	.....	.....	——	——	.....
Switzerland:												
Gymnasium, 3-year	.....	.....	.....	.....	.....	.....	.....	.....	.....	——	——	——
Realschule, 2-year	.....	.....	.....	.....	.....	.....	.....	.....	.....	——	——	.....
United States of America:												
Secondary, 1-year	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	——

TABLE 6.—*Years of study of analytic geometry and the calculus.*

	Analytic geometry					The calculus.					
Age	14–15	15–16	16–17	17–18	18–19		14–15	15–16	16–17	17–18	18–19
School year	9	10	11	12	13		9	10	11	12	13
Austria:											
Gymnasium, 2-year	.....	.....	——	——	.....	1 year.	.....	.....	.....	——	.....
Realschule, 2-year	.....	.....	——	——	.....	2 yrs.	.....	.....	——	——	.....
Realgymnasium, 2-year	.....	.....	——	——	.....	1 yr.	.....	.....	.....	——	.....
Belgium:											
Athénée, 2-year	.....	.....	.....	——	——	.....	.....	.....	.....	——	.....
Denmark:											
Gymnasium, 1-year	.....	.....	.....	——	.....	1 yr.	.....	.....	.....	——	.....
England:											
Private preparatory, 1-year	.....	.....	.....	——	.....	1 yr.	.....	.....	.....	——	.....
France:											
Lycée, 2-year	.....	.....	——	——	.....	1 yr.	.....	.....	.....	——	.....
Germany:											
Gymnasium, 2-year	.....	.....	——	——	.....	.....	.....	.....	.....	.....	.....
Realgymnasium, 2-year	.....	——	——	.....	.....	1 yr.	.....	.....	.....	——	.....
Oberrealschule, 3-year	.....	——	——	——	.....	2 yrs.	.....	.....	——	——	.....
Holland:											
Gymnasium, 1-year	.....	.....	.....	——	.....	.....	.....	.....	.....	.....	.....
Hungary:											
Realschule, 1-year	.....	.....	——	.....	.....	.....	.....	.....	.....	.....	.....
Gymnasium, 2-year	.....	.....	——	——	.....	.....	.....	.....	.....	.....	.....
Bürgerschule, 1-year	.....	.....	.....	——	.....	.....	.....	.....	.....	.....	.....
Roumania:											
Lycée, 1-year	.....	.....	.....	——	.....	.....	.....	.....	.....	——	.....
Russia:											
Realschule, 1-year	.....	——	.....	.....	.....	1 yr.	.....	——	.....	.....	.....
Sweden:											
Gymnasium, 1-year	.....	.....	.....	——	.....	1 yr.	.....	.....	.....	——	.....
Realgymnasium, 1-year	.....	.....	——	.....	.....	.....	.....	.....	.....	.....	.....
Switzerland:											
Gymnasium	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
Realschule, 2-year	.....	.....	.....	——	——	2 yrs.	.....	.....	.....	——	——
United States of America:											
[Analytic Geometry is seldom taught in secondary schools.]											

IV. Certain Important Points of Difference Between the Work in Mathematics Abroad and in the United States

The European schools are doing certain kinds of work that we are not doing, some that we can not hope to do under present conditions, and some that we might not care to do if we could. They are also doing some work that we wish we could do, and some that we shall probably do before many years have elapsed.

In every country of Europe the secondary school period extends over at least six years. Austria, Finland, Hungary, Italy, Roumania and Russia devote eight years; and France, Germany, and Sweden nine years to secondary education. The teachers have a margin of scholarship that is not common among teachers in the United States. The academic requirements for secondary teachers are higher in France than in any other country but the professional requirements are not as severe as in Germany. Most of the head professors in the French lycées possess mathemtical attainments equalled only by some of the best of our American college professors. Germany insists less on brilliancy and decidedly more on theoretical and practical professional training. Throughout Germany the attractiveness of the teacher's position is sufficient to induce some of the best mathematicians to become teachers in the secondary schools.[2]

✓ Abundant provision is made for daily drill in mathematics. The educator of Europe realizes that this daily drill is absolutely necessary in order to give the pupil a real mastery of number facts and relations. A little smattering of the subject will not suffice. The pupil is expected to know thoroughly certain facts and principles, and to this end daily drill is provided. No small part of the thoroughness in detail, which is so characteristic of most of the schools of Europe, may be traced to this drill. The American pupil has some information on a great variety of topics, but much of his knowledge is vague and indefinite, rather than clear-cut and definite.

✓ Everywhere algebra is introduced earlier than in the United States. In certain of the German schools some work in algebra is introduced during the sixth school year and in no country, except the United States, is this introductory work postponed later than the seventh school year.

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[2] For a more extensive discussion of the training required of teachers of mathematics in foreign countries, see Chapter XIV.

Some instruction in constructive, observational, or intuitive geometry is always offered during the sixth, seventh, and eighth school years. This instruction is always of a propædeutic nature. Much emphasis is placed upon estimates and constructions.

In all of the schools of Europe, algebra and geometry are studied simultaneously during a considerable number of years. The various mathematical subjects are more closely correlated than in this country. A pupil who is studying geometry can use his arithmetic and his algebra more readily than is the case with the average American boy. The introduction of the trigonometric functions while the pupil is studying similar figures in geometry has the sanction of most of the best teachers abroad. The distinction between plane and solid geometry is much less marked than in this country. This is due, in part at least, to the fact that models are very extensively used in the study of geometry. Thorough grounding in geometry is one of the characteristics of Italian schools.

Everywhere the attempt is being made to find genuine applications of mathematics that are really within the experience of the pupil and to link the subject of mathematics as closely as possible with the activities of real life. Drawing and physics are frequently taught by the same teacher, and the correlation between these subjects is found to be to the advantage of each.

European school men believe that a course in mathematics should be planned by those who know some mathematics rather than by educators who are practically ignorant of the subject. The reports do not indicate that the schools of Europe are hearing a demand for weak algebra and anemic geometry, or even for no work in these subjects. If any pressure of this sort exists, it has hitherto produced no modification of the course of study.

V. Bibliography

All of the reports submitted to the International Commission on the Teaching of Mathematics were consulted in the preparation of this report.

The reports listed below contain material of especial value for such a study as this. They may be secured from Messrs. Georg & Co., Geneva, Switzerland.

AUSTRIA.

Kraus, K. Volks-und Bürgerschulen. Heft I of the Austrian reports.
Bergmann, F. Realschulen. Heft I of the Austrian reports.
Dintzl, E. Gymnasien. Heft III of the Austrian reports.
Konrath, Th. Mädchenlyzeen. Heft IV of the Austrian reports.
Adler, A. Der Unterricht in der darstellenden Geometrie an den Realschulen und Realgymnasien. Heft IX of the Austrian reports.
Müller, E. Der Unterricht in der darstellenden Geometrie an den Technischen Hochschulen. Heft IX of the Austrian reports.

BELGIUM.

Rapports sur l'enseignement des mathématiques, du dessin et du travail manuel dans les écoles primaries, les écoles normales primaries, les écoles moyennes, les athénées, les collèges belges. Vol. I of the Belgian reports.

DENMARK.

Heegaard, P. Bericht über den Mathematikunterricht in Danemark. Vol. I of the Danish reports.

ENGLAND.

Newbold, W. Higher mathematics for the classical sixth form. No. I.
Filon, L. N. G. The relations of mathematics and physics. No. II.
Ballard, P. B. The teaching of mathematics in London public elementary schools. No. III.
Spencer, H. J. The teaching of elementary mathematics in English public elementary schools. No. IV.
Godfrey, C. The algebra syllabus in the secondary schools. No. V.
Palmer, G. W. The teaching of arithmetic in secondary schools. No. XIII.
Carson, G. St. L. The educational value of geometry. No. XV.
Barnard, S. The teaching of algebra in schools. No. XXII.
Kitchener, E. Mathematics in the preparatory school. No. XXIX.
Jones, L. M. Course in mathematics for municipal secondary schools.

FINLAND.

L'enseignement mathématique dans les écoles de Finlande. (Prepared by a commission.)

FRANCE.

Enseignement primaire. Tome I. (Prepared under the direction of Ch. Bioche.)
Enseignement secondaire. Tome II. (Prepared under the direction of Ch. Bioche.)
Enseignement technique. Tome IV. (Prepared under the direction of M. P. Rollet.)
Enseignement des jeunes filles. Tome V. (Prepared under the direction of Mlle. Amieux.)

GERMANY.

Lietzmann, W. Stoff und Methode im mathematischen Unterricht der norddeutschen höheren Schulen auf Grund der vorhandenen Lehrbücher. Erster Band. Heft I.

Thaer, A., Geuther, N., and **Böttger, A.** Der mathematische Unterricht an den Gymnasien und Realanstalten der Hansestädte, Mecklenburgs und Oldenburgs. Erster Band. Heft IV.

Wieleitner, H. Der mathematische Unterricht an den höheren Lehranstalten sowie die Ausbildung und Fortbildung der Lehrkräfte im Königreich Bayern. Zweiter Band. Heft I.

Witting, A. Der mathematische Unterricht an den Gymnasien und Realanstalten nach Organisation, Lehrstoff und Lehrverfahren und die Ausbildung der Lehramtskandidaten im Königreich Sachsen. Zweiter Band. Heft II.

Geck, E. Der mathematische Unterricht an den höheren Schulen nach Organisation, Lehrstoff und Lehrverfahren und die Ausbildung der Lehramtskandidaten im Königreich Württemberg. Zweiter Band. Heft III.

Cramer, H. Der mathematische Unterricht an den höheren Schulen nach Organisation, Lehrstoff und Lehrverfahren und die Ausbildung der Lehramtskandidaten im Grossherzogtum Baden. Zweiter Band. Heft IV.

Schnell, H. Der mathematische Unterricht an den höheren Schulen nach Organisation, Lehrstoff und Lehrverfahren und die Ausbildung der Lehramtskandidaten im Grossherzogtum Baden. Zweiter Band. Heft V.

Zühlke, P. Der Unterricht im Linearzeichnen und in der darstellenden Geometrie an den deutschen Realanstalten. Dritter Band. Heft III.

Lietzmann, W. Stoff und Methode des Rechenunterrichts in Deutschland. Fünfter Band. Heft I.

——— Stoff und Methode des Raumlehreunterrichts in Deutschland. Fünfter Band. Heft II.

Der mathematische Unterricht an den Volksschulen und Lehrerbildungsanstalten Süddeutschlands. With an introduction by P. Treutlein. Fünfter Band. Heft III.

Dressler, H. Der mathematische Unterricht an den Volksschulen und Lehrerbildungsanstalten in Sachsen und Thüringen. Fünfter Band. Heft IV.

Umlauf, K. Der mathematische Unterricht an den Seminaren und Volksschulen der Hansestädte. Fünfter Band. Heft V.

Lietzmann, W. Die Organisation der Volksschulen, gehobenen Volksschulen, Präparandenanstalten, Seminare, usw. in Preussen. Fünfter Band. Heft VI.

HOLLAND.

Rapport sur l'enseignement mathématique dans les Pays-Bas. Vol. I. (Prepared under the direction of J. Waltman.)

HUNGARY.

Beke, E., and **Mikola, S.** Im Auftrage der Mathematischen Reform Kommission des Landesvereins der Mittelschulprofessoren nach dem ungarischen Original deutsch herausgegben. No. I.

Rados, G. Der heutige Stand des mathematischen Unterrichts am Königlich Ungarischen Josefs-Polytechnikum. No. III.

Havas, M., and **Bogyo, S.** Der mathematische Unterricht an den Handelsschulen. No. 7.

Volensky, J. Der mathematische Unterricht an den Bürgerschulen No. 8.

Beke, E. Der mathematische Unterricht an den Mittelschulen. (Gymnasien und Realschulen.) No. 9.

ITALY.

Conti, A. Scuole infantili ed elementari. No. I.

Scarpis, Fazzari. Scuole classiche. No. III.

Lazzeri, G. Scuole industriali, professionali e commerciali. No. 5.

JAPAN.

Report on the teaching of mathematics in Japan. (Prepared by a commission.) Vol. I.

ROUMANIA.

Tzitzeica, G. L'enseignement mathématique en Roumanie. Enseignement secondaire.

RUSSIA.

Vogt, K. W. Bericht über den mathematischen Unterricht an den russischen Realschulen. No. III.

M. S. L'enseignement mathématique dans les écoles primaires et les écoles normales. No. IV.

Ministère de l'instruction publique. L'enseignement mathématique dans les gymnases garçons.

SWEDEN.

Der mathematische Unterricht in Schweden, d. by Dr. H. von Koch and Dr. E. Göransson.

SWITZERLAND.

Stocklin, Just. Der mathematische Unterricht an den schweizerischen Primarschulen. Vol. I. Part II.

Baderstecher. Der mathematische Unterricht an den schweizerischen Sekundarschulen. Vol. I. Part II.

Gubler, E. Der mathematische Unterricht an den höheren Mädchenschulen der Schweiz. Vol. I. Part III.

Brandenberger, K. Der Mathematische Unterricht an den schweizerischen Gymnasien und Realschulen. Vol. I. Part IV.

UNITED STATES.

(All of the reports were prepared by committees, and published by the Bureau of Education, Washington, D. C.)

I. Mathematics in the elementary schools of the United States, Bulletin, 1911, No. 13.

II. Mathematics in the public and private secondary schools of the United States, Bulletin, 1911, No. 16.

III. Mathematics in the technical secondary schools of the United States, Bulletin, 1912, No. 4.

IV. Training of teachers of elementary and secondary mathematics, Bulletin, 1911, No. 12.

CHAPTER XII

Experimental Courses in Secondary School Mathematics

BY

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Introduction

Purpose and content of this report.—The purpose of this study is to make generally available the experience of teachers conducting experimental courses in mathematics. Teachers will desire to know what is done by other teachers who have had greater freedom in the choice of materials and methods than is feasible in the typical public school. The value of this report is enhanced by the fact that the teachers in these schools, considered as a group, have wide experience, sound scholarship in mathematics, and unusual professional training.

The material includes a detailed description of one or more courses in each of fifteen schools. Not all schools designated as experimental schools feel that they have made a real contribution to the teaching of mathematics. The search for new materials by the National Committee has been widespread. It is reasonably certain that no work considered unique by the persons doing it has been omitted.

Form of the report.—The report is presented in three parts. Part I aims to meet the needs of the reader who is interested only in a general survey of the important conclusions reached, but who does not desire to examine the evidence given in later parts of the report. Part II is for the reader who may wish to consider illustrations without going into the full details which constitute the basis for the statements in Part I. Part III is for the reader who has a genuine interest in considering more carefully some of the problems involved in improving the mathematics instruction of secondary schools. This constitutes the source material of the report. A list of names together with addresses is given on page *277* to whom credit is due for furnishing the facts. Presumably, those interested in a particular phase of the problem may get further information by writing directly to the individual concerned.

Part I—Important Statements and Conclusions

1. A considerable number of schools in the United States claim to have undertaken some experimental work. Of these very few (probably not over ten) feel that they have made a significant contribution to the reorganization of secondary school mathematics.

2. There is very little, if any, work in mathematics that is experimental in a *scientific* sense. In general, the word "experimental" is used in a vague sense to indicate the use of materials or methods *differing* from those in current practice.

3. There are public school systems in which the administrators believe it highly desirable to incorporate experimentation as a regular and systematized feature of their programs of education. These officials believe that directed experimentation contributes to the efficiency of a school system.

4. The teachers of mathematics in the schools represented in this report constitute a group unusually well trained in mathematics and in professional courses.

5. Most of the efforts to improve mathematics at present are directed toward grades seven, eight and nine. Particular attention has been given to the mathematics of the ninth school year. There are relatively few innovations under way in the later years of the senior high school.

6. No general agreement as to the details of a reorganization program is found in the practice of these teachers. There is an especially wide difference of opinion concerning the grade placement of materials of instruction.

7. In spite of a wide divergence in both theory and practice there is a marked agreement concerning important issues. These may well serve as a basis for a number of practical and constructive steps in completely reshaping the teaching of mathematics in American secondary schools. These generally accepted principles will be stated in the paragraphs that follow.

8. There is dissatisfaction with the results achieved in the arithmetic of grades seven and eight. It is felt that the time and money spent should result in (1) a higher degree of accuracy and speed in applying fundamental principles to fractions (common and decimal), (2) greater skill in dealing with percentage relations, and (3) more significant information about common business forms.

9. The pupils of the seventh and eighth grades lack the maturity and the social experience for much of the social-economic arithmetic that now is taught in these grades. Mortgages, stocks and bonds, bank discount, cost of plastering and the like, are in general not vital problems to twelve and thirteen year old girls and boys. The time spent on the application of arithmetic to business may well be decreased in the early years and the work continued later so as to capitalize increasing maturity, social experience, and mathematical power.

10. The mathematics of the early high school years must be considered worth while by the student on the basis of what it does for him here and now. His experience in other school subjects and life out of school should begin to mean more to him because he is taking a course in secondary mathematics.

11. The algebra of the ninth school year as found in current practice is characterized by an indefensible amount of mere manipulation of symbols. The thinking processes involved are far removed from the way children get mathematical meanings. Spending from four to six weeks on factoring and special products, three weeks on "apartment house" fractions and all similar abstract and questionable units of work must cease. It is significant that everyone of the schools herein represented is concerned with the problem of modifying the mathematics of the ninth grade.

12. Ninth grade algebra has delayed to later courses the teaching of much material that is relatively simple, worth while and interesting to children. Graphic, statistical and trigonometric methods of solving problems are illustrations of these materials. So seldom are they employed in introductory courses that most high school graduates never learn to use these important tools.

13. The teaching of demonstrative geometry is recognized as one of the most serious problems. There has been much dissatisfaction with results generally secured. The teachers of mathematics in these schools are still convinced that the technique of thinking used in analysis (the method of attack in problem solving in demonstrative geometry) is of vital importance. But they also know that plane geometry was originally developed by and for adult philosophers. Current practice which devotes the seventh and eighth grades to arithmetic and the ninth grade solely to algebra, plunges the pupil into demonstrative geometry with inadequate preparation. The solutions proposed vary widely but they indicate a clear recognition of the need for changing current practice.

14. The course for grades seven, eight, and nine should include arithmetic, intuitive geometry, algebra, and numerical trigonometry. Some schools advocate the addition also of a unit of demonstrative geometry. The important fact is that the courses for these years are to be enriched by new materials through the use of intuitive geometry and trigonometry.

15. It is possible and desirable to ignore preparation for college entrance requirements. These schools have an excellent record in the matter of preparing pupils for college, but they do not permit college requirements to shape their courses much before the twelfth school year.

Part II—General Discussion

This part of the report aims to amplify and to furnish illustrations for the corresponding statements in Part I.

1. *Only a Few Experimental Schools Have Made a Systematic Attack on the Mathematics Problem.* ✓ At the beginning of this study, a list of schools which were giving courses more or less experimental in nature was prepared. It was desired at the outset to include every institution in which the teacher (or teachers) of mathematics felt that their experiment had progressed sufficiently for other teachers to profit by a detailed account. On this basis the list of experimental schools to be included was reduced to a brief one.

An experimental school must of necessity confine its constructive efforts to relatively few problems. Moreover, such schools are sometimes more concerned with becoming exponents of some philosophy of education or some educational ideal than in attacking a definite problem in a specific field. At any rate, there were very few experimental schools in which the teachers thought that they were teaching mathematics by other than conventional methods. Furthermore, it became clear that the report could not be complete without describing courses that are being taught in schools not generally considered experimental, as for example, Stuyvesant High School (New York City), East High School (Rochester, N. Y.), and the Cass Technical High School (Detroit, Mich.). The National Committee through its numerous cooperating organizations has solicited accounts of experiments in all sections of the country. The report includes a description of experimental courses in fifteen different schools.

2. *Experimental Work in Mathematics Not Scientific.* ✓ The word "experimental" is ordinarily used in a vague sense to indicate the use of materials or methods differing from those in current practice. But there is seldom an attempt to define the problem with precision or to measure the results; and rarely does anyone know when the experiment ends.

This leads one to say parenthetically that, viewed from an historical perspective, much of our accepted practice in high school mathematics has in this sense been experimental. The teaching of algebra alone in the ninth grade, and the teaching of demonstrative geometry in the tenth grade without propædeutic training and the like, have at best been experiments in this sense and, perhaps, very doubtful ones. In scientific education the word "experimental" has come to mean the study of a precisely defined problem under controlled conditions, and implies measurement of results. To have confined this report to such a definition would have excluded much material here presented which is nevertheless of great value. It represents at least the cumulative experience of critical, competent teachers. Accordingly, the reader will need to keep in mind that we are concerned in this report not so much with scientific experimentation as with purposeful innovations.

✓ 3. *Increased Emphasis upon Experimentation.* We have reason to believe that the school world is putting a new emphasis upon investigation and controlled experimentation. It seems no longer sufficient to advocate an innovation on the basis of educational theory alone. Many school officials believe it highly desirable for a city or a school to incorporate experimentation as an important part of their program of education. The study of a problem by a group of teachers in a school system offers a means for stimulating discussion and genuine professional growth. S. A. Courtis includes controlled experimentation as one of the important duties of a supervisor. Such cities as Berkeley (California), Denver (Colorado), Tulsa (Oklahoma), Chicago (Illinois), Rochester (New York), Detroit (Michigan) and many others are engaged in setting up specific problems in mathematics and making it possible for groups of teachers to give them serious study. To secure widespread coöperation in an experimental study that is clearly defined is no longer difficult. This increased interest in experimental work in mathematics is of great significance and promise.

✓ 4. *Teachers of Experimental Courses Well Trained.* The teachers of the schools described in this section have as a group an amount of professional training greatly exceeding that of mathematics teachers generally. Not only are they well prepared in mathematics, but also in general courses in education, educational psychology, measurements and special methods. The reasons are quite obvious.

Some of these schools are in proximity to or directly connected with universities, thus affording opportunities for graduate study; in other schools the opportunity to do a constructive piece of work is the attractive feature; and in a few cases the salary is sufficient to command a choice of teachers.

The training of the teachers of the Cass Technical High School (Detroit) may serve as an illustration. In the year 1919-1920, the Cass Technical High School had thirteen full time teachers of mathematics. These teachers averaged 31½ semester hours preparation in college mathematics, and 13.8 semester hours in education. Roughly, this means that more than one-fourth of a four-year college course was devoted to mathematics, and a little more than one-ninth of a college course was spent in courses in education. In addition, five teachers had completed courses in mechanical drawing during vacant hours in regular day school classes, seven had taken machine shop courses to gain a knowledge of machine operations and vocabulary, one had sufficient training in the industries so that he was considered one of the best machine shop instructors in the evening classes, and eight had been given temporary approval as teachers of trade-improvement courses under the Smith-Hughes law.

Not only are these teachers well trained in mathematics, but they are specially trained for a special task. It is a well known fact that corporation schools connected with large industries have great difficulty in securing teachers who are sufficiently well trained in education to do satisfactory teaching, and that in attempting to meet occupational needs the public schools find it difficult to obtain teachers who possess the necessary practical experience in the industries to use real problem material. It is safe to predict that the notion that a department head in an industry is a "boss" will be supplanted presently by the idea that he is first of all a teacher. And it is equally clear that the training for a teacher of mathematics in a technical school needs to give increasing emphasis to trade vocabulary, processes, and the real problem material. The Cass Technical High School suggests a solution to the problem, applicable alike to industry and to the public school.

The averages given for the teachers of the Cass Technical High School are probably somewhere near the median for the group as a whole and serve to make clear the fact that this report is concerned with the work of teachers with very unusual training. It is important to keep this fact in mind in considering the principles concerning which there is substantial agreement in this group. The courses are unique not because they are taught by inadequately trained teachers recently released from college and restless for something new but, on the contrary, because they exhibit the mature judgment of experienced, well trained, and successful teachers. In this group of teachers we may reasonably expect practice to approximate theory. This should be kept in mind in considering the results stated in Articles 8 to 15 inclusive on which there is substantial agreement, and which are sufficiently far reaching to reshape completely the teaching of mathematics in American secondary schools.

5. *The Emphasis on Grades Seven, Eight and Nine.* Most of the efforts to reorganize mathematics are being directed at grades seven, eight and nine. The coming of the junior high school is only partly responsible for the especial attention that is now being fixed upon these grades. At the beginning of the seventh grade, the pupils may be expected to possess the fundamental skills in arithmetic. Hence, this is the time for new materials and an extension of number and space relations. This may explain why there appears to be greater variety in seventh grade courses throughout the country than in the fifth and sixth grade courses. The statistics of elimination and the experience with general intelligence tests show that we are rapidly becoming a ninth grade nation. Whatever the causes may be, the schools are beginning to hold the children through the ninth grade. The ratio of pupils entering the ninth grade has shifted in a decade from about one out of eleven to about one out of three. This fact may partially explain the unusual effort to recast ninth grade mathematics.

Grades seven, eight, and nine are now available for inculcating common ideas, ideals, attitudes and habits. Hence the numerous efforts to build basic courses in these grades. All of the schools in this report which include seventh grade classes teach courses that differ widely from the conventional ones. On the other hand, these schools teach many senior high school courses that are frankly conventional. The early reports of the National Committee were particularly devoted to grades seven, eight, and nine, and undoubtedly the discussion of these reports gave additional emphasis to these grades. It is perhaps logical to await the day for improving the later grades until the work of the earlier grades on which these rest shall have become more satisfactory.

It should not be inferred, however, that the senior high school field is wholly neglected. Presumably, the Ethical Culture School's plane geometry course can be considered a senior high school project. Miss Blair's advanced course at the Horace Mann School for Girls is a marked innovation. Mr. Reeve at Minnesota and Mr. Breslich at Chicago have each developed courses covering the senior high school. Miss Sanford of The Lincoln School teaches a course of college grade[1] to eleventh grade pupils, who have had but three hours of mathematics per week during the junior high school years. Both Mr. Schlauch's work (High School of Commerce) and the material of the Cass Technical High School touch the senior high school. It is of particular interest to note the efforts to teach the introductory notions of the calculus as is being done in at least three of the schools described, namely, The University High School (Chicago, Illinois), The Horace Mann School for Girls (New York) and The Lincoln School (New York).

6. *No General Agreement Concerning the Details of the New Program.* The strongest evidence that much detailed and exacting work remains to be done in organizing a sound curriculum in mathematics is the fact that there is no general agreement concerning the details of the reorganized courses to be found in the practice of these teachers. In the content and in the grade placement of the materials in various junior high school books, we have clear evidence of distinctly different hypotheses for the organization of the curriculum of grades seven, eight, and nine. Some schools, for example, believe that the seventh grade should be given entirely to arithmetic. One school gives about one-half of the year to arithmetic and the remainder to intuitive geometry. Several are working out courses in general mathematics for all three grades of the junior high school. The Ethical Culture school gives most of the ninth school year to geometry. There are almost as many kinds of courses as there are schools.

Disagreement concerning grade placement is to be expected, for very little careful work has been done on this problem. Nor have we attempted to measure to what extent the alternative schemes are effective in forming desirable habits, giving useful information, and increasing mathematical power, the reason being that we do not now possess adequate tests. It is important to recognize that the issues involved in improving the teaching of mathematics cannot be wholly met by a committee report or by widespread discussion. As Professor Young has stated: "The work for which the National Committee was organized truly begins when the recommendations of its report are being translated into practice by the detailed and exacting efforts of the classroom teacher."

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[1] In this course Young and Morgan's Elementary Mathematical Analysis (Macmillan) is used. In the tenth grade the pupils study a conventional plane geometry text.

7. *Agreement Concerning Fundamental Issues.* It would, however, be wrong to infer that all is chaos in the reorganization movement in mathematics. There is agreement concerning issues that are sufficiently far reaching to reconstruct completely our high school mathematics. This important evidence is to be found in the fact that Articles 8 to 15 inclusive of Part I were submitted to each one of the schools concerned with the request that the teachers indicate their acceptance, desired modification, or rejection of these articles. All but one of the schools responded. (It was seemingly not possible to reach Professor Meriam.) With two exceptions, to be stated presently, there was unamimous endorsement and acceptance of these principles. Mr. Betz, speaking for the Rochester teachers, takes exception to Articles 11 and 13, not because of any objection in spirit but because he believes that courses are already being modified along these lines. Miss Auerbach of The Ethical Culture School thinks "twelfth school year" of Article 15 should read "tenth school year." The far-reaching effect of some of these principles will be noted in the sections that follow in this part.

Another striking illustration of agreement concerning important issues is found in an experience of the National Committee. It will be remembered that the Committee published in Secondary School Circular No. 6 (page 10) five possibilities for the arrangement of material in the way of suggesting to teachers possible courses for grades seven, eight, and nine. Each one of these plans provides that at some time or other during grades seven, eight, and nine, the pupils shall be taught arithmetic, algebra, geometry, and trigonometry. But a more startling fact is that in no one of these plans is it proposed that a whole year be given to any one of these topics, say arithmetic, algebra, or geometry. According to the plans proposed, there are but two alternatives: (1) the teaching of general mathematics throughout these years, or (2) the breaking up of each year into various units, algebra, intuitive geometry, trigonometry, and the like. Is it not remarkable that, after a widespread discussion of junior high school courses involving nearly one hundred coöperating organizations and in spite of the fact that current practice devotes all of the seventh and eighth years to arithmetic and all of the ninth year to algebra, there should not be enough sentiment in the country to support at least one plan for the junior high school that would continue conventional practice?

8. *The Norms of Performance in Arithmetic Are Exceedingly Low.*—Teachers of mathematics have in recent years had an opportunity to study the findings resulting from numerous school surveys. One outstanding conclusion to be drawn is that the results obtained in the arithmetic of grades seven and eight are very unsatisfactory. We assume that in the first six grades the child shall attain a reasonable degree of accuracy and speed in the fundamental operations with integers and fractions. The result of tests shows that this assumption is not even realized in the seventh and eighth grades. We have sufficient evidence to make us sympathetic with the complaints of the industrial and commercial world resulting from its contact with pupils who have achieved only 50% to 80% accuracy. This failure is probably not due to an insufficient amount of drill; it seems to be due rather to the fact the drill is not directed at specific disabilities. The results of available tests do not in general enable us to analyze the pupils' difficulties. For example, one may give the Courtis Test in subtraction and learn that a pupil or a class is weak in subtraction. But the tests do not tell us what steps in the learning are in need of remedial treatment.

The foregoing conditions seem (without evidence based on standard tests) to exist with reference to percentage relations. This topic is generally considered important and it receives emphasis in text books. But the degree of mastery of percentage relations seems low even among college students. The experience of the teachers in these schools suggests that a satisfactory skill in computation cannot be assumed as initial equipment even among ninth grade pupils, and that adequate drill must be provided from time to time. In short, computation must be "kept going."

9. *Delay of Social-Economic Arithmetic.* Although there is an increasing emphasis upon material that can be defended on a basis of social usage, it does not follow that a problem which involves principles that are widely used in everyday affairs is either a real or desirable

problem for children. Much emphasis is ordinarily placed on mortgages, stocks, promissory notes, cost of plastering, papering, and the like. These may at times be of interest to parents and to teachers, but it does not follow that these problems are vital to all the pupils or to any considerable portion of pupils in a class of twelve to fourteen year old boys and girls. There is a marked decrease in the amount of textual space devoted to these social-economic materials to be found in all eight of the junior high school seventh grade books. It is generally believed by teachers of these experimental courses (with the possible exception of the Trenton group) that this type of material calls for more maturity than is ordinarily possessed by seventh grade pupils. This does not mean that time should not be devoted to the arithmetic of the home and to a growing familiarity with common business forms, checks, personal accounts, and the like. But it is held that the applications of arithmetic to business may well be decreased in the early years and continued later in grades eight and nine so as to capitalize increasing maturity, social experience and mathematical power. If we are getting to be a "ninth grade nation," the proposed suggestion is not only desirable but feasible. The delay of this material to later grades is advocated because of the demand of modern secondary education that social needs be more adequately met. These materials should be taught at a time when they are of the greatest profit to the pupils, keeping in mind that they must be taught before the pupils leave school.

10. *The Appeal of Mathematics.* There is a desire on the part of the teachers of these experimental courses to construct courses in mathematics that shall make a direct appeal to the pupil as being worth while here and now. If the new courses in mathematics are to respond to the modern demands in education, they must "reveal higher activities and make them both desired and to an extent possible." This suggests that there is to be less pyramiding (the practice of including elements to be used in problems occurring in later courses), and a more direct attack on the chief objective: an increased understanding and appreciation of mathematics. It is claimed that there are schools in this country giving rigorous courses in mathematics where the question "Why study mathematics?" or "Why study this topic?" is no longer asked in the spirit of dissatisfaction. Much thought is being given in these experimental courses to this phase of the problem.

11. *Formalism of Ninth Grade Algebra.* Extreme emphasis upon the mere manipulation of symbols which does not have meaning and significance to the pupil, together with poorly selected content and material of little value even in later courses of mathematics, constitute the two pronounced indictments of the standard algebra of the ninth grade. This indictment has often been expressed, even by conservative mathematicians.

The standard ninth grade course in elementary algebra is in theory and symbolism substantially what it was in the seventeenth century. Fully 70% of the work has been shown to be mere manipulation of symbols and drill on non-essentials set up fifty years ago. Ninth grade algebra has remained static, while the quantitative needs of high school children have changed. To be sure, we no longer teach as many as seventeen cases of factoring. But the Brinton[2] study makes it reasonably certain that ordinarily schools spend from four to six weeks on special products and factoring, three weeks on simplifying fractions and something like eighteen more weeks of the school year on material equally useless. Experienced teachers and psychologists alike agree that pupils can learn to do this manipulation without associating it with meaning. Much of the manipulation of symbols in the ninth grade leads nowhere, not even in later mathematics. Professor Crathorne, who has had the opportunity each year to inventory the mathematical equipment of hundreds of Illinois freshmen, asserts that many of these students can factor with more facility than their instructors, and yet have but little understanding or appreciation of the simplest mathematical principles. It is significant that every one of the schools herein represented is concerned with the problem of simplifying the algebra of the ninth grade. Indeed the Las Vegas School went so far as to eliminate for a time all mathematics from the ninth school year in order to avoid objections to ninth grade algebra.

12. *The Exclusiveness of Ninth Grade Algebra.* Not only are the teachers of these experimental courses concerned with the formalism of ninth grade algebra but they are determined to break up its exclusiveness. We have noted that the teachers of the Las Vegas School began their experiment by teaching no mathematics at all in the ninth grade. They chose to do this rather than to teach useless ma-

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[2] See Lee Brinton's question blank study, Relative Importance of the Topics of Algebra, reported in Supplementary Educational Monographs, Vol. II, No. 1, The University of Chicago Press.

terial. But they have come to believe that it is more important to strive to develop the right kind of course for the ninth grade. The exclusiveness of algebra has delayed to later courses the teaching of material that is simple, worth while and significant to children. The most patent illustration of delayed material is found in numerical trigonometry. In the high schools of our large cities, most teachers have sufficient preparation to teach trigonometry, or at any rate some one in the department has. But even the simplest notions in this subject are seldom referred to until we reach the eleventh and more frequently the twelfth school year. This is in spite of the fact, that the introductory notions of trigonometry, covering possibly three or four weeks' work, are exceedingly simple and are often the means of awakening a genuine and abiding interest in mathematics. While the case of trigonometry is more familiar to mathematics teachers, other topics are perhaps even more important and more extensive. The graph is seldom introduced early or, if included, is not consistently taught as a vivid method for solving problems. The arithmetic involved in the elementary notions of statistics, the simple principles of mechanics, practical devices such as logarithms, the slide rule, and the like are equally neglected. So seldom are these topics taught that most high school graduates are denied the opportunity to obtain even elementary notions of these important tools. It is not necessary to argue the early introduction of these topics on the basis of practical usefulness. It is sufficient to point out that these constitute most effective ways of presenting quantitative data and relationships to children.

Presumably the aims of reorganizing the mathematics of the ninth school year are identical with those for grades seven and eight. These aims seem to be well formulated in a report by the Chicago Principals Association published in full in the September, 1921, number of the Chicago School Journal. According to this report, the aims of teaching mathematics in these early years are: "(1) to apply computational arithmetic to the most useful phases of business practice and social applications, (2) to study space forms and relationships arising in linear, surface, volume, and angular measurements, (3) to study and to represent quantitative data and relationships by means of simple graphs, (4) to express quantitative relationships in the language of the formula and the equation and to evaluate and solve the same, (5) to study positive literal numbers and the necessary forms and operations to evaluate simple formulas and to solve simple equations, and (6) to practice at times on the fundamentals of computation until a norm of proficiency is acquired."

In the discussion of Section 7, it has already been pointed out that the five plans proposed on page 10 of Secondary School Circular No. 6 nowhere provide for a year to be devoted exclusively to algebra. The suggestion is that American Secondary Schools have experimented with the exclusiveness of algebra in the ninth year and have pronounced it a failure.

13. *The Special Difficulties of Demonstrative Geometry.* Teachers of mathematics have not surrendered the opinion that plane geometry offers possibilities for practice in scientific thinking. It is still believed that this field is rich in possibilities for giving a method of attack in problem solving and for analysis and, especially, for inculcating ideals and norms of precision and of rigorous reasoning. It is also quite clear that the time and money spent on plane geometry cannot be justified on the basis of social utility. It must be defended for its contribution to a systematic technique of thinking or it must be omitted. Insofar as conclusions are valid from questionnaire studies, it must be admitted that men and women of affairs are still guided by the conviction that plane geometry offers much of value in addition to information.

One may accept the argument presented by both groups as correct and yet insist that plane geometry as taught is an intolerable failure, because it falls far short of its possibilities. In his presidential address Professor Moore[3] pointed out the unwisdom of forcing children prematurely into this type of thinking. To be sure, a certain number in a geometry class who are chronologically quite young may succeed in doing plane geometry well. But this only furnishes additional evidence for Professor Moore's contention, for these children invariably have, as far as known, high intelligence quotients, and are therefore of high mental age.

Recognition of the special difficulties of plane geometry is clearly shown in the material of Part III of this chapter. To be sure, there is a very wide difference of opinion as to how the problem is to be solved. The Ethical Culture School, which perhaps has devoted most attention to plane geometry, offers one solution. The University of Chicago

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[3] Science March 13, 1903.

High School offers one that differs markedly. But the various plans are characterized by this important agreement that pupils shall no longer be permitted, much less required, to take a plane geometry course in the tenth or later grades wholly unprepared. If geometry is to be required, we can at least give pupils with fair (or greater) ability a chance not only to pass the course, but also to get a considerable amount of satisfaction from it. In these schools, plane geometry appears to be a popular course, even with some pupils whose intelligence quotients are as low as 95-105. In every case, the pupils have been given the vocabulary, the symbolism and a preliminary organization (achieved through intuition and experiment) in the years that precede the course in plane demonstrative geometry. In some of the schools this preparatory geometry is given as a separate unit. For example, the Horace Mann School for Girls offers a plane geometry unit in the last half of the seventh school year. In the Lincoln School this geometry takes the form of a study of space forms and relationships arising in linear, surface, volume and angular measurements, and in the representation of quantative data and relationships by means of simple graphs. In the measurement work of this school, it forms a basis for building up the concept of an algebraic number. It leads at once to the formula and the equation, to the evaluation of algebraic numbers, and to the substitution of special values in the formula. In this school the geometry is continued throughout the three years that precede demonstrative geometry. Most of the schools herein described are found somewhere between these two practices. The purposes in these various forms are identical, namely, to give the pupil the usable values much earlier, and to increase greatly the probability that he can later pursue a course in plane demonstrative geometry with interest and profit.

14. *Content in Grades Seven, Eight and Nine.* There is agreement that the work of grades seven, eight and nine should include arithmetic, intuitive geometry, algebra and numerical trigonometry. Although some schools advocate the addition also of a unit of demonstrative geometry to the ninth school year, this issue is still debatable.

In the Horace Mann School for Girls will be found a provision for a half year unit of demonstrative geometry. The teachers in that school argue that the junior high school years must give all an opportunity to know what is meant by a *proof*. The Ethical Culture School gives a more extensive unit of plane geometry in the ninth year. In the work of the University High School of Chicago, there is an attempt to have a gradual transition from the intuitive to the demonstrative. At no time, however, does the student study demonstrative geometry as such.

We see from the foregoing that the tendency is to make the work of these three grades more inclusive. Still further evidence is found in the tendency to introduce earlier than is now customary certain topics and processes which are more closely related to modern needs, such as the use of logarithms and other simple tables, the use of the slide rule, simple work in arithmetic and geometric progressions, the elementary ideas concerning statistics, the technique of graphic representation as a tool in problem solving, and the like. While these topics and others were listed in a preliminary report of the National Committee (Secondary School Circular No. 5) as optional, their appearance in such a report dealing with the work of the ninth and tenth school years shows the tendency of the times. More material concerning these topics may be found in Part III of this chapter in the sections dealing with the Stuyvesant High School, The Lincoln School, the High School of Commerce, the English High School (Boston), and others.

15. *College Entrance Requirements in the Early Years of the High School.* The mistaken notion that it is necessary to formalize the teaching of mathematics throughout the years of the high school in order to make a satisfactory showing on college entrance examinations needs to be overthrown. There is sufficient evidence in the reports that follow in Part III to show conclusively that it is possible to give worth while courses in the early years of the high school with little reference to college entrance examinations. The reader will learn that the graduates of the English High School (Boston) do creditable work at the Massachusetts Institute of Technology. The graduates of the University High School of Chicago leave little to be desired in their subsequent record in mathematics. The Lincoln School (New York) devotes but three hours per week to mathematics in grades seven, eight and nine; five in grades ten and eleven; and one hour per week (review work) in the twelfth grade. No special lessons have ever been given, nor has tutoring been permitted. The graduates of this school not only are able to pass college entrance examinations, but are given advanced credit at colleges of the first rank. The Cass Technical

High School (Detroit) has an excellent record at the University of Michigan to which it sends many of its pupils to take later courses in mathematics.

It is possible that the indictment that college entrance examinations are not the stimulus toward valuable work in mathematics, which they might well be, can be substantiated. But bad as they may be, they are not the obstacles to worth while courses in mathematics that they are frequently believed to be.

Part III—The Work of the Individual Schools

This part furnishes a description of the work in each of fifteen schools. An effort was made to include sufficient detail to reflect the spirit of the work as seen by those engaged in it. The schools are listed alphabetically using the most important or identifying word of the name.

The Cass Technical High School, Detroit, Michigan

WHAT MATHEMATICS IS NEEDED BY INDUSTRY?

A school for the city at large.—Cass Technical High School is located within a few blocks of the downtown district of Detroit. It is the only technical high school in a large and important industrial center. A survey of the location of the homes of the pupils shows a remarkably even distribution over the map of the city. On this basis the school makes the claim that it belongs to the city at large.

Object and organization of the school.—The object of the school is to offer such courses as will serve the needs of Detroit as an industrial center: (1) to give courses which are broad in general training but which will allow the graduate to specialize without a distinct break in his training; (2) to offer opportunity to those whose education has been cut short to continue study while earning a living; and (3) to give to the industries a place to send their employees for part-time continuation and evening classes.

The school has been organized on the 8-4 plan, drawing its students from the grade schools and the Junior High Schools of the city. At the beginning of the second semester (February, 1920) a group of seventh and eighth grade students was formed into an Intermediate Technical School in connection with the school and under its supervision. This intermediate technical school tries to train for the industries and conforms to the high school course with credit in mathematics.

Enrolment and the use of the school plant.—The total enrolment of Cass is over 6000 day and night students. Classes run continuously from 8:00 in the morning until 10:30 at night six days in each week, twelve school months in the year.

There is much discussion of reorganizing mathematics that seems to keep in mind only the need of the academic group. Whether or not early differentiation is desirable is a debatable question. Choices are in fact being made. There are great groups in our large cities that have specific needs. The following tables tell a vivid story of Detroit's needs. There is no reason to assume that the needs of other cities are different in kind though they undoubtedly are in degree.

The student body during the year 1919-1920 was divided into six distinct groups.

1. Regular high school students. (8.30 A. M.—3.00 P. M.) five days per week	1100
2. Disabled soldiers sent for training by Federal Board for Vocational Education, five days per week	250
3. Junior Continuation students, one half day per week, sent while working under permit under the compulsory education law	650
4. Senior Continuation students, one half day per week, under pay from employers	100
5. Evening school students, from four to twelve hours per week, those working on shift coming mornings while working nights	3000
6. Intermediate technical students, seventh and eighth grades, preparatory department for technical training. (This group was formed in January, 1920, and is an experiment in the school. At present, no data are available)	150
Total	5250

The nationality of the day school students (1919-1920) based upon the birthplace of the father, is as follows:

American	47%
Canadian	10
Russian Jew	10
German	9
English	5
Miscellaneous (25)	19

The organization of the mathematics department.—The mathematics department is directed by a department head in charge of both day and evening work. He is responsible for the choice of teachers, the courses given, teachers' programs, and changes of program involving mathematics. The department is administered by committees of teachers in charge of different courses, meeting once each month with the department head. General department meetings are held four times each semester to discuss topics of general interest and to report on committee work. It has already been shown (see p. 183) that the mathematics teachers are unusually well prepared for their particular task.

A technique for determining content.—The present mathematics course has been in the process of construction during the past seven years. It is the result of a study of the needs of the other school subjects. It was started after the drawing, shop and laboratory courses were well developed. As these correlated courses are formed to meet the industrial needs of the community, the mathematics courses are fitted to the same needs.

The wood shop, machine shop, forge shop, electrical laboratory and chemistry laboratory, were surveyed so as to define the knowledge and skill to be gained in each course. The drawing and mathematics requirements for each course were arranged in parallel columns, thus timing them in each course. When the data thus gathered were combined, the mathematics requirements for each semester were listed and timed as to when they would be needed. The department felt that the combined requirements followed a definite sequence easily followed, but not adapted to any series of texts then published. It seemed to them that the applications furnished by the existing texts were too far from the conditions of the shop and laboratory to allow correlation with any of those subjects. The development of the present course was the result.[4] The mathematics course does not attempt to teach shop and laboratory mathematics completely, as most of such problems require actual shop and laboratory conditions for demonstration. The object is to furnish the mathematics necessary for such problems *at the time* when they are needed. Such illustrations are used from the correlated courses as can be easily understood by students without definite shop and laboratory experience. The ideal, for example, is to train a student so that he can make a working drawing of his idea, make a pattern, a casting, perhaps use the casting finished in the machine shop as a part of the repairs on a dynamo, and have the mathematics required for the necessary computations at each stage.

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[4] This material may be found in the Cass Technical High School Series (John Wiley & Sons): Mathematics for Shop and Drawing Students, Keal and Leonard; Mathematics for Electrical Students, Keal and Leonard; and Preparatory Mathematics for use in Technical Schools, Ray and Doub.

The teachers concluded that the natural sources for applied problems were geometrical. Hence the drawing and mathematics were first correlated. Since the most important algebraic need is the formula, the first course is given to the development of the algebraic equation and the algebraic formula with geometric illustrations wherever possible. Easy shop and laboratory problems are used to correlate and are chosen at such times as the student needs them, in the desire to convince the pupil of the usefulness of mathematics. Since the course as taught is available in a series of three textbooks, the details of this correlated course need not be given here. It may however be helpful to state in a general way the units covered by years.

UNITS COMPLETED BY YEARS

First Year.
- Formulas involving perimeters, areas, and volumes.
- Equations in one unknown through quadratics.
- Proportion with applications to shop and laboratory.
- Ratio and its applications.
- The right triangle.

Second Year
- Literal equations.
- Geometry through similarity.
- Right triangulation.
- Graphs of simultaneous equations.

Third Year.
- Solid geometry.
- Trigonometry.
- Logarithms.
- Slide Rule.

Fourth Year.
- Higher algebra.
- Mechanics with necessary mathematics.

The preceding outline attempts to show only the time of completion of the various units and does not imply a break or division of units mentioned.

Timing instruction of mathematics material.—To the mathematics teachers of Cass Technical High School must be given the credit of being the first group to attempt a study of the mathematical needs of the shop and laboratory courses of a technical high school and to initiate a readjustment of the mathematics courses characterized by an aim to so *time* the teaching of the mathematics material as to be most effective in shop instruction. The many occupations represented in a modern technical high school makes this undertaking very difficult. The seven columns in the adjoined table are intended to illustrate the instructional needs of only one group of students as determined by the coöperation of the shop, drawing laboratory and mathematics teachers. The reader will need to keep in mind that the topics included from 1 to 7 run parallel and are taught simultaneously at such times as the teachers believe to be most effective.

"Standard" requirements similar to the preceding have been developed for a considerable number of the various occupational groups taught and are published in manual form by the printing department of the school.

There is an obvious economy of time inasmuch as the mathematics teacher can use problems that are real problems without taking the time to teach the situation out of which the problem grows.

The sequence of industrial mathematics.—The order of topics throughout the mathematics course is based upon the development of the algebraic formula. The department takes the position that the introduction of geometrical material is justified only as it applies to the formula or the equation. All constructions, with the exception of the drawing of circles, are done with a ruler and protractor. The formulas for perimeter, area and volume furnish abundant material for equations and are enriched by the measurement of models and drawings with calculations of area, perimeter, volume and weight. Equations involving angles offer opportunity for drawing and checking with a protractor. All these require little development and form natural material for use. They are based upon definitions rather than proof and do not raise the question as to their truth. Care is taken to see that problems demand constant use of common and decimal fractions and that problems from correlated subjects are in use in these classes at about the same time. The first half of the ninth grade aims to carry the pupil far enough so that he will be able to manipulate the equation well sufficiently to substitute in the formula for any letters and solve the resulting equation for the remaining letter. In the latter half of that year he learns to solve the formula for any letter before evaluating.

The mathematics of the tenth grade is devoted to the demonstration of the truth of geometric formulas, the development and proof of new formulas including proportionality of lines, functions of angles and formulas of trigonometry. There is here an attempt to correlate deductive geometry with trigonometry and algebra. In particular, the course includes more trigonometry than the conventional courses.

There are no geometrical developments separate from algebra until the use of the formula begins to demand a formal proof. This comes after the manipulation of the equation has been covered and the formula developed to the place where unfamiliar formulas need to be introduced. The geometric formulas used up to the last half of the tenth school year have been within the pupils' experience or else are treated as clearly intuitive. At that point when more complicated concepts arise with constructions whose development demands a formal proof, deductive geometry is introduced.

The ninth and tenth grades.—The resulting courses are not far different in outline of principles from traditional courses. The difference lies in the choice of applications from the field of geometry and correlated subjects so that no real break in the sequence comes when the formal geometric proof is introduced. The recognition of that sequence allows the development of the algebra to proceed throughout all courses instead of endeavoring to finish algebra as a unit before geometry is developed. For example, the teachers of this department would insist that the natural place or development of radicals, radical equations, complicated formulas and other stumbling blocks in algebra is when the problems of geometry, trigonometry, and higher courses demand their development. In general, it is the position of this group of mathematics teachers that the change demanded by the needs of the industries is not that the geometry be crowded between steps of the algebra but that all courses be recognized as part of a definite sequence for which the algebraic equation furnishes "backbone."

The required courses.—Seven semesters' credit in mathematics is required for graduation in all mechanical and scientific courses. Students in special courses in commercial art, nurses' courses, etc., are required to take two years or more of mathematics.

A good record in the university.—Graduates from technical courses are admitted to the engineering department of the University of Michigan without examination. Most of the graduates who go into university or college work enter that university or the Michigan Agricultural College, and no data are available as to those who enter other colleges. About one-half of the graduates have taken college work. The last time Cass graduates in the engineering department at Ann Arbor were checked as to their standing in college mathematics (about fifty of them) the mathematics record of the group was entirely free from failures, with the exception of one student who had a condition in a course, which later was removed. Their graduates have maintained a record there which is higher than that of most of the other schools sending students to that University.

Some special courses.—The preceding curriculum refers to the day high school department and does not apply to the special courses developed in connection with the other departments. The effort in these other departments is to specialize in the applications of mathematics peculiar to those situations and to define interchangeable units in the mathematics which can be transferred as credit from one department to another. The evening mathematics classes numbering 1400 students (1919-1920) were divided into the following groups:

1. *Preparatory Mathematics.* A review of common and decimal fractions. This course is required of all students wishing to enter any shop or laboratory course. Credit may be secured by examination.

2. *Mathematics for Machine Shop Students.* To be taken one evening per week by men who enroll in machine shop. Required before entering advanced shop courses. This course corresponds to the first year and a half of the day school with applications taken from the shop.

3. *Mathematics for Machine Shop students. Advanced.* Including trigonometry. This course can be used for regular credit in high school units if the student does additional work in theoretical trigonometry.

4. *Mathematics for Electrical Students.* Mathematics to correspond to that for shop students except that applications and formulas are taken from electrical courses. This course is followed by a course in trigonometry along the same plan.

The shop and laboratory instructors are giving the fullest coöperation in living up to the requirements and in furnishing the applications needed for the work. The drawing classes appear to need practically the same applications as those used in the shops and require the same credits for enrollment. The shop courses in automobile work, electrical construction, etc., which do not demand the amount of mathematical skill as do the other shops and laboratories, use the course in Mathematics 1 as a classification test in the formation of classes. They have found that men who cannot master arithmetic with the simple shop applications are poor material for anything except mechanical production work requiring very little mental ability.

The whole mathematics course for the evening school is based upon the same units and upon the same mathematical skills as defined by the course for the day school.

The budget system.—In evening classes the student is checked by subjects and receives credit for small units of work. The system is called the budget system and is being tried with success in a part of the day classes. The skills are measured by simple tests given at stated intervals in the text. As the evening and continuation classes cannot be timed by semesters the completion of a course is not marked by the time taken but by the progress made in the course. Progress cards are used and are checked by subjects completed. As each test is completed, the corresponding item is credited upon the progress card. The work of the instructor is largely individual and the student can progress as fast as his ability will permit. The items included in each group are covered in about two weeks' work and the student is then checked by a short test taking not over ten minutes. This allows the instructor to be very definite as to work to be made up by students and permits the student to make up back work in small units. It divides the work of each semester into about ten parts and if these are all credited to the student, he is excused from final examination. The examinations are covered in from three to five questions.

Supervised study.—The courses at Cass Technical High School include so many shop and laboratory courses that mathematics is restricted to one forty-five minute period per day. Of this time it is recommended that the instructors use a definite part for development and a part for supervised study. There are four periods devoted to special classes so that students in need of special help can be assisted during their vacant hours. Students enrolling after the semester's work has begun, or transferring from other schools are placed for adjustment in a double period special class until they are ready to be placed in a regular class.

Home work. Method of the recitation.—For work of the highest grade, pupils are required to devote to home study an amount of time equal to the length of the recitation. This varies with the course, but during the first year instructors are advised to plan the home work so as to require a maximum of about that amount of home study. A typical recitation divides into three equal parts, the first for checking work, assigning new work, and developing new principles, the second for drill work, and the third for supervised study. Whenever possible, laboratory work in the use of the protractor, measuring solids, investigating certain machines illustrating equations and formulas, is undertaken by the class or by committees appointed by them. The work in the shops and laboratories furnishes abundant material for such problems. A class, for instance, may measure the diameters of cone pulleys in a pattern shop and calculate the different speeds of the machines with the cutting speeds of work of a certain diameter. Patterns are used involving well known solids and these are measured and their areas and volumes calculated together with the weights of castings to be made from them.

The University of Chicago High School, Chicago, Illinois

CORRELATED MATHEMATICS

The aim of the school.—The University High School is a part of the School of Education of the University of Chicago. It aims to perform a double function. For its pupils it seeks to provide the best possible opportunities for education in the secondary period. For the School of Education it serves as a laboratory for the solution of educational problems. There is no conflict between these aims. A school whose officers and teachers are students of the educational problems connected with their work is likely to furnish the most favorable conditions for the development and training of its pupils.

The development of a course in correlated mathematics.—It is nineteen years ago that Professor E. H. Moore delivered his presidential address[5] before the American Mathematical Society which has so profoundly stimulated progressive thinking on the teaching of high school mathematics. In the following school year (1903-1904) a tentative program for mathematics was formulated in conferences of the teachers of mathematics of the College and of the University High School. Professor G. W. Myers of the School of Education then acted as chairman of the mathematics department. Under his direction assignments of various chapters were given to the members of the high school department of mathematics. These studies were published by the University Press (Chicago) in two volumes, called "Myers First Year Mathematics" and "Myers Second Year Mathematics." Though these books were never used widely, much credit is due to Professor George Myers for this pioneer effort; in particular to Professor Myers must be given the credit for attempting a thorough breaking-up of the "water tight compartment" method of teaching algebra, arithmetic, geometry and trigonometry.

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[5] Science, March 13, 1903.

The second stage of the experiment was the trying out of these courses in the classroom and the recasting of the material into greater uniformity. This part of the work was directed by Mr. Ernest R. Breslich who became head of the high school department of mathematics in 1912. The result of this work was published in two volumes.[6] This was followed by a third year course, which completes the traditional high school course under the form of correlated mathematics.

The department of mathematics then took up the problem of organizing the "Junior College" work. The first course has been published. This will in time be followed by a second year course. Later we shall discuss the junior high school courses that are now being developed.

Aims and values.—Among the purposes of secondary mathematics the following are emphasized:

1. A knowledge of the fundamental facts and principles of mathematics, needed by students because of the correlation of these facts and principles with other studies.

2. Development of spatial and pictorial imagination; i. e., the ability to visualize objects, relations, and conditions. This training, which cannot be secured from other high school subjects, is a real need in life, especially for those who cannot go beyond the high school.

3. Development of functional intuition, i. e., of the appreciation of the dependence of one magnitude upon another. The pupil learns to attack new problems with the fertile question: "What relations are here involved?" rather than the sterile: "Where is the unknown here?" The existence of this dependence falls within the experience of every pupil.

4. The disciplinary value: e. g., training in accurate thought and power of concentration; the acquisition of orderly habits of precise oral and neatly written expression and, not at all the least, of honesty, i. e., of saying exactly what one means and meaning exactly what one says; training in setting out to

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[6] Breslich's First Year Mathematics; Breslich's Second Year Mathematics; Breslich's Third Year Mathematics; Breslich's Correlated Mathematics for Junior College, (University of Chicago Press.)

do a specific thing: i. e., training in the habit of "making good." The disciplinary value of mathematics always has been considered one of its principal values.

5. Improving the study-habits of the student. Since supervised study is easily introduced in mathematics, this subject is especially valuable as a means of teaching students how to study.

Four years of correlated mathematics.—It is seen from the preceding that the course covering four years has been developed in the form of correlated mathematics. The details need not be given here for the courses as taught are available in text book form.

The second year course.—The following extended statement concerning the methods of the second year, prepared by Mr. Breslich, is included for the reasons that (1) the issue of the feasibility and desirability of correlating deductive geometry is debatable; (2) there is little verifiable information concerning this issue.

> "In the first-year course the student has gained a thorough understanding of the fundamental notions of geometry. Hence in the second year somewhat formal methods are introduced from the start. Even before this, the advantage of the reasoning process over the process of measuring has been recognized. Mathematical fallacies and optical deceptions are now used to make the need of a logical proof still more apparent.
>
> "To develop in the pupil a sort of geometrical strategy, i. e., to attack, to take possession, and to exploit a geometrical problem, methods of proof are discussed and emphasized, not once for all, but throughout the course. To cultivate versatility and system, students are taught to choose between various methods of proof and to follow some definite plan rather than to trust the chance of stumbling upon a proof. Many model proofs are therefore given, while in other proofs, statements or reasons that should be apparent to the student are omitted with the purpose of making him independent and of developing his powers of argumentation.
>
> "The old method of dividing the subject of geometry into a few books has been abandoned as being of only traditional value.. The course is divided into a number of short chapters, each dealing with one or a few central topics. This arrangement is far better adapted to study by high-school students, since the aims and purposes of the several chapters are easily seen. It is more economical of the student's time and energy than the old method.
>
> "Since the usefulness of a study is what always appeals most strongly to a beginner, this phase is emphasized throughout the course. The importance and the significance of geometrical facts in the affairs of everyday life are impressed upon the pupil.
>
> "By the employment of algebraic notation and by the continued application of the equation to geometrical matters, a firm hold is kept on algebra. New algebraic topics are developed when opportunity and need arise. Thus, elimination by comparison and by substitution, so frequently needed

in proofs and in the solution of exercises, is taught very early. The solution of the quadratic equation by means of the formula, the operations with fractions, and factoring, are all reviews or further extensions of topics whose study was begun in the first year.

"A number of geometric proofs have been simplified by the use of trigonometry and algebra."

Fourth year mathematics.— The outline of the fourth year course is of special interest; for, few schools attempting a reorganization have extended the new work to the last year of the high school.

The course presupposes the essentials of high school algebra, of plane and solid geometry, and of trigonometry. It aims to combine the work which is commonly covered in separate courses in college algebra and in analytic geometry. The course is therefore suitable either for the first year of the junior college or the fourth year of the high school.

This combination tries to motivate each topic, to show the student more clearly the meaning of the subject by means of geometric representations, and to develop in a natural way the important concept of functional correspondence. There is an effort here to give unity to a course in which the subject is usually presented as a number of isolated topics.

The idea of the derivative is presented early and is used in discussing the slope of the tangent to a curve, and maximum and minimum values of a function.

Among the other topics treated, are location of a point, number system, linear functions, determinants, algebraic solution of the general cubic, limits, infinite series, combinations and permutations, conic sections, the general equation of the second degree, diameters and loci.

A junior high school course.—At present a Junior High School course is being worked out by the department. The materials are tested by means of analytic tests, in the effort to retain only topics which the pupil can fully master. This has been carried on for three years. As a result of this testing some of this material was moved to the senior courses, some material was moved down from the senior courses, and in many cases the treatment of a topic was changed with a purpose of adapting it to the pupil's ability to understand and to attain mastery.

It is the desire of those responsible for the course that when completed it will be characterized by the following:

(1) The establishing of mathematical principles inductively by experiment.

(2) A wide initial experience with each new principle.

(3) Sufficient time exposure to make a lasting impression.

(4) No topic retained only because tradition has placed it in the curriculum.

(5) The teaching of much material commonly taught in the traditional senior high school course.

(6) Elimination of much of the commercial arithmetic ordinarily found in the seventh and eighth grades.

(7) Emphasis on experimental geometry bringing in arithmetic and algebra only where they are believed to be helpful.

(8) The use of measurement to derive geometric facts.

(9) Emphasis on the formula.

(10) No formal demonstrative geometry but much reasoning of the informal type.

(11) Complete mastery of the equations of the form, $ax = b$, $ax + b = c$, $x^2 + a = b$ and others which reduce to these forms.

Units of Instruction in Seventh-Grade Mathematics[7]

1. The Geometric Conception of Lines.

a. Measurement of line-segments. Line-segments are found in boundaries in maps, scale drawings and graphs, polygons, surfaces. The pupil learns to measure segments found in objects in the class room. He learns to use compass, ruler and squared paper as instruments of measuring. He studies the metric system. He appreciates the fact that measurement is at best only approximate and that the degree of approximation is expressed by the number of significant figures.

b. Drawing of line-segments. Facts and relations between facts are represented in various forms: by means of tables, by graphs, and by means of formulas. Many scale drawings and graphs are made. There is an effort to get appreciation of geometric forms.

c. Geometric concepts. The intersections of lines are points. The names of polygons are learned.

d. Algebraic concepts. Points are denoted by capital letters, lines by two capital letters. Literal numbers are used to denote the number of units in line-segments which have not been measured. The symbols $=$, $>$, $<$ are used. The study of perimeters leads to simple algebraic polynomials and equations.

e. Arithmetic operations. In measuring the pupil meets and uses the operations with common and decimal fractions.

2. The Properties of Angles.

1) Angles are seen in the classroom, in models, and in drawings. They occur in problems in surveying and navigation. An angle is formed by the rotation of a line.

2) Angles are measured with the protractor. The degree is used as the unit of angle. Angles are classified.

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[7] The first two units are outlined in detail to indicate the character of the type of work and of the methods used. For a more detailed discussion of the other units see School Review, May, 1920, and Jan., 1921.

3) Angles of given sizes are drawn with the protractor. The protractor is used to draw parallel lines, perpendicular lines, bisectors, triangles from given parts. This leads to some discussion of congruent figures and the establishment of basic theorems.

b. The properties of angles studied by indirect measurement. Various methods are used to determine unknown distances.

1) The congruent triangle method.
2) The scale drawing method.
3) The similar triangle method.
4) The tangent ratio method.

c. Measuring angles leads to many functional relations, such as $x + y = 90$, $ax + bx + cx = 180$. Many arithmetic applications are given.

d. *Approximate measurement.* Here, as in the case of line arguments the pupils appreciate the fact that measurement at best is only approximate.

3. THE PROPERTIES OF THE CIRCLE.

a. Uses of the circle, meter, longitude, time, circular graph.

b. Use of the compasses in constructions and designs.

4. EQUATIONS AND FORMULAS.

The aim is to master equations of types which occur in the previous units, the formula being an assimilative application.
Circumference, uniform motion, percentage.

5. AREAS.

a. Rectangle, square, parallelogram, trapezoid, triangle, circle. The theorem of Pythagoras.

b. Equations of the second degree. Square root.

c. Evaluation of algebraic expressions.

d. Operations with positive literal numbers.

6. THE USE OF THE EQUATIONS IN THE INTERPRETATION OF MATHEMATICAL PROBLEMS. FORMULAS.

Units of Instruction in Eighth-Grade Mathematics

1. PROPERTIES OF THE COMMON SOLIDS.

a. The cube. Algebraic functions of the second and third degree. Evaluation. Arithmetic computations.

b. Rectangular solid.

c. Prism, cylinder, pyramid, cone, sphere.

2. Directed Numbers.
   - *a.* Uses of directed numbers.
   - *b.* Operations with positive and negative numbers.
   - *c.* Problems involving positive and negative numbers.
   - *d.* Equations in one unknown involving positive and negative numbers.

3. Equations and Applications.
   - *a.* Equations of the first degree in one unknown.
   - *b.* Equations of the first degree in two unknowns.
   - *c.* Quadratic equations.

The work outlined above is covered in the first semester of the eighth grade. In the second semester contact is made with senior high school mathematics, Book II, for those who continue to study mathematics. In terms of traditional courses these pupils need a half year's work in demonstrative geometry and a half year's work in algebra to complete the 2½ year requirements of the colleges.

Special course: preparation for college-entrance examinations.—A course of two periods a week for the second semester of the last high school year is required of all candidates for recommendation for college entrance examinations. Class periods are conducted entirely as supervised study periods, no home work being required. The work of the first two and one-half years is summarized and reviewed.

Time required of students.—Five one-hour class periods (55 minutes in the clear) a week are given to mathematics. (The college preparatory course is an exception.) The major part of the period is used for supervised study where special attention is given to the individual differences existing among students. An average of about fifteen minutes a day is expected of home work. This work is generally of the same type as the work done in the class room, all advanced work being developed in the school.

✓ **Results.**—The question will be asked whether the results attained with a course in correlated mathematics are as good as, or superior to, those attained by pupils trained in traditional courses. From the point of view of the teachers engaged in this experiment, the following gains are made:

1. A saving of time. The work usually done in three and one-half years in separate courses is being done in three years, without noticeable loss to the pupil.

2. A large registration in mathematics. In the University High School 70-75% of all pupils are taking regularly courses in mathematics. In view of the fact that only one year is required for graduation this must be considered a high election.

3. Failures are reduced. The average number of failures in the course in mathematics for the last 5 years is less than 10%.

4. Successful college work. University High School pupils who have completed four years of mathematics take calculus as their first course in the University of Chicago and are able to carry the work successfully.

Reports from other colleges are also very favorable, thus, for the graduates of 1919 the following average grades were received for the work in 1919-20:

Pennsylvania	B	Illinois	B, B—
Oberlin	B—	Smith	B—, C
Dartmouth	C	Alabama	A
Wisconsin	B—, B	Vassar	A, B, C, C

5. The teaching of material relatively simple in the introductory courses, which in the traditional procedure are delayed so as to be taught in the later years of the high school when but few students are afforded an opportunity to master them. An examination of the materials used in this school show it to be characterized by shifting of elements rather than the creation or introduction of new materials.

6. Greater power in problem solving through the use of the notion that geometry constitutes concrete material in which the situation of a problem is graphic and relatively easy. It may be noted that the early courses are so constructed as to contain a considerable amount of intuitive geometry. The desirability of this feature has long been advocated by competent leaders.[5]

The High School of Commerce, New York City

What Mathematics is Demanded By Commerce?

History of the school.—The High School of Commerce is a part of the New York City school system. It was founded in 1903 by the Board of Education to provide vocational education along commercial lines to pupils of high school age, more thorough than that given in the so-called "commercial colleges." The purpose was to provide a high school training which should give the students less "cultural material" as that expression was then understood, than was given in the conventional high school, and to give instead the technique of business—stenography, bookkeeping, typewriting, etc.,—and considerable outlook in the commercial world through the medium of economics.

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[5] Incidentally it is of special interest that Professor Paul H. Hanus as early as 1893 advocated systematic instruction in intuitive geometry remarkably similar in content and method to that being introduced in progressive schools today. He urged the use of measurement as a basis for the course. Concerning method, his statement that "the teacher and the pupil must work together in discovering the facts and their relations" is of significance in the present tendency to develop a laboratory type of recitation. Progress appears to be achieved very slowly. The writings of Fehr, Nunn, D. E. Smith, and J. W. A. Young, produced many years ago, include numerous suggestions for improvement in mathematics recommended in the reports of the National Committee on Mathematical Requirements and those found in the practice of progressive schools. In particular Moore's Presidential Address (1903) and Brown's Curricula in Mathematics (U. S. Bureau of Education Bulletin, 1914, No. 45) indicate that considerable progress remains to be achieved.

This object has been broadened with the progress of the years, till now it is essentially a *vocational school,* rather than prevocational, and its courses in business management, advertising, transportation, business statistics, marketing, business mathematics, foreign trade, and the like, fairly entitle it to its name—High School of Commerce. It studies many phases of commerce, domestic as well as foreign. The school serves the Metropolis with its varied interests and peoples. It has grown to a school of 3200 pupils, with 135 teachers. The school curriculum covers the ninth to the twelfth school year, although students are admitted into the sophomore class from junior high schools. The great majority of the students are Jewish. Being a part of the public school system, the tuition is free.

The mathematics teachers.—Ten teachers besides the Chairman of the Department are now (1921) giving full time to teaching in the department. (The work of the department includes "Mercantile Accounts I," a combination of Arithmetic and Bookkeeping.) Six other teachers give part time to the department. All these teachers are college graduates, and have specialized in mathematics. Five of them have also studied accounting, while all but one have had courses in bookkeeping. The average amount of graduate work in mathematics is over one year. All of the teachers have had special training in education courses, varying from one to five years' work.

Time devoted to mathematics.—The recitation periods are 42 minutes in length, with 3 additional minutes for passing between class rooms. Each course in mathematics, as for example "Mathematics 3" consists of a half year's work, 20 weeks of 5 recitation periods a week. Two courses constitute a year's work, and together make up a "unit" for graduation credit. A unit is thus 200 periods of 42 minutes each.

Equipment.—The equipment of the department includes a large 2-meter model slide rule, 60 10-inch Mannheim slide rules, a Monroe calculator, corporation reports, business barometer services (Babson's and Brookmire's) as well as such conventional equipment as a typewriter and filing cabinet.

The curriculum.—The curriculum in mathematics as now constituted, gives the following in the eight terms of a four year course:

First Term—Mercantile Accounts with Arithmetic.
Second Term—None given.
Third Term—Commercial Mathematics.
Fourth Term—Commercial Mathematics.
Fifth Term—Commercial Algebra.
Sixth Term—Mathematics of Finance and Investments.
Seventh Term—Principles of Statistics.
Eighth Term—Business Statistics and Business Barometers.

The work of the first term is required while that of the terms from three to eight is entirely elective.

The first term's work.—This is a combination of bookkeeping and arithmetic drill work. The work in arithmetic is introduced only as the work in accounts calls for calculations which the boys show some inability to perform readily. Thus, when "Bills and Inventories" is the topic, the need for ability to multiply rapidly and accurately is soon shown to the student, and a drill in aliquot part multiplying and distributive law multiplying is taken up. It is believed that when the boy has realized the need for such drill, his attitude toward it is very different from that found in the ordinary business arithmetic class. Only 7 aliquot parts are taught, 1/2 to 1/16 and combinations of these.

When treating "Trading and Profit and Loss Statement," trade discounts in connection with finding the cost of goods sold is taken up as a drill topic. The theory of such discounts is given in a more concrete setting because it comes into the discussion of accounts such as are kept in the retail stores with which the boys are familiar.

The subject of "Negotiable Paper" is introduced in the work in accounts because the proprietor needs to convert notes into cash by discounting. The entries appropriate to checks, notes, and acceptances are part of the course, and the whole subject of negotiable paper is thoroughly treated with discussions of the business situations, the technique, use of sample documents, etc. Naturally this topic calls for bank discount and the methods of reckoning, such as the "bankers' 60 day rule," etc.

The work of this course on the bookkeeping or accounting side is shown by the following list of topics: Bills and Inventories; Listing Purchases and Sales in Purchase and Sales Journals; Trading and Profit and Loss Statements; Negotiable Paper; Listing Cash and Notes in Journal and Ledger; Personal accounts, Customer and Creditor; Preparation of the Balance Sheet and Profit and Loss Statement from the Trial Balance; Interest and Bank Discount.

It is easy to see that the aim of the course is to give in skeleton form the major problems met by a business man in keeping accounts. These are treated simply, with a minimum of detail, and lead naturally to the clarifying and extending of the student's knowledge of the arithmetic involved.

The elective work.—The aim of the elective courses in mathematics, offered in the second, third, and fourth years, is to give the student possession of the following:

(1) A knowledge of algebraic symbols and generalizations, in so far as they help to give a greater power in arithmetic and business calculations.

(2) The subjects leading to a thorough mastery of simple and compound interest, equation of payments, the theory of discounts, annuities, amortization, sinking funds, depreciations reckoning by the five standard methods and bond valuation calculations. These subjects involve for their thorough mastery, logarithms, series, exponents, and equations.

(3) Possession of those concepts and methods of calculating involved in the processes of collecting, tabulating, graphing, and interpreting facts of advertising, marketing, and adjusting the internal relations of a business. Estimating business conditions and the business environment involve the ability to interpret the facts and figures of commercial reports, economic reports, trade journals, and financial reviews.

The basis for selecting subject matter.—The basis on which subject matter was selected is thus briefly indicated. In particular, it was attempted to find out by personal interviews with bankers, brokers, retailers, wholesalers, managers, advertising managers, sales managers, purchasing agents, foremen, and the like:

(1) Just what figuring, calculating, and planning they have to do that involves mathematics.

(3) What calculations are involved in routine positions, and what in managerial and administrative positions.

The course aims to retain such mathematical material (largely arithmetic and algebraic) as may best be used in making such calculations and estimates, and such as will give an understanding of these calculations. Since the basis for selecting subject matter is so in accord with modern methods of curriculum construction it is to be regretted that no record of the interviews previously referred to is available.

Mathematics III.[9]—In Mathematics III, algebraic symbols, formulas, equations, the fundamental processes of algebra, graphs, and fractions are taught.

The applications of these subjects in developing short methods of calculating and deriving arithmetic processes are stressed. For example:

$$\begin{array}{rr} \text{adding} & 4ax \\ & 3ax \\ & 7ax \\ & 5ax \\ \hline & 19ax \end{array}$$

leads to the short method used by bankers in finding the month's allowance of interest on daily balances by adding the daily balances, and taking one day's interest on the total, instead of extending each day's interest and totaling, for

$$\begin{array}{rr} \text{adding} & 45{,}000 \times .02 \times \frac{1}{365} \\ & 87{,}000 \times .02 \times \frac{1}{365} \\ & 90{,}000 \times .02 \times \frac{1}{365} \\ \hline \text{gives} & 222{,}000 \times .02 \times \frac{1}{365} \end{array}$$

as indicated in the above algebraic addition.

Again, the equation $i = prt$ is solved for p, r, and t, thus giving the converse cases of interest all derived from one generalized expression. Substituting in it, for $r = .04$, $t = \frac{1}{360}$, we have

$$i = p \times \frac{4}{100} \times \frac{1}{360} = \frac{p}{9000}$$

The student is taught to interpret this as meaning:

(1) For one day's interest at 4%, point off 3 places and divide by 9.

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[9] The reader may find the material used in the elective courses of the second and third years in Commercial Algebra, Book I and II, by Wentworth-Smith-Schlauch (Ginn & Co.).

(2) For 9 days' interest at 4%, point off 3 places.
(3) For 90 days' interest at 4%, point off 2 places.

Useful bankers' discount and interest rules are thus derived from the generalization $i = prt$.

Examples might be multiplied indefinitely, but these show the spirit of the work.

Mathematics IV.—In this course the following topics are covered: Simple equations applied to business problems, mixing and diluting, ratio and proportion, graphs, trade discount and net cost rate factors, selling price cofactors, generalization of interest, simultaneous equations and their application to ratio of solvency, dietaries, commercial interpolation graphs, and interest graphs.

The following illustrates this work:

After the pupils have been taught the idea of estimating profit as well as overhead as a percentage of the selling price and after concrete problems have been solved, a formula is derived algebraically, as follows:

Let x = the selling price
c = the cost

and $\frac{sx}{100}$ = overhead and profit

$$\text{Then } x = c + \frac{sx}{100}$$

$$\text{or } x = \frac{c}{1 - \frac{s}{100}} = c\frac{1}{1 - \frac{s}{100}}$$

The expression $\frac{1}{1 - \frac{s}{100}}$ gives the factor by which we must multiply the cost to get the selling price. Thus, if goods cost \$100, and overhead charges are 15% of the selling price, and profit is 10% of the selling price,

$$s = 15 + 10 = 25$$

The cofactor

$$\frac{1}{1 - \frac{s}{100}} = \frac{1}{1 - .25} = \frac{1}{.75} = 1.3333$$

Then x, the selling price $= 100 \times 1.3333$ or \$133.33.

Mathematics V.—This course is a preparation for investments and management mathematics given in the sixth term. The topics covered are: Powers and roots, exponents, logarithms, the slide rule, calculating machines, series, compound interest, discount and present value, equation of payments, annuities.

To illustrate the type of mathematics take the following: Find the compound amount on $800 for 3 years at 4%.

Solution: Since $(1 + i)^3 = 1 + 3i + 3i^2 + 1^3$,

$$(1.04)^3 = 1 + .12 + .0048 + .000064 = 1.124864$$

$$A = P\,(1 + i)^n$$

$$A = 800 \times 1.124864 = \$899.89.$$

Mathematics VI.—This course is given the second half of the junior year. The topics are: Amortization, sinking funds, depreciation, bonds, and bond valuation, insurance calculations, alignment charts, investment yield.

Mathematics VII.—Statistics. This course gives the general principles and methods used in handling statistics: Modes, medians, averages, correlation, tabulation, graphing, and approximation are briefly considered. Experiments and reports are involved. The school statistics are collected, analyzed, graphed, and bulletined, (20 weeks, 100 hours).

Mathematics VIII.—This course applies the knowledge gained in Mathematics VII to mastering the theory and reliability of such business barometer services as Babson's, Brookmire's, Dun's, Bradstreet's, The Times Annalist, as well as to the problems of internal management and control of a business, such as sales and performance records and standards, advertising test, discovering causes of failure, and the like. This course is also concerned with stock exchange and marketing statistics, trade reports and journals, and like sources of business information.

The aim of the course throughout is vocational. It is desired that only topics lending themselves as tools for business calculations be retained.

Arithmetic—covering the fundamentals, interest, percentage, commercial paper—is given twice a week in the eighth term to enable the graduating class to recover a high degree of speed and accuracy in these routine calculations.

Real problems.—The work of the High School of Commerce of New York City reveals a special effort to collect "real" problems. High school teachers of mathematics have always been handicapped by a lack of sufficient problems that may be used for drill in the application of algebraic principles. The difficulty is not the scarcity of such problems but the fact that many real problems require a background of social experience unfamiliar to the student. Frequently it requires more teaching to give this social or economic experience than the teaching of the algebraic principles. There can be no doubt that for the students which the High School of Commerce instructs, the mathematics department furnishes an unusually large amount of real problem material. This school, in proximity to the country's greatest business district, can be of further help not only to all schools of its type but also to schools including only the academic courses. Undoubtedly there is a minimum amount of such commercial experience and information that may be expected of all high school graduates. What is this minimum information of social-economic arithmetic which all citizens should possess? It is precisely on this question that the High School of Commerce with its well trained staff and its great laboratory of business practice can be of service.

The English High School, Boston, Mass.

An Early Effort to Construct a Course With Little Reference To Later Courses

The English High School was established by the City of Boston in 1821. It holds an important place in the history of American secondary education as an early exponent of a liberalized education for boys. It is the oldest public high school in America. At present it offers commercial, college preparatory, and general academic courses. The school serves a cosmopolitan community enrolling (during the year 1919-1920) 2200 boys. The school is organized on the basis of both the 8-4 and 6-3-3 plans.

Time devoted to mathematics.—There are devoted to mathematics 245 minutes per week, divided into three single and two double periods. A part of the double periods may be devoted to supervised study.

All mathematics courses are elective, though college and technology students are advised to take four years of mathematics.

Preparation of the teachers.—In the year 1919-1920 there were 7 full time teachers and 5 part time. Of these, 6 had training beyond the M. A. degree.

The courses.—The outline of the courses in mathematics, including a time schedule, is given below:

1. MATHEMATICS 1.[10] (First high school year)	Periods (Approx.)
Simple equations derived from easy problems	40
Computation	20
Formulas	5
Fractional Equations (Problems)	15
Addition, subtraction, multiplication, division, and factoring	15
Quadratic Equations	30
Pair of linear equations. Graphs	10
Linear quadratic pair	5
MATHEMATICS 2. (Second year)	
Intuitive geometry	17
Syllabus of plane geometry arranged under the following topics: Parallels and parallelograms; area of rectangle, parallelogram, triangle, trapezoid; geometric ratios; Pythagorean theorem; oblique triangles; symmetry; similar polygons; remarkable points in a triangle; inequalities; angles of polygons; secants, chords, and tangents; proportionate section; regular polygon and circle. About 80 theorems in all. Ratio; proportion treated as a fractional equation in connection with the regular polygon; literal equations; simple surds treated in connection with the regular polygon.	153
MATHEMATICS 3.	
Solid geometry theorems from Harvard Syllabus	110
Review of plane and solid geometry	60
Some incidental algebra	

MATHEMATICS 4.

Formulas for solution of plane triangles derived from geometric figures; solution of plane triangles with natural functions, using the methods of approximate computation.

Theory of exponents and logarithms.

Solution of plane triangles by logarithms.

Trigonometric functions of any angle, radian measure.

Fundamental relations between the functions.

Addition theorem generalized by fitting a model demonstration to a figure formed from angles of any size, with corresponding points lettered as in the model demonstration.

Trigonometric identities.

Trigonometric equations, general values, inverse notation.

Graphical representation of functions, radian measure for angles near 0 or 90 degrees.

Algebra. Review and advanced work to prepare for college entrance examinations.

Development of geometry by years.—The following statements indicate the development of geometry through the four years work:

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[10] See *First Year Mathematics,* by Evans and Marsh, The Macmillan Co.

1. In the first year (grade nine) geometric facts are used as material for algebraic and arithmetic problems (rectangle, rhombus, triangle, stripe, etc.) and enough measurements are made with ruler and protractor to bring home the facts used.

2. The second year begins with intuitive geometry, constructions, and the development of assumptions. The idea of demonstration is gradually developed by use of such assumptions and theorems as seem best to serve this purpose.

3. In the third year the course in solid geometry is followed by a final review of plane geometry. Here for the first time some consideration is given to college entrance requirements.

4. In the fourth year all formulas used in solving plane triangles are developed from geometric figures.

Under the 6-3-3 plan geometric construction and measurement are begun in the seventh grade.

Development of algebra by years.—The following outline indicates the development of algebra through the years:

1. In the first year the equation applied to the solution of problems, to the abbreviation of rules, and to the statement and proof of theorems; so much of the fundamental processes and of factoring as is useful in transforming these equations. The minus sign as used in equations for a sign of operation, and to indiccate a "shortage" and the "opposite sense".

2. In the second year algebra is applied to the proof of theorems and the solution of geometric exercises. The new work includes fractional equations with binomial denominators (informal), literal equations, and such surds as are used in connection with triangles and regular polygons.

3. In the third year algebra is used rather than taught. Very little new work is undertaken.

4. In the fourth year algebra is reviewed. Such advanced topics as are required for college entrance,including some taught solely because "required."

Under the 6-3-3 plan, algebraic symbols and very simple equations and formulas are used as early as the seventh grade.

Results.—The success of these courses may be inferred to some extent from (1) the percent of the student body electing mathematics and (2) the degree of success in college classes in mathematics.

The following table indicates degree of election in mathematics for the year 1918-1919.

Mathematics Election for the Year 1918-1919.

62% of the First-Year students elected first year mathematics.
63% of the Second-Year students elected mathematics, of whom 17% took M1.
83% took M2.
68% of the Third-Year students elected mathematics of whom 2% took M1.
22% took M2.
76% took M3.

65% of the Fourth-Year students elected mathematics of whom 10% took M1.
6% took M2.
16% took M3.
68% took M4.
64% of all students in school elected courses in mathematics during this year.

The data of students pursuing college courses in mathematics are available for three institutions only. Twenty-nine students of the class of 1919 took mathematics in the fall of 1919 at Harvard, the Massachusetts Institute of Technology, and Tufts. Twenty-six passed the courses of which nineteen are known to have received better than a mere passing mark. Faculty opinion in these institutions seems to be that the boys coming from the English High School have a very high average of mathematical competence.

Constructing a course with little reference to later courses.—The mathematics course at the English High School represents an early effort to apply the organizing principle stated in italitics on page 14. The first two years aim to give the material that will prove most valuable to the pupil with little reference to later courses and disregarding wholly college entrance requirements. This aim led the teachers of the English High School very early to include certain elements characteristic of a course in general mathematics. For a considerable number of years the first year course has included intuitive geometry as material for algebraic and arithmetic problems, as for example the area theorems and the Pythagorean theorem. Measurements with ruler and protractor were made to give the pupil conviction of the geometric facts. Cajori gives Mr. George Evans, one time head of the department, credit for being the first to introduce graphing into secondary school algebra. The second year has included much algebra material correlated in a natural way with the geometry as for example proportion in the proof of theorems, the consideration of such surds as are used in connection with regular polygons, and the like.

Approximate Computation.—In particular the courses have placed great emphasis upon approximate computation used to develop common sense in handling numbers. A very practical kind of computation is carried through all the courses. Decimal fractions usually neglected elsewhere seem here to get proper emphasis. Both in the emphasis on the approximate character of measurement and on the continued consideration throughout all courses of the degree of precision as expressed by the number of significant figures this school affords an excellent illustration of what is meant by these topics as listed under B (a) on page 22 and under J, page 28.

The school has from time to time undertaken investigations and experiments with the aim of improving the results in mathematics instruction. Of recent interest is the study[11] made by Mr. John Marsh in a graduate course with Professor Inglis at Harvard University which has a bearing on the degree of transfer of training of instruction in mathematics and possibly also on the value of the type of course offered in the first year at English High School.

The Ethical Culture School, New York City

The Problem of Plane Geometry

General aims and purposes of school.—In considering the mathematics work of the Ethical Culture School it is helpful to keep in mind some of the unique purposes which have characterized the school since its foundation in 1878. The emphasis has been on the social and ethical as may be seen in the following aims: First, that of "inculcating the democratic spirit"; it furnishes opportunity for rich and poor of different races to meet together and learn to respect one another. Second, that of "awakening serious intellectual interests and enthusiasms in order to counterbalance the pleasure loving and self-indulgent tendencies fostered by the life of a great commercial city." Hence, while trying to use modern devices and methods for making subject matter clear, interesting, and stimulating, it endeavors not to lose sight of the fact that the study of each subject should require persistent effort. Third, that of "awakening a spirit of social service". Fourth, that of "building up a humanitarian and ideal purpose". Its ideal is not the adaptation of the individual to the existing social environment; it is the development of young men and women who will be competent and will have the desire as well as the courage to change that environment into closer conformity with moral ideals. In other words the school desires to train people who believe that "their salvation consists in reacting beneficently upon their environment".

It is implied in these general purposes that the mathematics teachers have to date been singularly free in developing their material. The improvement in the results of the teaching of demonstrative geometry has received most attention. However, it seems desirable to consider first the mathematics of the Junior High School.

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[11] Published in Educational Administration and Supervision, November, 1921.

Application of the school's aims to junior high school mathematics.—The school is administered on the 7-2-3 plan. The mathematics of grades eight and nine are the only courses required in the high school. The seventh grade is a transitional course. The work of grades eight and nine is exploratory in character and is planned, not only to give all that is considered essential for the boy or girl who either leaves school at the close of the ninth year or desires not to elect the more advanced courses, but is planned also with the idea of arousing an interest in the subject per se. That the mathematics teachers have the distinctive aims of the school in mind in planning the work of these two grades is shown by their effort (1) "to inculcate the democratic spirit" in the junior high school pupil by introducing him to the mathematics used in as many phases of life as possible, with the desire to make the pupil sympathetic with all classes (this does not mean that the mathematics is made wholly and solely practical, for that might exclude that type of individual who loves it for its beauty and theoretical power); (2) "to awaken serious intellectual interests and enthusiasms" by incorporating sufficient of the theoretical to let the pupil sense a gain in power and a love for the development of the science as a science; (3) to awaken "a spirit of social service" by showing how the great mathematicians of all ages have rendered service through their devotion to their tasks in the development of the subject; (4) "to help in building up a humanitarian, ideal purpose" by showing students the function of mathematics in the human progress of the world.

A new seventh grade course.—It is to be understood that previous to 1920-1921, only grades eight and nine were considered as exploratory. In that year a reorganization of the seventh grade work introduced much new material into this grade. The second half of the year now includes simple accounts, intuitive geometry, and algebra.[12] The main topics of algebra are (1) the graph (statistical, circular pictogram); (2) the formula (from mensuration), its evaluation and translation; (3) the equation (limited to positive numbers). The work in intuitive geometry offers opportunity for the use of compasses and ruler in simple constructions, drawing to scale, similarity, and field work.

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[12] The algebra is only incidental to the geometry and arithmetic.

The work of the eighth grade.—In the first half there is an extension of the algebra begun in the seventh grade. The work still centers about (1) the formula, its derivation and graphic representation; (2) simultaneous equations, solved graphically and algebraically; (3) the number system extended to include negative numbers; (4) fundamental operations with simple algebraic expressions.

The second half takes up the problems in percentage relations, which are widely used in the business world and which seem to be beyond the pupils' experience while in the seventh grade. The usual social-economic[13] topics of interest, insurance, taxes, discount, and the like, are used as applications.

The spirit of junior high school mathematics.—The spirit of the junior high school work is shown in the constant effort to avoid types of problems that go to make a perfect technique and to replace these by work rich in human interest and opportunity for problem solving. The entire object of the junior high school course is naturally not to perfect technique but to enable the pupil to read with some degree of intelligence the more or less scientific articles which he is apt to run across in his general reading.

Plane geometry in the ninth year.—The heart of the course in the ninth year is plane geometry, finally demonstrative in its nature. The transition from intuitive geometry is made most gradual. There is an effort to make a close connection with art, applied science, and algebra. The course in outline is as folows: (1) Introductory work in (a) the history of geometry, (b) observation of some of the uses of geometry, (c) application of the simplest constructions to making designs, pupil experimentation leading to the desire for a proof; (2) algebraic fractions, ratio, and proportion; (3) areas; (4) similarity; (5) trigonometric functions and simple applications; (6) logarithms; (7) locus problems; and (8) the circle (measurement of angles and mensuration of the circle).

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[13] In the year 1921-1922 this social-economic arithmetic was removed from the eighth grade. It will probably be offered as an elective, parallel with solid geometry and trigonometry in the twelfth year,—surely a significant change.

The aim is to make this course truly practical but it explores not only the applications of geometry but also the power and dignity of its logic. The pupil who has taken it knows whether he is fitted and desires to delve more deeply into the realms of demonstrative geometry.

The later years.—In the tenth and eleventh school years the student may elect a rigorous course in plane geometry. In this course the emphasis is constantly on the technique of thinking employed in plane geometry, as is also the case in the last half of the twelfth school year in which the pupil may elect solid geometry.

The need for adequate preparation for plane geometry.—From the foregoing it will be seen that the Ethical Culture School has realized the difficulties of a plane geometry course and has set about developing with great care a course that is a gradual transition from the ninth year. Whether the reorganization effected for the seventh and eighth school years which shifts some of the intuitive geometry to these grades, will decrease the amount of intuitive geometry to be taught in the ninth school year, remains to be seen. The important fact is that it is the opinion of the mathematics teachers in the Ethical Culture School that plane geometry is scarcely worth doing unless it is done well, and that it cannot be done even in mediocre fashion by plunging children of the tenth or eleventh school year into it without adequate preparation during the junior high school years.

Mathematical library.—The library consists of about 300 volumes of which more than one third are reference books. Of those most helpful to the pupils might be mentioned:—

Cox: Manual of the Slide Rule.
Heath's Monographs on Famous Problems.
Klein: Famous Problems of Elementary Geometry.
White: Scrap Book of Elementary Mathematics.
Ball: Mathematical Recreations.
Abbott: Flatland.
Henrici: Congruent Figures.
Clifford: Common Sense of the Exact Sciences.
Becker: Geometrisches Zeichnen.
Smith: Euclid, his Life and System.
Conant: Number Concept.
Fine: Number System of Algebra.
Manning: Non-Euclidean Geometry.
Cavendish: Recreations With Magic Squares.
Dudeney: Canterbury Puzzles.
Hill: Geometry and Faith.

Schubert: Mathematical Recreations.
Abrens: Mathematische Unterhaltungen und Spiele.
Ball: A Primer of the History of Mathematics.
Ball: A Short History of Mathematics.
Allman: Greek Geometry from Thales to Euclid.
Cajori: A History of Mathematics.
Gow: A History of Greek Mathematics.
Boyer: Histoire des Mathématiques.

Equipment.—The equipment that has meant most to pupils and teachers alike has not been the purchased apparatus so much as that which has been made by the pupils. The home-made list includes quadrants, diagonal scales, pantograph, slide rule, astrolabes, plumb levels, theodolite, proportional dividers, parallel rules, and models for use in solid geometry. The helpful purchased list includes large black globe on pulleys, coördinate boards, pantograph, proportional dividers, transit, sextant, set of mechano, plane table, knitting needles and five wire needles of different lengths for building figures for use in the study of solid geometry, lantern slides, and solid geometry models.

Rejected experiments.—For a time the attempt was made to teach some of the eighth grade material through the use of the project method in an extensive study of such matters as the city water supply system. The teachers feel that such projects are artificial in this year and that they begin and end in names and confusion.

The department has also discontinued the course in mathematics specially designed to meet the needs of non-mathematical students of fine arts. The facts seem to indicate that students failing in the exploratory course of the ninth school year could not study fine arts with profit nor could a geometry course be designed for them which they could pursue in a satisfactory manner.

The Horace Mann School for Girls, New York City

A Course For Girls In The Eleventh School Year

The Horace Mann School was established in 1897 as a department of Teachers College. For a time this school was used as a practice school for students in education. The chief function of the school now is to serve as a demonstration school for Teachers College. Up to 1914 the school was co-educational in all grades. The growth of the school and of Teachers College made it desirable at that time to transfer the boys of the six upper grades to a new building at Fieldston. In this report, the secondary department which is housed in the Teachers College group of buildings, will be referred to as the Horace Mann School for Girls. The high school is organized in two sections, grades seven, eight and nine constitute the junior high school, whereas the last three years make up the senior high school.

The teachers.—The mathematics of the six high school grades is taught by four teachers. These teachers are well trained, both in mathematics and in education courses. In fact, each teacher has had training somewhat beyond the masters' degree. Their experience ranges from five to twenty years in various secondary schools throughout the country. The department is particularly fortunate in having as advisors Professors David Eugene Smith and Clifford B. Upton, Provost, of Teachers College.

The equipment and the library.—The school has still another unique advantage in that it has access to the mathematical equipment of Teachers College, and to many of the valuable books and instruments belonging to Professor Smith. The teachers assert that this mathematical equipment is used by high school pupils for the following purposes:

(a) To provide opportunity for the pupils to make generalizations, definitions, and to discover laws for themselves.

(Models of geometric solids, metric weights and measures, Fahrenheit and Centigrade thermometers, spiral springs, pulleys, and other appliances illustrative of the law of the lever.)

(b) To show how the world has applied the principles of mathematics to solve some of its problems.

(The abacus, quadrant, sextant, astrolabe, transit and rod, and slide rule.)

(c) To encourage pupils to solve their own problems.

(The pantograph, squared black board, yard and meter stick, fifty-foot tape, and large protractor. Other equipment necessary for this purpose the pupils make for themselves, applying the principles they have learned, just as the world has done.

As for the library, it is extensive but many books are used only once or twice during a year. A few are in nearly constant use. Chief among the latter are:—

Number Stories of Long Ago. David Eugene Smith. Ginn & Company.

A Brief History of Mathematics. Fink. Translated by Beman and Smith. Open Court Publishing Company.

Story of Arithmetic. Susan Connington. Swan, Sonnenschein & Co., London.

The Teaching of Geometry. David Eugene Smith. Ginn & Co.

Flatland, by A. Square. Little Brown & Company.

Easy Mathematics. Sir Oliver Lodge. Macmillan Company.

A Source Book of Problems for Geometry. Mabel Sykes. Allyn & Bacon.

Famous Geometrical Theorems and Problems. N. C. Rupert. D. C. Heath & Co.

For history and biography the encyclopedias are used quite as much as histories or mathematics.

Four important problems.—The mathematics teachers are given much freedom in their demonstration work. They not only strive to employ the best practice but venture from time to time to undertake distinct innovations. During the last ten years materials and methods have been created which center around the following four problems. (1) to construct mathematics courses for the junior high school grades; (2) to place a unit of demonstrative geometry in the minimum requirement (the mathematics for citizenship); (3) to improve the results of teaching plane geometry; and (4) to teach a more meaningful course to the girls in the senior high school. Each of these will now be discussed briefly.

A junior high school course.—The junior high school curriculum tries not to lose sight of the fact that there are two classes of pupils to be provided for: those who drop out sometime during the junior high school period, and those who continue through to the end. An effort has been made to open the door of mathematics to all in order that they may be able to decide intelligently whether or not to elect further work in this field. In consequence, no year is devoted exclusively to arithmetic or entirely to algebra. The first half year of the 7th grade is devoted to arithmetic,[14] the last half year to geometry. In the 8th grade, the work of the first half year is devoted to algebra, and the second half to arithmetic. In the 9th grade the first semester consists of algebra and the second semester of geometry.

A special feature of the junior high school course is the intuitive geometry, taught in the last half of the seventh school year. The seventh grade geometry unit aims to familiarize the pupils with the beauty and utility of geometric forms. This is done by drawings, constructions, and by class discussions leading to the discovery and testing of geometric truths. This part culminates in design work,—in the reproduction of model designs and, later, in original designs. The idea of similarity is approached by drawings to scale, objects actually measured—as bookcovers, desks, the school room floor, etc. Illustrations of the idea are sought by the class and found in photography, sculpture, map drawing, etc. The pupils learn to use a pantograph at this point. Similarity leads naturally to symmetry, the study of which brings in the solid figures and their cross sections. In the work on areas, squared paper and paper cutting are used. The aim is to get the pupil to discover how to obtain the area of a rectangle, parallelogram, triangle, trapezoid, and circle, and to become familiar with the use of the formulas, as a short-hand method of writing a rule, thus laying the foundation for the algebra work in the next year. Volumes are taught from models. This topic affords a good opportunity for a stiff arithmetic drill in a form not unattractive to the pupil. Outdoor work of a simple kind is done, such as measuring distances by taping and pacing, finding heights of trees in various ways and the areas of plots of ground.

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[14] The school follows the work as outlined in Wentworth-Smith-Brown, Junior High School Series, Books I, II and III.

This geometry is correlated with household arts by special drill on the type of fractions and mixed numbers needed in the cooking and sewing classes; with geography, by teaching arc degrees and applying them to longitude and latitude; also, in showing the connection between conic sections and the orbits of the heavenly bodies; with history, by studying the historical beginnings of geometry in Egypt.

A unit of plane geometry in the ninth grade.—A second special feature of the Junior High School is the teaching of a unit of demonstrative geometry. In this work, covering the last half of the ninth year, the pupils prove those propositions necessary to the proof of the Pythagorean theorem. They begin the work by reviewing the geometric constructions and certain ideas acquired in intuitive geometry. They get some notion of the use of geometric proof outside of geometry. This is followed by proving such propositions as lead directly to the proof of the Pythagorean theorem. More than half the time is spent in doing exercises. This unit of work also includes at opportune times such history of geometry as helps to interest the pupils.

The Horace Mann School for Girls is one of the schools that has faith in the desirability of teaching a unit of demonstrative geometry as one of the requirements in the mathematics for citizenship. It is believed that no boy or girl should leave the school unless he or she has become familiar with the technique of setting up a valid proof. In short, the argument which this group of teachers advances for demonstrative geometry is like the argument that some teachers advance for a unit of trigonometry.

Improving the results of plane geometry.—Plane geometry as we ordinarily think of it is here taught in the fifth high school year, that being the year which corresponds to the junior year in the standard high school. The teachers are firmly convinced that their pupils do very much better in plane geometry in recent years on account of the preparation which they have had in the junior high school course. While this is what common sense would lead one to expect, the notion that junior high school mathematics, which includes intuitive geometry, removes many of the burdens from the teacher of plane geometry, is one that is being accepted but slowly. It is entirely possible that there will presently be a vast improvement in the teaching of plane geometry in this country, because of the foundation that is being laid in the junior high schools independent of any notion of preparing for plane geometry.

A senior high school innovation.—A fourth special feature of this school is an attempt to work out a course in Senior High School mathematics which will be more successful than the traditional one in developing desirable habits and attitudes. Since relatively little experimental work has been done in senior high school mathematics, this work seems particularly interesting.

The units in the experimental course are not topics of algebra, but are problems which arise in life (though possibly not in everyday affairs) and which require for their solution certain mathematical processes.

The unifying element is the function notion expressed in the formula, by use of which correlations are made with most of the departments of knowledge having a quantitative aspect. The table, the formula, and the graph are an inseparable trio throughout the course, each being chosen for use when it seems best suited to the purpose in hand. An effort is made to show that the history of mathematics is the history of the race, and that mathematics has always been occupied with the discovery of existing laws and the invention of symbols for their expression.

The course includes the following topics:

1. Practice in making and testing formulas.

2. The use of factors as a means of expressing laws concisely and of shortening computations.

3. Ratio and proportion as a case of fractions, and without the usual formalities.

4. Logarithms as an application of exponents (with the slide-rule as an application).

5. Binomial expansion with applications to compound interest.

6. Series, with applications to physical laws and to annuities; introduction to the concept of infinity, and the notion that analysis may obtain measurements which cannot be obtained by practical means.

7. Something of the history of algebra and of its contributions to the progress of the world; and, for the human side, something of the lives of great men whose major interests have led them to discovery and invention in the field of mathematics.

8. The introduction of methods of trigonometry, analytic geometry, and the calculus.

It is necessary here to give somewhat detailed descriptions of a few units, in order to convey something of the spirit which dominates this senior high school course. The first unit of the course is drawn from trigonometry and surveying. The demand for a method of finding the height of the room or of any other vertical distance serves as a means of arousing discussion as to possible ways of finding inaccessible distances. It is only a short step from the notion that the corresponding sides of similar triangles are in proportion, to that of a tangent of an angle and its use in solving the problem in hand. Tangents of various angles are found by different pupils and, from the table formed from these results, a tangent graph is made. Tangents of acute angles can be read from it with a fair degree of accuracy, and these can be used in solving a number of real problems.

The work connects also with industrial arts. Instruments for measuring heights by the aid of angles are made by the class and compared with fine old brass astrolabes from Professor Smith's collection, instruments which were in actual use some hundreds of years ago. The disadvantages of all these instruments for the purposes for which they were intended are discussed, a modern instrument is introduced, and its advantage over the others noted. Outdoor work with the transit and rod, and with the improved improvised instruments follows.

The second unit is statistical, the problem being how to express the dependence of one variable upon another by means of the equation of a line. Descartes is looked up and a period given to reports from the class of what they found interesting in his life, his philosophy, and his contributions to mathematics. The problem is made clear to the students. Cases so simple are taken at first, that the equation of the line can be found by inspection of the table or of the graph. Natural dissatisfaction with the limitations of a trial and error method is used as an incentive for the development of the method of obtaining the equation of the straight line analytically. In this work a real use of linear equations in two unknowns appears, and also a first notion of the meaning of a constant, when in the equation $y = mx + k$, m and k become the unknowns and are shown to be constant for all values of x and y.

Perry's "black thread" method is used. Experiments giving data which are connected by the straight line law are made by the girls. For example, data obtained with Centigrade and Fahrenheit thermometers are tabulated and graphed. The points are found to lie more or less in a straight line to the extent that the data are correct. The notion gradually develops that if it were not for unavoidable inaccuracies in measurement, these points would lie in a straight line, and the judgment of the girls is appealed to in drawing that line and in finding its equation. Very clean-cut notions of the meaning of generalization come up here when a page full of simultaneous linear equations is reduced to the form, $ax + by = c$, and again when, for convenience, the equation $y = \frac{c}{d} + \frac{x}{5}$ assumes the form of $y = mx + k$.

Very important notions connected with statistical methods arise here, as it becomes evident that no assumption is to be made that any one measurement is more accurate than another but that as many will tend to be too small as too large, and that consequently the line must be chosen which shall have the same number of points on both sides of it, and, moreover, that the neighboring points on both sides of the line shall be equally distant from it. In the comparison of the different equations obtained by different members of the class, the great advantage of decimal over common fractions becomes evident.

In connection with this work, graphs of statistics are made and the point brought out that, while the graph is of great value in studying all sorts of data respecting variables, it is particularly valuable when the curve is such that the law can be expressed mathematically and that then a sure prophecy can be made as to the value one variable will take when the other is known.

Another topic is algebraic and is concerned with the making of formulas for various uses individual or otherwise. Manipulations and evaluations of formulas made by others, and relating to as many fields of knowledge as possible and to interests in life having a quantitative side, are discussed, used, and judged. The daily paper is drawn upon for problems which may be answered by formulas already set up. In particular, thrift, savings, and investments are treated and formulas expressing aesthetic, physical, and astronomical laws are interpreted.

In the year 1919-1920 opportunity was taken of the awakening interest in the heavens to extend notions of the immensity of space and to teach the equations of the circle, the ellipse, and the parabola, and how to plot them.

The principles of proportion and variation are introduced as a means of understanding such expressions as "the proportions" of an object of beauty, and are applied to such practical matters as choosing widths of tucks and hems, and distances between them. Photographs of objects of which a chief claim to beauty lies in their proportions are shown, and the algebraic method of discovering the so called "Divine Proportion", or golden section, is taught. Applications to physics and geometry of the laws of proportion are then made, and the notion of variation and the algebraic expression for it are introduced.

The fourth unit deals with maxima and minima and takes the pupils beyond the scope of algebra into the field of the calculus. The work begins with the problem of finding the maximum areas with given perimeters solved graphically. The graphs make clear the fact that the highest point on a graph is reached when the line whose slope measures the rate of growth is parallel to the x axis; that is, when the tangent of the angle is zero. A method of solving this problem by analysis appears to be desirable because of the impossibility of getting more than a rough approximation of the turning point in many of the problems. It is then pointed out to the class that the calculus provides a means of finding an algebraic expression for the slope of a tangent at any point on a graph. They next learn that when this expression is set equal to zero and solved for x, the exact turning point can be located. The rule for finding the first derivative in simple cases is discovered and stated. The problems solved previously by the graph are then solved by the calculus. Thus, for an experimental method a more exact method is substituted. The so-called "parcel-post problem" requires work with problems similar to that done with areas. The question arises as to the shape and proportions of the parcel which will provide the maximum volume while its dimensions satisfy the local requirements. The problem is solved both by the graph and by the calculus. Many other problems in maxima and minima are solved by the pupils, some of which require more extended work in the finding of the derivative.

There are, of course, other topics in the course in addition to those listed above.[15] It appears that the course has been built by skimming the cream from a considerable number of crocks. The course aims to be genuinely cultural and at the same time retain the spirit of rigorous scholarship. Though unusually attractive the course cannot be stigmatized as "sugar coated" for it offers insight into many problems which many well trained adults would find very difficult.

The Lincoln School of Teachers College, New York City

A Program of Coöperative Research

Aim.—The Lincoln School is an experimental school administered by Teachers College and financed by the General Educational Board. The first group of students was admitted in October, 1917. At that time classes were organized in grades from one to nine inclusive. Advancing classes have been retained so that now all twelve grades are represented in the school's organization.

The school is trying to construct a liberal curriculum, deriving its material from the world in which the student lives. This purpose compels the school to take as its point of departure the best in current practice and to improve materials and practices by controlled experimental efforts.

Organization.—The school is organized on the 6-3-3 plan. The following time schedule for the junior high school grades as administered during the year (1921-1922) has an important bearing on the mathematics situation.

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[15] The reader will find a complete description of this course together with other features of this experiment in the September, 1921, number of the Teachers College Record. The test material used is presented in the Teachers College Record of May, 1920.

JUNIOR HIGH SCHOOL SCHEDULE

Hours Weekly

Grade	Eng.	German French or Spanish	Math.	Science	Phys. Edu.	Music	Ind. Arts Boys	H.A. Girls	Fine Art	History and Civics
VII	3	2½	3	3	5	2	3	3	3	3
VIII	3	2½	3	3	5	2	3	3	3	3
IX	3	2½	3	3	5	2	3	3	3	3

The numbers in the table refer to hours not periods, an hour being fifty minutes net time in the class room, laboratory or playground.

Mechanical drawing is required in grades seven and eight and vocational guidance in grade nine for one hour per week.

The preceding table shows a distribution of time that affords an opportunity to the pupil to take a foundational course in each large field of human activity. Since these introductory courses (each considered as a unit for grades seven, eight, and nine) consist of the simple and significant principles it is possible that the "exploratory" function may be realized within the subjects. To confront a boy (or girl) at this early age with the necessity of making a choice of a few subjects from an extended list of "electives" such as algebra, biology, civics, general science, and the like, may be leaving too much to *chance* since he has only very vague notions of the content of these courses. Under the suggested schedule the pupil does not face the question of algebra or no algebra. On the contrary, the introductory course aims to give him some notion of algebraic methods, of geometric relations, of statistical representation, of trigonometric solutions, etc. Similarly for the other subjects; e. g., the sciences (social, biological, and physical) and the arts (fine, household, and industrial). Liberal time is assigned to the "activity" subjects without decreasing unduly the time of the "academic" subjects.

The preceding discussion is of great importance because a considerable number of schools are placing algebra (or mathematics) as an elective in the ninth year.[16] The reason is easily found. Administrators find the bucket full and in attempting to pour in new materials something is spilled. But it is more difficult to discover the educational principles which justify this policy. It is possible that the material worth while teaching in grades seven to nine is not so extensive as it now appears in some curricula. There is a considerable amount of duplication in the various school subjects. It is not uncommon to find the same project running in several school subjects without the teachers being aware of the fact. It should be noted that this is different from the teaching of a project to which instructors in the different departments are contributing. The one usually means a wasteful duplication while the other may signify an economical correlation. Still another suggestion for economy of time is found in the organization of materials of instruction along problem solving lines so as to make possible a laboratory form of recitation in junior high school subjects. There is evidence to justify the hope that there will be time sufficient to teach all the material that is left when we shall have (1) made a proper evaluation of individual items in mathematics (and particularly in other school subjects), (2) reorganized the material so as to get more economical learning, (3) eliminated much wasteful duplication, and (4) secured a natural correlation so as to give sufficient strength to such bonds.

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[16] See Koos, *the Junior High School,* Harcourt, Brace and Howe.

The most important task.—While mathematics may not have been the worst offender in its emphasis on mechanical formalism, it nevertheless furnished an excellent example of the need for the particular type of reorganization which brought The Lincoln School into existence. The Lincoln School has attempted from its beginning to develop a mathematical curriculum that would teach pupils things worth knowing and discipline them rigorously in things worth doing. The school has assisted in creating a type of mathematical material suitable for the average intelligent citizen or the general reader, without regard to subsequent specialization. Hence, it was found necessary to eliminate, or to reduce considerably, much of the traditional material, to adopt a new sequence of topics, and to lay reasonable stress on motivation and probable life situations. In adopting this policy, there has been no desire to curtail in any way the inherent values which are ordinarily associated with mathematical studies. On the contrary, it is believed that the new courses, while making a far greater human appeal to the pupil, are capable of yielding all such disciplinary results as the old courses may have possessed.

The chief function then of The Lincoln School mathematics teaching is to so perfect the technique of experimentation in this field that a liberalized curriculum may be achieved more efficiently.

A general mathematics course for the ninth grade.—When The Lincoln School was established, the mathematics department attempted to take as its point of departure the best practice then found in a number of other schools. Much of the material which guided the initial efforts is included in other sections of this report. It was clear that the crucial need at that time was the reorganization of the mathematics of the ninth school year. The reasons are fully stated in the part of this chapter which concerns the University High School, Minneapolis (see p. 262) One of the members of the present mathematics staff had for several years previous to the opening of the school been engaged with Mr. William D. Reeve of the University of Minnesota in developing a basic course for the ninth school year. During that time, the material was used in mimeograph form in various types of public and private schools. On the basis of this extensive classroom experience, modifications were made in the course. After two years of further reshaping of material in the classrooms of The Lincoln School and of the coöperating schools, this material was published as a textbook for the ninth grade.[17]

Since the material is now generally available, only a brief description is here necessary. The course consists of the simple and important principles of arithmetic, algebra, and geometry. It also includes about four weeks' work in numerical trigonometry and an extensive unit in graphic representation. In addition to this, the course offers certain optional topics, as may be illustrated by small units on logarithms, slide-rule, mechanics, and the like.

Particular efforts were made to collect the desirable correlations quite generally used by good teachers. On the other hand, it was desirable to avoid correlation for correlation's sake. Both these purposes seem rather easy to attain when there is no effort to correlate with demonstrative geometry.

A junior high school course.—At this time the school is particularly concerned with determining and trying out the materials for a unit course for the junior high school (grades seven, eight, and nine). The teachers believe that it is important to recognize that much detailed and exacting work is necessary before the mathematics problem of these grades is satisfactorily solved. The strongest evidence of the fact that much work needs to be done is found in the marked variation in the existing series of texts for these grades. This variation has been pointed out in another part of this chapter. Scarcely any critic would argue that these alternative schemes are equally effective in forming desirable habits, giving useful information, and increasing mathematical power. Experimentation involving a very careful testing of methods and results should eventually disclose the most effective use and organization of material. It is urgent that the various courses now being used be evaluated.

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[17] *General Mathematics*, by Schorling and Reeve, Ginn & Company.

Still another need for careful experimentation is found in dealing with the bases for the selection of subject matter. The validity of proposed subject matter will probably be determined by the degree to which it serves worthy individual and social needs, and to which it meets the requirements of a sound psychology of learning. When the materials have been fairly well determined, the arrangement (grade placement and sequence of topics) becomes a psychological problem, the solution of which will depend upon critical analysis of children's responses to these materials.

The desirability of organizing subject matter in accordance with known principles of learning.—The modern psychology of learning states that children learn the concepts and principles of mathematics as they learn other things—through abundant, varied, and purposeful experience. In the junior high school grades numerous attempts to improve the curriculum take the form of "shoving down" the mathematics from the senior high school grades. Much of the present dissatisfaction is due to the fact that these courses were shifted downward without an adjustment to the needs, capacities, and interests of less mature minds. This temptation is particularly noticeable in those communities which are instituting junior high schools without a clear philosophy of the junior high school curriculum and with teachers who are held by traditional practice. The new movement clearly requires that a new spirit and method shall be developed in these grades. The details of this technique can be discovered and arranged only on a basis of extensive classroom studies.

The guiding principles.—The guiding principles for this entire period that are listed below are conditioned by the growing belief that the three-fold purposes of teaching junior high school mathematics are (*a*) to make school studies and life out of school mean more to a boy or a girl than they otherwise would; (*b*) to induce the pupil to react to a variety of human activities, which yield greater satisfaction; and (*c*) to give to the pupil more ready and accurate control of the numerical and spatial relations of human life.

1. The course in each year should give the pupil the most intrinsically valuable mathematical information and training which he is capable of receiving at that time, with little consideration for the needs of subsequent courses.

2. The content of these courses must be selected from the point of view of the children and from the consideration of social needs, and not solely from the logical requirements of mathematics.

3. The general aim stated at the outset will necessitate the inclusion of certain elements of arithmetic, intuitive geometry, algebra, numerical trigonometry, and statistics.

4. Throughout the courses the idea of relationship or of the dependence of one quantity upon the other is to be emphasized. From the mathematical point of view, this notion of function is the unifying principle; but from the point of view of teaching, the basic, guiding principle is not found within the science itself, but within the children's learning.

5. In organizing the course the usual emphasis upon the special divisions of mathematics and the customary time allotment should be replaced by the introduction of the topics in such a way as to insure a maximum of direct and intensive application, flexibility, and significant interrelations.

6. The mathematics of the junior high school should mark the end of the required mathematics, and, hence, it must throw into bold relief those general mathematical ideals, tools, and habits which are now regarded as of maximum importance.

7. The unity of space and number should persist throughout, for geometry furnishes a concrete source, setting, and illustration of significant number relations. In consequence, measurement is one of the fundamental processes by which the pupil may discover the significant number relationships directly through the senses.

8. Manipulation as an end is to be eliminated. Mechanical work can be justified only when necessary for understanding and using fundamental principles.

9. Attention should be directed toward a better appreciation of the part that mathematics has played and is playing in the progress of civilization, in order to give a better understanding of common situations.

10. The material should be vitalized[18] through the early introduction of principles that are commonly delayed until the later courses, as, for example, numerical trigonometry; and through a closer correlation with other school subjects, as, for example, elementary science, mechanics, industrial art, fine art.

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[18] A concrete illustration of the Junior High School work may be had in a separate booklet entitled "Illustrated Mathematical Talks by Pupils of the Lincoln School." It is a program that was given at a meeting of the Parents-Teachers Association.

11. The material is to be socialized through the extension of units of instruction from classroom exercises and topics through a series of activities, projects, or problems requiring (1) coöperation and (2) sharing of interests, efforts, and results. The course should aim to give rigorous discipline in things worth while. The ability of children to undertake and carry through purposeful activities should be recognized. The course should capitalize the rigorous discipline that comes in engaging and carrying through purposeful activities, instead of trusting to the discipline of activities which are often uninteresting ends in themselves.

12. Throughout junior high school mathematics common sense in computing with approximate data should be exercised.

13. Through observation, measurement, intuition, and the consideration of elementary properties of geometric figures, the course should lead to control of symbols, vocabulary, and the principles of space relations which common experience requires.

14. Junior high school mathematics should teach the necessary social and economic uses of arithmetic. The more technical forms of business practice, such as insurance, brokerage, stocks, bonds, etc., should be placed late in the course to utilize the greater maturity, experience and mathematical knowledge of the pupil.

15. A marked increase in the accuracy of computation with integers, fractions (common and decimal) and per cents is imperative in these grades.

Two types of investigations.—The preceding general principles constitute the point of departure for the more detailed investigations which are necessary to translate this program into schoolroom practice. Two types of investigation appear to be involved: (1) A series of researches and investigations designed to determine a body of curriculum material which can be defended on a basis of social worth. These are for the most part individual in character. (2) A series of classroom studies (psychological studies) calling for coöperative efforts which have been referred to in the preceding section and which will be discussed in detail later.

Studies in content.—The following studies, bearing on the determination of a valid content for these grades have been initiated.

I. *What do the first six years of arithmetic instruction achieve?* The existing tests show fairly well the ability of children to compute with integers and fractions, and they tell us something about their ability to reason. But we have little information concerning the knowledge of symbols, processes, terms, principles, and relations which sixth-grade children may possess.

In consequence, an inventory test has been developed. This, together with the compilation of results from widely used existing tests, will yield a reasonably complete account of mathematical instruction at the close of the sixth grade.

II. *A social survey (industrial, commercial, and professional) to determine the mathematical materials that can be justified on basis of their social usage.*

This is an investigation of the quantitative aspects of economic and social needs. Certain efforts to inventory social needs have been made in several doctors' theses, as, for example, the study by Wilson, but the technique used in these studies is challenged on two counts: (1) the data were secured in devious ways, and (2) it may be questioned whether discovering "what is" is a valuable index to "what should be." An inventory of various citizen groups showing that they think inaccurately about things mathematical in life and that they dodge efficient methods in treating things quantitatively does not prove that people should not be trained to think clearly and accurately about mathematical things. For example, the fact that relatively few people in the United States know the metric system is not sufficient evidence for concluding that the whole nation, or indeed the whole world, should not be trained to use this system.

This part of the investigation is very difficult, but its importance is great. We can get some help in finding out what the general reader needs by a quantitative study of the frequency of occurrence of mathematical symbols, terminology, and facts in newspapers, popular magazines, advertisements, articles, and the like. Probably the solution of this problem lies in bringing together a sufficient number of job analyses from which specific common elements may be selected.

III. *What material should be taught in these grades?* The materials which have been assembled by the survey should be evaluated in the light of certain considerations other than frequency of use. For example, the survey may show that relatively few people know the marks of a good investment, or know how to compute compound interest, yet both of these elements might be included in the curriculum because of other considerations. There are ten or more of these considerations before which the elements of the new curriculum must "pass in review." Of these five are given as illustrations.

1. *Complete analysis of texts.*
    - (*a*) Arithmetics for grades 6-8, inclusive;
    - (*b*) At least two or three French and German texts covering grades 6-9;
    - (*c*) Junior high school texts.

This analysis of texts will result in a series of charts, one showing the material that was taught in the arithmetics ten years ago, a second one showing what recently published texts include, and a third the material that is contained in the junior high school texts. The practice of our public schools is, of course, determined almost wholly by the material and organization of text-books. A comprehensive survey in graphical form of the material that is now in use raises many questions, as, for example, relative emphasis and reasons why certain topics appear and disappear. Obviously such a study facilitates a critical review of the wide range of proposed materials. It does not determine what ought to be taught, nor will it be weighted heavily.

2. *An analysis of materials used in schools undertaking innovations in mathematics.*

It has already been pointed out that the innovations described in this part of the chapter are undertaken by a group of teachers unusually well prepared in mathematics, both in content and in professional courses. In consequence, the composite opinion of this limited group of teachers is a most significant guide to those who desire to improve the teaching of junior high school mathematics.

3. *A collation of all committee reports* on the mathematics of the junior high school grades.

Writers of text-books have been affected by committee reports. Hence, it is essential to make a collation of committee reports to see to what extent elements in the different reports have persisted. Committee reports at their best represent the composite opinion of competent teachers.

4. *A survey of the literature, articles, and studies which bear on the problem.*

The material listed above needs to be studied for the opinions of those who have given especial thought to this problem. The literature of high school mathematics has a number of very thoughtful papers with which teachers are not generally familiar. These papers need to be collected and studied to see whether practice can be improved by the counsel which this literature includes.

5. *Mathematical needs of other subjects.*

This takes the form of a preliminary survey of junior high school mathematics to see to what extent it contributes toward making pupils intelligent in other school subjects. We need to know what materials in other school subjects can be brought into the mathematics classroom and used to illustrate mathematical principles, and we need to know to what extent mathematics can be used in other school subjects. The question of transfer to other subjects will not be so important as the question of direct use in making these subjects more meaningful.

A program of coöperative research.—*Studies of Learning.*—It has been stated that one of the methods to be used in developing a valid curriculum is that of coöperative experimentation. The study of children's responses to proposed materials of instruction must be made by many teachers with many types of children and under a variety of school conditions.

A group of teachers, working with mathematics teachers in The Lincoln School, are engaged in making detailed classroom studies of this character. In general, this program takes the form of choosing one or more seventh-grade sections for intensive study. The detailed steps in the program include the following:

1. The taking of a complete inventory of the mathematical equipment of beginning seventh-grade pupils. One phase of this inventory refers to the first investigation listed under the preceding studies. This test consists of two parts. Each part can be completed by all pupils in a thirty-minute period. The remainder of the inventory consists of a compilation from existing standard tests. Any teacher who makes a special study of mathematics with seventh-grade children is primarily interested in the mathematical growth of her children, but no valid conclusions can be achieved without complete tests before and after the subject is studied. In consequence, this inventory test has been designed to be given at the beginning of seventh-grade work.

2. The teaching of at least two distinct types of instructional material, one of which is constructed especially for experimental purposes. The other may be chosen from one of the junior high school series that is now available, or it may be any standard arithmetic in use in the particular school system. It should be noted that the many types of material now available indicate distinct hypotheses concerning the selection and placement of material in these grades. One outcome of this experiment will be a comparison, under controlled conditions, of the effectiveness of these hypotheses.

3. A critical observation of the children's responses to the type of material taught. A record will be kept of pupils' successes, failures, interests, the sufficiency of explanation, the adequacy of practice material, the clearness of terminology, the degree to which the subject matter is taken from the experiences of the children, and the suitability of the projects which are included. Special blanks are provided for these records in order that the teacher may not find this technique an added burden.

4. A systematic use of measurement to test the pupil's mastery of each unit of material and growth in skill, information, and mathematical power. In this the order is teaching, testing, and further teaching by means of practice material especially selected to supplement instances of inadequate learning, and finally a comprehensive inventory test of the outcomes of teaching each year's work. From these inventories comparative data on the effectiveness of the two types of material will be secured.

5. A similar intensive study for grades 8 and 9. The printed materials[19] used are now available for use and can be secured by writing to the school.

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[19] Mathematics for the Seventh Grade, Schorling and Clark.

An Outline of the Program as a Whole.[20] The two types of investigations will be used in carrying out the following program:

I. What mathematics do children know when they come to the seventh grade?

II. A social and economic survey to determine the mathematics materials that can be justified upon the basis of social usage.

III. The evaluation of possible materials of instruction for grades 7, 8, 9, resulting in a tentative curriculum of what ought to be taught.

IV. The organization or arrangement of these materials in accordance with known principles of learning.

V. The teaching and *testing* of materials to ascertain the degree to which the objectives have been attained.

VI. A final revision of the materials to embody any needed changes of order, of emphasis, or of method which may have been suggested by the teaching-testing step.

A considerable number of schools will participate in this coöperative program. It is, of course, desirable that representative types of pupils and teaching conditions be included. In particular, it is desirable to have the constructive help of superintendents and teachers who have a keen interest in improving the effectiveness of instruction in these grades. During the present year (1921-1922), 33 schools are participating in the seventh grade study. The experiment involves over 1300 children using the new materials and more than that number in the control groups.

Materials to be used by the coöperating schools.[21] The following types of material are included in the program: (1) the inventory tests, (2) experimental teaching material for pupils in each of the junior high school grades, (3) test and record blanks which will economize the time of the teacher and assist in helping to analyze pupils' learning. The seventh grade material includes the following units:

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[20] For a detailed statement of the program of investigation and experimentation, see The Mathematics Teacher, May, 1921.

[21] These materials will be supplied at 80 cents per pupil in class lots.

1. Making accurate measurements.
2. How graphs are used to picture numbers.
3. How to locate places; direction; measurement of angles.
4. The use of lines and angles in designs.
5. The use of geometry in architecture.
6. The measurement of area; formulas.
7. The measurement of volume; formulas.
8. How to solve problems by the use of the equation.
9. The percentage relations; profits and loss; applications of percentage.
10. Projects in business practice.

The effort is to place great emphasis upon (1) the fundamental operations with integers and common fractions; (2) percentage relations and such business forms as are simplest and most commonly used in and out of school. Each unit of work ends with a three-fold review. The first part of this review gives the pupil a summary of the unit which he has just finished. The second part is a "cumulative review" of the work that goes before. It is now known that relearning at the proper intervals is of the greatest importance. This relearning of units previously taught is provided in entirely new problem situations to the pupil. The third part of the review is made up of standardized speed and accuracy practice drills. These are "setting up exercises" designed to help the pupil "keep fit" in arithmetic skill.

The content of the eighth grade.—

1. Practical measurements.
2. How to find unknown distances.
3. Using the right triangle.
4. Four ways to express a relation between numbers.
5. The use of geometry in architecture.
6. Practice in problem solving.
7. Interest.
8. The secret of thrift.
9. Making money earn money.
10. The nature of insurance. Taxes.
11. The use of positive and negative numbers.
12. Household measurements.

The material, like that of the seventh grade, is in the form of *general* mathematics. Considerable advance is made in algebra, the topics show that the arithmetic is "kept going," whereas the basis of the material is geometric or graphic. Since the material used in each of the junior high school grades is available further comment is not necessary here.

A Special Learning Study Concerning Common Fractions. This study involves among others the following steps: (1) The construction of a number of tests of equal difficulty in addition, subtraction, multiplication, and division of common fractions; (2) the analysis of learning involved in applying the particular operation; (3) the designing of certain practice training series; (4) the application of these practice cards together with measuring the effect which such practice has on the test scores of the individual or of the class; (5) the study of the effectiveness of a graphic device which focuses the pupil's attention on his particular disabilities.

The senior high school.—*College Entrance Examinations.* The question may well be asked whether the new courses such as are being recommended for grades seven, eight, and nine can be given in these grades without losing the chance to meet college entrance requirements. Other parts of the report answer this question in the affirmative. The experience of The Lincoln School supplements this evidence. As has previously been stated, the junior high school mathematics courses recite only three hours a week. In the first senior high school year (tenth grade), the pupils study a traditional plane geometry book. The next year classes take up subject matter[22] that is ordinarily considered college material. The twelfth grade review classes recite but one period a week. This work consists of a series of problems designed with the definite purpose of getting pupils ready for college entrance examinations. It will be seen that the time devoted to mathematics is very much less than is commonly the case. To date such pupils as have completed the courses outlined above have not only been able to meet college entrance requirements, but have been given as much as a year of credit by institutions of the first rank. All have so far succeeded in carrying their college mathematics successfully. The number of graduates of this school is of course as yet too small to be significant. It is here reported not as an achievement but by way of quieting the fears of those who hesitate to give a course that differs from those dictated by college entrance considerations. College entrance requirements need not be the obstacle to progress in high school mathematics that they are generally believed to be.

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[22] Young and Morgan, *Elementary Mathematical Analysis*, The Macmillan Co.

Plane Geometry. The school has as yet made no special studies of the very important question of demonstrative geometry. The intuitive geometry that is being taught in grades seven, eight, and nine seems to contribute greatly to the success and enjoyment which pupils get from a geometry course. This appreciation and profit probably extends to a lower level of intelligence than is commonly supposed. A fractional part[23] (not at all small) of students with intelligence quotients under 105 have taken a traditional plane geometry course with what seemed to be genuine enjoyment. Was the effort and time spent worth the cost? Might these pupils not have gained more from some other school subject? The school has no evidence on these important points. Surely the content could not itself justify the expense of time and effort. If demonstrative geometry is to be shown worth while it must be in the thinking outcomes. And that suggests that the correlation of demonstrative geometry with material from algebra, trigonometry, and other mathematical fields is not likely to prove to be the correct solution.

The New Mexico Normal University Experiment, Las Vegas, New Mexico

No Mathematics in the Ninth Year

The department of Mathematics of the New Mexico Normal University undertook in 1913, under the leadership of Professor Tom Gladstone Rodgers, to contribute toward the solution of the following four problems: (1) to vitalize the teaching of secondary mathematics; (2) to reduce the number of failures; (3) to vitalize the review work; and (4) to improve the examinations.

The characteristics of this experiment were (1) omission of mathematics from the first year of the high school, (2) introduction of the "Parallel Method", (3) increase in the project and problem work, and (4) adoption of the system of "Bank Problems."

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[23] For example, 54% of the pupils in the course in the eleventh grade analysis have intelligence quotients less than 108, the lowest being 95. This group is probably lower in general ability than typical senior high school groups and certainly lower than pupils ordinarily found in advanced mathematical classes.

No mathematics in first year high school.—It seemed to the teachers in the department that the student whose course terminates with the end of the first year would gain more that would be of value to him in later life from a study of English, History, Science, and the like, than from a brief study of mathematics. It seemed to these teachers that the methods of thinking involved in algebra were foreign to children and difficult for them. Hence, they desired an additional year in maturity. The somewhat radical implication that all which is offered by the school subjects substituted is of greater value than some of the mathematics which might be given during this first school year has not been generally admitted, not even by school officials and teachers of these other school subjects. Here there is still a persistent conviction that some of the mathematics may contribute toward making other school subjects and the first year (considered as a unit of worth while life) more meaningful than it can be without the study of mathematics.[24]

It will presently appear that the number of failures[25] in mathematics (now given in the second year) is greatly reduced. No doubt one of the chief contributing factors is the weeding out of the immature and those of low intelligence or mathematical ability. Concerning these it is asserted that "this class of students gained more that will be of value in after life from the study of English, history, general science, and vocational work than it would have gained from the permission to fail in mathematics." The experiment does not furnish data to answer such questions as: To what extent would failures be decreased by delaying the beginning of mathematics until the third year? the fourth year?

The parallel method.—In the beginning mathematics courses, algebra and geometry are taught side by side. It is believed that acquaintance with each of these great methods of thinking can be secured only by teaching the algebra and synthetic geometry separately. Confining algebra to two or three days a week limits the work to the simpler parts of algebra. Hence there is no floundering in the abstractions of manipulation during the latter part of the year which so often causes failure at the end. As a substitute for the difficult parts of algebra we find in the course the simpler parts of geometry which (so the teachers assert) the pupils "travel with ease and delight."

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[24] In fairness to Dr. Rodgers it is necessary to quote from a recent communication (Nov. 30, 1921) concerning this point as follows:

"I have never been in full agreement with this change (he refers here to omitting mathematics in the first year of the high school) and under the present simplified courses in mathematics I believe it to be a mistake. I feel that the proper kind of mathematics is of vital importance to first year students."

[25] See New Mexico Normal University Bulletin (1919, No. 62)

The plan provides that the pupil spend two years on each subject, thus gaining another year of maturity, for the more difficult parts. Still another advantage is claimed in that one subject is not forgotten while the other is worked up.

Correlation projects used in reviews.—The careful organization of relearning is important. One desirable form of review is the presentation of a new situation involving the application of principles that one desires to strengthen. It must be admitted that opening a course with from four to ten weeks of review work in the same form as in the preceding text, prevents "selling the subject" to the child at the outset. In the experimental course the new work of the whole course in both algebra and geometry is covered more rapidly and of course less in detail than is customary. The aim is to get a working knowledge of the essential parts of each subject. The time saved is given to the solution of a carefully selected set of miscellaneous problems. Many of these problems call for the combined use of arithmetic, algebra, and geometry. There is here an increased number of projects involving simple investigations. Correlation comes in the form of applications and relearning through which the pupils get a better appreciation of the relation of the two subjects to each other.

Bank problems.—Some thirty problems of a special character (called "Bank Problems" because they add to the student's scholastic "reserve" or mental bank account) are assigned to each class. These problems are not like any discussed in class. Usually they lie just beyond the pupil's previous study but within his power to solve. One week is given to the solution of each problem. The solution is entirely voluntary. Not only a passing but a creditable mark can be obtained without even attempting the "Bank Problems". Experience seems to show that even the weaker students are greatly interested in this kind of work and acquire greater power in problem solving.

The following type of "bank problem" is given the first week to a Mathematics I class: "A ten inch square is rolled, without slipping, along a straight level path. Construct the curve described by one corner of the square during a complete revolution and find its length."

After a few weeks the following is given: Three thin circular plates of the same size are on a plane each tangent to the other two, and a string is drawn tightly around the three. Construct a figure representing these data and, given the radius of a plate, find the length of the string; or given the length of the string, find the radius. By the middle of the year pupils are expected to find the area of the figure bounded by the string. In this work as in the regular work, knowledge gained by measurement and intuition is gradually replaced by that gained from algebra and geometry.

Examinations are made up in two parts. One part covers the minimum essentials of the courses. A passing mark may be gained on this list. A second list, "Star Problems," is made up of problems which have the characteristics of the bank problems. Thus there is an effort to reserve the highest mark for those of real ability.

Results.—The following table (Table A) shows the per cent of pupils failing to make a passing grade in Mathematics I before and after the beginning of the experiment.

TABLE A

Percent of Pupils Failing in Mathematics I.

1910	1911	1912	1913	1914	1915	1916	1917
44	41	40	56	no class	11	17	16

It is clear that before the experiment, from 40 to 56 per cent of the pupils failed in Mathematics I. Under the new practice the failures are reduced to from eleven to seventeen per cent. It has already been pointed out that these comparisons are made with different compositions of student groups.

Table B shows that a greater number of pupils elect Mathematics II under the reorganized course.

TABLE B

Percent of Pupils Electing Mathematics II.

1910	1911	1912	1913	1914	1915	1916	1917
50	30	32	25	28	65	69	70

This table tells us that in four of the five years preceding the change, the per cent of pupils electing was under 32 per cent. In the remaining year (1910) the election was as high as 50 per cent. Since the change the elections run from 65 to 70 per cent. Here again it must be pointed out that 32 per cent of the pupils finishing mathematics at the end of the first year in high school means something quite different from 32 per cent finishing the same course at the end of the second year.

Not only do a large number elect mathematics but the per cent of failures in Mathematics II is greatly reduced as is shown in Table C.

TABLE C

Percent of Pupils Failing in Mathematics II.

1910	1911	1912	1913	1914	1915	1916	1917
50	38	35	41	12	25	6	5

The table shows that before the change from 35 to 50 per cent of the pupils failed in the advanced course. Now it is as low as 5 or 6 per cent and only once has it been as high as 25 per cent. Possibly the statistics concerning withdrawal should be included here. However, the figures for the year before and the year after the beginning of the experiment may be sufficient to convince the reader that the facts of withdrawal would not seriously disturb any conclusions he may have drawn from the preceding tables. The withdrawals are rather high due to the shifting population of a newer state. All students who withdraw before a credit examination or after taking a credit examination and failing are counted withdrawals. Those who take such examinations successfully are counted passes. All who remain to the end of the year are either passes or failures. With this basis the total per cent of withdrawals and failures for the year 1913-1914 for Mathematics I was 51%, for Mathematics II, 31%; during the year 1915-1916 for Mathematics I was 12%, for Mathematics II, 29%. It would also be possible to present facts (not given here) to show that the per cent of students who elect the final year of mathematics work has about doubled. There is evidence that the plan makes for a more sustained effort and thus prevents the tendency of pupils to get a smattering of many subjects.

A simplified course is now the important factor in the experiment.—At first the dominant idea was to extend algebra and geometry throughout the two years with a radical change in the introduction to geometry but not much change in the complete courses. The idea of simplification from the old formal work and the introduction of new topics has evolved from year to year until at present it is the important factor in the experiment.

In algebra they give during the first year the fundamental operations, rarely exceeding trinomials; the linear equation in one unknown; negative quantities; special products in which the factors are binomials; factoring the binomial, the simple four-term and the trinomial; simple fractions; fractional equations in which the denominators are small numbers; linear equations in two unknowns; the quadratic that can be solved by factoring; and, if time permits, the simple quadratic.

About six weeks, in which algebra enters incidentally only, is given to intuitive and constructional geometry. The ruler, protractor, dividers, and graph paper are freely used. Much construction to scale and reading results by means of squared paper, including trigonometric problems in heights and distances and areas, is included in this work. This is followed by the simpler theorems and exercises generally included in Books I and II in a conventional text book though, in this more formal work, construction, measurement, and intuition are never entirely abandoned.

The second year's work consists of the use of logarithmic tables; square, square root, cube, and cube root tables; a review and slight extension of linear equations in one or two unknowns; quadratics; simple radicals; simple radical equations and simple simultaneous quadratics, in which one equation is linear. In geometry, selections are made from theorems and exercises in Books III, IV, and V. These selections depend somewhat upon the class, but those theorems of widest application, and exercises which require construction and arithmetic, are given preference.

At the beginning of this series of experiments, Mathematics I was placed in the second year. Under the old formal, rigorous methods of teaching, this seemed to be a real advantage. It seemed materially to decrease failures and hence waste. But under the present reorganization and simplification of subject matter and more rational methods of teaching this advantage is greatly lessened. It will require a further inquiry to know of what value it is at present.

It is important to note that the decrease in per cent of failures, the increase in per cent of elections, and the obvious rejuvenation of the mathematics work in this school is not necessarily to be associated to any considerable extent with the omission of mathematics from the ninth grade. There is at least one other significant causal factor, namely, the organization of new introductory courses evidencing such modern tendencies as are represented by the elimination of heavy formal algebra and the introduction of many new attractive features.

Parker High School, Chicago, Illinois

EXPERIMENTAL TEACHING

The experimental teaching done by H. O. Rugg and John R. Clark at the Parker High School during the year 1918-1919 furnishes an illustration of an effort to define more clearly the problem studied and to gain more adequate control through systematic record.[26] On the side of curriculum study is found also the recognition of the importance of investigation as a means of revealing desirable changes.

The technique.—(1) The "paired-teacher" system of instruction was one of the important devices used. Each teacher taught one or more classes of ninth grade pupils in a public school throughout the year under the immediate supervision of the other.

(2) Records were kept (a) of important difficulties met by pupils; (b) of methods of presenting various types of subject matter; (c) of the results of drill as shown, for example, by the scores on practice exercises; and (d) of scores made on verbal problems.

(3) Frequent conferences were held covering records taken of class work, the distribution of time and emphasis, the choice of method, the selection of appropriate content, and the like.

Learning studies.—The work throws light on particular difficulties encountered by pupils in various operations, on methods of rationalizing operations, and on effective ways of developing skill. The data support a conviction often expressed by well trained teachers that much time is wasted by present methods. It is asserted that the "paired-teacher" device contributed to this study. It is difficult for a teacher, however experienced, to carry out a well conceived plan, to readjust this plan and to maneuver his teaching materials to the particular reactions encountered, to extemporize new materials, and to keep a sufficiently reliable mental record of learning difficulties. A trained observer can make a needed record at the time the pupil meets the difficulty.

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[26] *Scientific Method in the Reconstruction of Ninth Grade Mathematics*, a monograph published by the University of Chicago Press.

Economy of time.—Some of the studies attempted to measure the economy of time in teaching specific processes as, for example, in teaching factoring. In this problem the experimenters tried to compare the relative efficiency of teaching factoring by the "case" method and the method of analyzing the general trinomial $ax^2 + bx + c$. It is generally recognized that pupils have difficulty in classifying problems when taught by the case method. Often these cases appear to be arranged so that the learning of one tends to inhibit the learning of some other type. It seems that this difficulty of classification can be avoided to some extent by beginning the study of factoring (after the monomial factor has been disposed of) by analyzing the nature of the expression $ax^2 + bx + c$. This of course is done by a drill on getting the product of two general binomials of the form $(ax + b)(cx + d)$. In the teaching, importance is attached to the middle term. The pupil can readily be taught to make classifications and to recognize the relationship between product and factors. Finding the product of two binomials does not build up a set of habits which inhibit learning to find the product of two other binomials. All this seems to be in accord with the important principle of learning that the natural method of learning proceeds from the undifferentiated, vague, complex mass of experience to the differentiated, detailed, classified and logically related. This means that we do not recognize the details, classifications and logical divisions and subdivisions until we have had adequate experience with the larger class or more general type. It is reported in this study that the experimenters were able to teach with satisfactory results in six lessons all the special products and factoring presented in a standard textbook (omitting the sum and difference of two cubes) and that the material and method of a standard algebra would probably require from four to six weeks.

Time gained in the use of practice exercises.—In this experiment much time was gained in the teaching of formal processes by the use of standardized practice exercises constructed to give facility in the specific skill taught. This part of the work indicates clearly the need of developing suitable practice material for all specific skills desired.

Comparative achievements of two sets of classes.—Two sets of classes were compared by standardized tests as to formal skill in the four operations, simple equations, special products, factoring, and exponents. The first set of pupils (55 in number) had spent about 20 per cent of their time on formal instruction whereas the second set (112 pupils) had spent at least 70 per cent of their time on such formal work, since they had followed a traditional textbook page by page with practically no deviation. The first set had, of course, spent more time on what is commonly considered more meaningful material and presumably had gained considerably in outcomes that as yet defy measurement. The following table shows the class achievements, by sections A, B, etc., on the four formal tests mentioned.

Achievements of Two Types of Classes.

TEST	Median Number of Examples Correct					
	Experimental Classes		Traditional Classes			
Operation	A	B	C	D	E	F
Special products	13.	12	8	8	9	8
Exponents	16	20	8	9	8	8
Factoring	8	9	5	6	5	5
Simple equations	8	7	6	6	6	5

It appears from this table that the "experimental" classes developed a greater skill in manipulation of these four operations than the traditional classes, though they had been subjected to far less drill.

Economy of time through elimination of non-essentials.— A second method used to economize time was the elimination of non-essentials. As a result of this work it is the opinion of Rugg and Clark that from one third to one half of the present course can be eliminated. It is suggested that at least four weeks can be gained on special products and factoring, two to four weeks on addition, subtraction, multiplication, and division, a week on highest common factor and least common multiple, at least one week, perhaps two, on fractions between one and two weeks on quadratic equations, and from one to two weeks on radicals. Thus it is asserted that from 10 to 15 weeks can be gained by the elimination of material that has relatively little value.

There are teachers who are convinced that the program suggested by the National Committee in Chapter III is richer in content than the traditional courses, but who have difficulty in seeing how all the new material suggested for the introductory courses can possibly be taught in the time available. Such teachers probably have the traditional courses in mind and these already appear crowded to the limit. It does not appear to them that new material can possibly be added, however attractive it may be. The solution lies partly in eliminating the waste material. If from ten to fifteen weeks can thus be saved these weeks can be used for instruction on more worthwhile material. It is possible to economize time by improvement in teaching methods, by the application of psychological principles. If all the factoring presented in the standard textbooks (excepting the factorization of the sum and difference of two cubes) can be taught in six lessons instead of from twenty to thirty, there is obviously more time for other material. Further experimentation is necessary in order to determine what additional gains can be made.

A curriculum for the ninth grade.—It will have been correctly inferred by the reader that the experimental classes were not taught by the use of a traditional text. It was the aim to have the course include among other items the following:[27] (1) Training on rational grounds in the representation of numbers by symbols, (2) training in translation not only from verbal material to algebraic symbolism but also from algebraic material to English equivalents, (3) evaluation, (4) the equation, (5) automatism in collecting terms, removal of parentheses, and handling of fractions, (6) constructing of formulas, (7) skill in manipulation of formulas, (8) measurement, (9) graphic representation as one of the two fundamental methods of representing numbers, and (10) functional dependence.

The Stuyvesant High School, New York City

WHAT MATHEMATICS SHOULD A BOY STUDY WHO CAN STAY IN SCHOOL ONLY TEN WEEKS?

Radical reorganization in a large public high school.—Stuyvesant High School is one of New York City's large public high schools. It has a mathematics department of more than 20 members. Teachers of mathematics often feel that changes cannot be made in a huge machine with the numerous administrative difficulties accompanying any effort toward reform.

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[27] See Rugg and Clark, Fundamentals of High School Mathematics, The World Book Co., Yonkers, N. Y.

For the last four years Stuyvesant High School has been using for the first four terms (half year units) a course radically different from the old established arrangement of a year of algebra followed by a year of geometry.

The problem: An introductory course of ten weeks.—The problem is clearly stated in the following: "Suppose a pupil can remain in school only ten weeks. What mathematics can we give him that will be of most value to him? If he leaves school and goes to work, he may desire to go to an evening school and study mechanics or physics. He may wish to read a mechanic's hand book and he may want to know how to use the formulas in the daily work. Whatever he may do for a living, he will find it very useful to know how to use the practical parts of elementary mathematics."[28]

This effort on the part of the Stuyvesant High School teachers of mathematics to construct a special introductory course of ten weeks that aims to include all the mathematics that can be justified on the ground of possible utility is applying the implication of the organizing principle stated in italics on page 14 with a vengeance. The principle is extended to mean giving the pupil "the most valuable mathematical information and training which he is capable of receiving" not only "in the first year" but especially in the *first ten weeks*. This policy seems to them more worth while than the general practice of spending the early weeks in review or preparation for later work. Moreover the principle is being applied under conditions that seem most difficult.

The aim of reorganization.—The controlling aim of the new course is—*to give the pupil the mathematics that is best for him whatever the length of his school course.* To this group of mathematics teachers this seems to make provision for the boy who can remain in school only ten weeks, giving him what he needs most, and at the same time not to neglect the proper training of the boy who will continue his studies. Moreover, it is desirable that these two boys be taught in a single course rather than in separate courses, since the former boy may be persuaded to continue in school.

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[28] Quotation from Mr. Breckenridge, the head of the department.

Special features of the course.—It is believed that a graded series of real problems captures the interest of the pupil at the start and holds it throughout the course. It is asserted that from the first lesson in which the boy measures the height of his class room (by the use of a simple isosceles right triangle cut out of pasteboard and from this makes his own formula) to the more difficult problem of laying out a baseball diamond (involving a quadratic equation), the growth in correct mathematical habits is rapid. The student's interest in doing things at the same time that he is learning to use letters as numbers appears to be much greater than under the older method.

The problems used are of such a general nature that they can easily be taught in any type of high school or in the last year of the junior high school. The teachers assert that at the end of ten weeks the student has completed all of the mathematics that can be justified on the ground of practical utility.

A course for ten weeks.—The course includes:

1. The ability to make a formula, use it, solve it for any letter, and interpret the result.

2. The knowledge of how to measure ordinary plane and solid geometric figures.

3. Practice in computation including the checking of all operations, until reasonable accuracy is secured.

4. The use of graphs and the solution of right triangles by natural functions.

5. A natural introduction to algebraic work through the motivation of the real problem.

6. Preparation for demonstrative geometry through familiarity with the material of the subject found in the mensuration of practical problems.

The course for the rest of the year.—The next best thing has seemed to be an extended ability to use the equation in concrete problems. Hence, the second ten weeks is devoted to the equations of elementary algebra with applications to concrete problems through quadratics, omitting most of factoring and fractions, all of exponents, and all of radicals. This completes the first semester.

For the second semester, plane geometry with its training in forms of reasoning seemed more useful than the more abstract parts of the algebra. Hence, this term is devoted to demonstrative geometry. In the third term geometry is finished and the State Regents' examination taken. The fourth term is used in completing elementary algebra, including the Regents' examination. Beyond the fourth term mathematics is elective and includes the usual subjects required for college entrance, together with a course in surveying and the use of the slide rule in every class in trigonometry.

Results.—The results of this course as seen by the mathematics department are as follows:

1. There is much more mathematical insight and joy in the work than by the older method.
2. From a series of real problems properly organized, better training can be secured than from the ordinary abstract algebra.
3. The Stuyvesant Plan is an easy and natural method of introducing the pupil to the use of letters and numbers in algebra.
4. It is safe as far as Regents' examinations are concerned, as the records show.
5. Stuyvesant has tried to make the teaching of mathematics democratic,—that is, of the most use to the most people.

A desirable option: A course in surveying and the slide-rule.—This course is given in the fourth year, five periods a week for twenty weeks. Every student electing the course is obliged to take three hours per week of field work in the afternoon. This subject may be elected by boys who are taking trigonometry or have completed that subject.

The content of the course is equivalent to the first course in surveying in Columbia University, and is so credited by that university. The work in the slide-rule occupies about two weeks of time and includes a thorough drill in the operations performed on that instrument. The textbook in surveying is Tracy's Plane Surveying. For the slide rule, the manual published by Keuffel & Esser, New York City, furnishes sufficient material for the work of the course.

At the conclusion of this course a student is fitted for the usual field work, mapping, or computation of a surveyor's office. From this course entrance is easy into the City, State and Federal Civil Service in the capacity of chairman, rodman, or computer. Many of the graduates have found that this course has fitted them for good jobs in which they earned reasonable salaries.

The University High School, Columbia, Missouri

ALGEBRA WITHOUT THE USUAL TRAINING IN ARITHMETIC

Principles.—The University Elementary School and the University High School connected with the University of Missouri have been engaged in extensive efforts to improve their curriculum. In particular, an effort has been made to adjust the curriculum to the needs of children rather than to the needs of the community, the demands of the colleges, the interests of adults, and the like. A background for the work in mathematics which has been done in these schools can best be obtained by quoting from a recent book entitled "Child Life and the Curriculum" by Professor Junius L. Meriam. These quotations are used in the book as themes around which the chapters are built.

> Principle 1. The curriculum should contribute primarily to enabling boys and girls to be efficient in what they are now doing and only secondarily to preparing them to be efficient later.
>
> Principle 2. The curriculum should be selected directly from real life, and should be expressed in terms of the activities and the environment of people.
>
> Principal 3. The curriculum should provide for great scope and flexibility to meet the individual differences in needs and abilities.
>
> Principle 4. The curriculum should be so organized that it will admit of easy rearrangement of the schedule for any of the work for any grade, and even to the transfer of work from grade to grade.
>
> Principle 5. The curriculum should lead the pupil to appreciate both work and leisure, and to develop habits of engaging in both.

If these principles sound reasonable and commonplace it must not be assumed that they are often applied in practice, not even in the progressive schools. It is Professor Meriam's contention that the acceptance of these principles requires the abolition of the traditional subjects, reading, writing, arithmetic, languages, geography, history and the like. The Elementary School which is an exponent of this educational ideal proposes to organize its curriculum about four studies, namely *observation, stories, play,* and *hand work.*

No formal arithmetic.—The reader will want to know what form this philosophy takes in the teaching of arithmetic and secondary school mathematics. Concerning arithmetic, the atmosphere can be cleared by saying that it is believed that the best way to teach arithmetic is not to teach it! By which is meant that the most effective way to give boys and girls a real concept of arithmetic problems is not to teach arithmetic as a formal subject, but to teach it on the basis of everyday life in which arithmetic calculations play a part. Moreover, it is believed that it is of great importance that arithmetic be not kept in mind while doing that kind of work. The emphasis is to be placed on the problems of everyday life in which quantity plays a part. For example, in the games of the smaller children, and in such a project as "ways and means of transportation" for the fifth and sixth grade in their study of industrial activities, a considerable amount of quantative arithmetic calculation is involved. But there is an insistence that the emphasis should be upon the activity rather than upon the arithmetic operation.

In consequence, it is asserted that there is no formal training in arithmetic at the University Elementary School. The question naturally arises,—"How well do these pupils do their work in traditional high school mathematics if they have not had the usual training in arithmetic?" On this question it is possible to submit significant statistics. The tables that are about to be presented include the grades made by one hundred pupils who completed the course in the University Elementary School. Of these, fifty-five entered the Columbia City High School, and forty-five entered the University High School.

Table A shows the percentage distribution of grades received by graduates of the University Elementary School (line E) in later years as students of the Columbia City High School, in comparison with the distribution throughout the high school. Percentages in the lower line, C, represent the grades of the city school:

TABLE A

[29] DISTRIBUTION OF GRADES IN MATHEMATICS

	E	S	M	I	F
E	2.9	39.1	30.6	18.9	8.4
C	0.0	22.9	24.9	29.3	22.6

This table seems to show that the pupils trained in the University Elementary School are equal to carrying the traditional work of a public high school. Irrespective of any comparison with the standard set by the city school, it is easily read in the table that pupils from the University Elementary School get many high grades and comparatively few low grades. But the most satisfactory measure here is that by which comparison is made with the work of students with quite a different previous training. The comparison is strikingly in favor of the graduates of the University Elementary School. This situation suggests that arithmetic in the grades is not essential as preparation for mathematics in the high school.

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[29] The five marks indicate differences in quality of work with E (excellent), S (superior) at the high end of the grading.

Table "B" shows a similarly favorable comparison for the forty-five graduates of the University Elementary School who entered the University High School. Table B shows percentage distribution of grades received by graduates of the University Elementary School in later years as students in the University High School in comparison with the distribution throughout this high school. Percentages in the lower line U represent the general distribution of grades in the University High School:

TABLE B

Distribution of grades in mathematics

	E	S	M	I	F
E	16.8	21.5	55.1	5.6	0.9
U	8.5	19.0	52.5	15.8	4.4

Again it may be noted that these graduates do especially well in mathematics, without having had arithmetic in the Elementary School. The percentage of high grades is high and the percentage of low grades is very low.

A further inquiry might appropriately be made as to whether those pupils who had most of their grade work in the University Elementary School do better than those who transferred from the traditional city school. Such figures as were available seem to show clearly that those pupils who took practically all their grade work in the University Elementary School were better prepared for high school work than those who had done their work in the traditional schools involving arithmetic drill. It seems that one group of 29 pupils had their preparatory work in the Elementary School, whereas another group of 44 pupils came in at various stages. It also appears from the records of the high school office, that the 29 who had all their work in the Elementary School did on the whole better work than the 44 who were added from time to time.

To most teachers of mathematics this theory and evidence may seem strange and, perhaps, disturbing. Indeed, the conservative teacher will raise the question whether the children do not in fact receive a good deal of drill in spite of the efforts of those who are responsible for the philosophy of the school. But that need not necessarily be the case. At least three factors arise of which any one or all may contribute towards the making of good records in high school by children without formal instruction in arithmetic.

1. How much arithmetic do people learn incidentally through their social relations in school, in the home, at recreational affairs? The number of business men (self-made men), with practically no training in arithmetic, who nevertheless are keen and efficient in the arithmetic of life, is significant. This is the issue which Meriam and his followers raise. The problem needs careful investigation and critical analysis by an unprejudiced group.

2. The second question is how much of the material ordinarily emphasized in arithmetic is really needed? Many teachers insist that much time in arithmetic is taken up in wasteful reviews and in the teaching of meaningless processes. This matter, too, cannot be settled except by a carefully measured experiment. Some light may be obtained by giving a comprehensive inventory test to business men, professional men, and to freshman students in college, so as to ascertain what information, skills, and arithmetic powers are retained and what bonds have been weakened by disuse. If a considerable part of arithmetic is useless, then naturally there would be no great difference when pupils go to their algebra without this subject matter.

3. The third question is how much arithmetic (independent of utilitarian values) does a pupil need to know to succeed in algebra? It may turn out that success in algebra depends less upon success in arithmetic than is commonly believed. This also requires investigation.

The University High School, Minneapolis, Minnesota

A GENERAL MATHEMATICS COURSE

The University High School attempts to meet the three needs of a laboratory for the School of Education of the University of Minnesota, in that it (1) offers practice to teachers in training, (2) demonstrates what it accepts as desirable curricula and methods, and (3) strives through experiment to improve existing school practice.

The school's student body appears to be representative of the various social groups of the community to an unusual degree. It is fortunate also in that the various public and private schools in and about Minneapolis offer wide opportunity for coöperative experimentation. Both are distinct assets in work that a school of this type attempts to do.

The mathematics department of the University High School has been in charge of W. D. Reeve. In coöperation with Mr. Raleigh Schorling and many other teachers of mathematics, this group began in 1915 to determine the content of an introductory course for the first year of a standard four year high school course.

The most important problem.—It seemed that the mathematics of the first year was the problem that demanded immediate attention. Important considerations were: (1) The mediocrity of the content of a first year algebra course in the way of giving pupils an understanding and appreciation of what mathematics is for; (2) the insufficient mathematical training received by the large number of students whose course terminates with one year of mathematics, a condition which results from wasting time in teaching useless manipulation of symbols in algebra and delaying to the advanced courses the teaching of many important, interesting, and useful principles; (3) the indefensible number of failures in the first year; (4) the extent to which pupils fail to enjoy and appreciate (of course the unusual students and the classes taught by especially good teachers are exceptions) one of the fields of human learning which is rich and vitally concerned with a fundamental human interest; (5) the extent to which pupils fail in the plane geometry course when required to take it with inadequate preparation; (6) the probability that children learn mathematics as they do other things, through varied purposeful experience with something of a beginning in the technique of the investigator and the scientist and not by the acceptance of a system of rigidly organized and mature principles; (7) the failure of correlated courses which incorporate conventional algebra and demonstrative geometry to meet the needs of first year students; and (8) the demand that first year mathematics contribute here and now toward making more meaningful the pupils' experience in other school subjects and in life out of school.

Pupils and teachers.—The method used in constructing this course rested on the conviction that teachers had something to do with the selection and pupils something to do with the organization of the materials of instruction. In consequence a large amount of material, drawn from the simple and significant parts of algebra, intuitive geometry, arithmetic, trigonometry, together with certain "optional" materials, e. g., statistics, logarithms, and the slide-rule was prepared. This was used in mimeographed form (approximately a quarter of a million pages) for four years under many kinds of school conditions. On the basis of the "follow-up" conferences with teachers, not only problems but whole sections and chapter units were omitted, modified or added.[30]

These efforts resulted in a first year course from which it is thought the following results may reasonably be expected: (1) On the side of subject matter, (a) control of the simple and important parts of algebra and geometry, (b) an introduction to trigonometry and statistics probably adequate for common needs, (c) greater facility in the use of fractions (common and decimal) and percentage relations, (d) training in the use of a number of "optional" topics, e. g., logarithms, the slide-rule and tables; (2) the considerable reduction of the number of first year failures; (3) increase in the number of pupils taking a subsequent course in plane geometry with interest and profit; (4) more intelligent election of later courses in mathematics; (5) more adequate preparation for the mathematical needs of other school subjects, industrial arts, household arts, physics, chemistry, and the like; (6) a beginning on the part of the student in the technique of investigation (many parts of the material are organized in laboratory form); (8) greater power in problem solving through the use of more methods of attack (correlation is used as a means and not as an end); (9) a clearer notion of the relationship of various mathematical methods without confusion (there is no attempt to correlate plane geometry with material from other fields); (10) a better understanding of algebra as far as it goes (this is probably well within the limits of ordinary life needs); (11) an appreciation and understanding of the importance of the idea of relationship (function concept) and; (12) greater enjoyment in the study of mathematics.

The Trenton Junior High School, Trenton, New Jersey

The Project Method

The social-economic arithmetic of the Parker family.—Much of the social-economic part of arithmetic is taught to the seventh grade children through the activities of the "Parker Family" consisting of father, mother, two boys and one girl. The Parker family needs to arrange a family budget by assigning various per cents of an $1800 income to food, rent, operating expenses, clothing, and higher life. Their decisions need to be checked by comparison with standards. The daily and monthly expense account needs to be adjusted to fit the budget. Graphic representation is used in this study. The topics of rent, interest, and profits arise in the discussion of the feasibility of building a new home for the Parkers as also do taxes, insurance, and repairs. The plan of the house and the lot involves linear and square measurement and drawing to scale. The measurement of volume comes in the planning of a coal bin, potato bin, and cistern. Very much of the remainder of the arithmetic appears in connection with furnishing and decorating the children's rooms, reading the gas meter and the electric meter, planning a Christmas dinner, thrift talks, borrowing money, paying on the installment plan, buying in quantities, and the like.

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[30] For the list of co-operating schools and the outcome of this work see Schorling and Reeve, *General Mathematics*, Ginn & Co.

Real problems.—The course includes an unusual number of real problems. As yet it is only available in mimeograph form.[31] The outline of the course includes suggestions for the teacher which cover both content and method.

Need for a controlled experiment.—It is obvious that the philosophy of the course is markedly different from that of the conventional arithmetic and also from that of any new junior high school seventh grade book. The need is great for a controlled experiment to test the relative values of such a course which differs so widely in content and method of teaching. For example it ought to be possible to find out to what extent the Trenton course composed of material easily defended from a social point of view can be made interesting and vital to seventh grade children who usually lack the social background to appreciate and understand the importance of problems dealing with taxes, insurance, budgets, and the like. If the course does operate successfully, what contribution is made by the project method of teaching?

It may be helpful to the reader to read the series of problems which the pupils and teachers collected while studying the rent project.

Sample Lesson: Rent

1. Does your father own the house in which you live? If so, does it cost him anything? How much does he pay for taxes, fire insurance, repairs?

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[31] This course has been developed by Dr. William Wetzel and others.

2. Is it fair to say that this is all that it costs him? How much would he get for the house if he sold it? How much would this money earn for him annually, if he loaned it to some one?

3. What are the four chief items of cost in owning and maintaining one's home? (Taxes, insurance, repairs, and interest.)

4. If some one gave Mr. Parker a house worth $50,000, should he move into the house, or sell it? Give reasons for your answer.

5. In what respects does the man who owns his home really pay rent? (Interest, taxes, insurance, repairs.)

6. If your father owns the house in which you live, what must you know and what must you do to find out how much it costs per month to live in the house?

7. Mr. Parker rents a house worth $4200. Determine approximately what his monthly rent will be.

(Upkeep on a house averages about 2% per annum after the first five years. Insurance will average a little more than $1 per thousand dollars of insurance per year.)

8. Mr. Parker pays $30 per month rent. The landlord pays annually for taxes $105, for repairs and insurance $87. How much net profit does the landlord receive annually for his house?

9. The house cost the landlord $4800. What rate of profit does his house pay? (Use formula.)

10. A man who is paying $20 per month rent considers buying a house for $3000. He has $1000 in bank which pays him 3% interest. How much money would he have to borrow? If he borrows this amount at 5%, how much would this cost annually? What is the entire interest cost of his house annually?

What other cost items must he know before he can determine what it would cost him to live in this house?

11. The taxes on the house are $51 per year, repairs average $60 per year, and the insurance costs $2.75 per year. How much would it cost him per month to live in this house?

12. Would it be cheaper for him to own this house, or to pay $20 per month rent for the house in which he is now living? What do you think the man decided to do? Why? (Discuss advantages and disadvantages of owning one's home, aside from financial considerations.)

13. A man has $2000 in bank drawing 3% interest. He invests this in a house costing $5000. He borrows the balance of the money and pays 5% interest. The repairs cost $90 per year, taxes $85 per year, and insurance $4.40 per year. Should he buy the house?

14. A man is paying $15 per month rent. He can buy the house for $2000. Money is worth 5%. Taxes are $38.25, insurance $2.20, repairs average $40 per year. Should he buy the house?

15. Answer the same question for the following house: Rent $45 per month. Value of house $8000. Taxes, $153, insurance $6.60, repairs $125.

The Washington Junior High School, Rochester, New York

"PROGRESSIVE" CORRELATION

Organization of the school.—Many of the general features of the Washington Junior High School and of its mathematics work have been described in detail elsewhere.[32] The seventh, eighth, and ninth grade work for that section of the city is centered in the Washington Junior High School. There are seven contributing elementary schools with East High School as the outlet.

The pupils.—The problem of organizing the new institution was influenced in a unique way by the location of the school building and by the type of pupil whom it desired to serve. A vast majority of the children of the district come from working classes of foreign parentage. The 1673 children reporting during the year (1919-1920) represented as many as twenty-two different nationalities, distributed as shown in the following table:

Russian	633	37.8%
American	396	23.7%
German	236	14.1%
Italian	180	10.8%
Polish	87	5.2%
All others	141	8.4%

Economic factors.—It was found that the average family income of the school constituency was between $125 and $150 per month. This means some economic stress because of the large families usually found in that district. The bearing of that situation on the function of a junior high school is obvious. Before the Washington Junior High School was established, the percentage of elimination at the end of the eighth grades of the seven elementary schools of the district was as high as 50%. This fact, as well as the unfortunate home conditions discovered in numerous cases, suggested a type of curriculum that should give due weight to early industrial and commercial opportunities. It also became necessary to create a mechanism of supervised study, thus automatically introducing the long school day. Moreover, it was seen to be essential to adopt a most detailed plan of vocational guidance, and to provide "prevocational" and "ungraded" vocational instruction, through a system of "try-out" shops and laboratory activities.

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[32] Superintendent H. S. Weet's reports were published in The School Review, 1916, pp. 142-151, and in Educational Administration and Supervision, 1916, pp. 433-477. See also *A Study of the Junior High School Project,* Education, 1916, pp. 1-19, by J. Abelson. The most recent account is that by R. L. Lyman, The School Review, March, 1920, pp. 178-204. In particular see The Teaching of *Mathematics in the Junior High School,* The Mathematics Teacher, Dec., 1917, pp. 58-84, by William Betz.

Pupil intelligence and classification.—The intelligence of the children naturally revealed very great extremes. Mediocre pupils have been noticeably numerous, especially in the ungraded vocational work. A general intelligence test (Otis) given during the year 1919-1920 to the 6th A classes of the seven contributing elementary schools showed a high score of 127, a low score of 15, while the median was 63. It seemed imperative, from the beginning, to attempt a classification of pupils on the basis of ability and of vocational interests. A study-coach system also was developed.

Pupils who have failed in one or more subjects are placed in "non-promotion groups"; pupils who are in danger of failing are put in "failure prevention groups"; and retarded seventh-grade children coming from the elementary schools enter one of the "trial promotion groups." In short, the desire is to make the school a real educational center and to open the door of opportunity to "all the children of all the people."

The teachers and their training.—The important problem of getting junior high school teachers is facing many cities. In the Rochester plan we have a definite suggestion. First of all the city has a supervisor of junior high school mathematics. It then proceeds to train good grammar school teachers in content and method by courses given in extension work or Saturday institutes. There are seven teachers giving their full time to mathematics. All have had at least four years' experience. They were selected from those doing departmental work in the grammar schools, after a special year of preparation by means of Saturday institutes, supplemented by extension courses at the University of Rochester. All have had courses in intermediate and advanced algebra, plane and solid geometry, trigonometry, and the elements of the calculus, given at the University of Rochester.

Supervised study.—Supervised study has been in operation since the opening of the school. The organization provides for the divided period. There is no required home study. Pupils may request special assignments because of absence or illness or a particular interest in any topic. The period is divided into three parts: Review, 1/5; development of lesson, 2/5; study, 2/5. Papers are checked in class at the close of the study period, or privately by the teachers.

It is the opinion of the teachers that a fixed time allowance for study, even under supervision, does not seem to take care of individual differences. A differentiated assignment is almost imperative. The chief danger appears to be that slow pupils learn to depend on the teacher under all circumstances and that accelerated pupils become accustomed to a minimum standard.

Time devoted to mathematics.—There are five recitations in mathematics each week, during the entire term of three years, in the academic courses. Including supervised study, the normal length of each period is 80 minutes. The following table shows the weekly time allowance during each semester:

Seventh grade	320 minutes
Eighth grade	320 minutes
Ninth grade	400 minutes

Method of experimentation.—An outline of a course in "general mathematics" has been developed and directed by Mr. Betz, the head of the mathematics department of East High School (the senior high school of the district). He tried out new materials in one or two daily "observation" classes for three years. This work was always done in the presence of the regular classroom teacher, who furnished a written account of each recitation. These daily written reports were duplicated, and copies were given to the other teachers for criticism and subsequent revision. The material which was accumulated in this way served as a first basis for a tentative outline of a course of study. Each teacher was given the opportunity to repeat and to modify the "lessons" that grew out of these class room experiences. In general, the teachers were left free to use their own judgment as to the extent to which it seemed best to follow the typewritten "lessons" in any given class.

Teachers were urged to submit an accurate *time schedule* of the work done in their own classes. The time element was considered to be one of the most crucial items of the whole situation. These daily or weekly time budgets, furnished by the teachers during a period of two years, reflected actual school room conditions and made possible the distribution of emphasis.

Subject-matter. The problem of "general mathematics."—The mathematics teachers at the outset faced the problem pertaining to the manner in which the principal mathematical ingredients of the curriculum should be interwoven and correlated. The old system of water-tight compartments seemed out of the question to them. In any case, how soon, at what points, and for what specific reasons, is correlation desirable? It was believed that an intelligent answer to these questions presupposed, (1) a definite *tabulation* of the main topics of each subject; and (2) a set of pedagogic criteria for determining the most profitable *sequence* of topics.

The list of topics.—The preparation of a list of topics limited to the minimum essentials[33] resulted in the following:

Arithmetic	Geometry	Algebra	Trigonometry
1. Vocabulary	1. Forms and ideas	1. Vocabulary and use of symbols	1. Vocabulary
2. Computation (Technique and applications.)	2. Measurement (Segments, angles, areas, volumes.)	2. Formula (Function)	2. Metric relations
	3. Relations	3. Graph	3. Indirect measurement by computation
	4. Constructions	4. Equation	
	5. Informal proofs	5. Generalization of processes (Technique and applications.)	

Criteria for the selection and arrangement of topics.—At the Washington Junior High School the following guiding rules are considered valuable:

1. No principle or idea should be emphasized which does not have inherent mathematical or practical value.

2. A topic should not be introduced until there exists a proper apperceptive background.

3. A new principle should not be introduced if its actual use would be too long deferred.

4. Not more than one topic should be presented for *intensive* consideration at any given time.

5. Enough attention must be given to each important topic to secure a fair degree of assimilation. Certain processes must be made automatic.

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[33] Applied problems are to be introduced wherever possible, in all these fields.

6. Opportunity must be given for cumulative reviews and organic summaries.

7. The best moment for presenting a new topic is that which offers the greatest economy of time and effort on the part of teachers and pupils.

Some limitations.—These general considerations were subject to the following limitations:

(1) Due weight had to be given to the immediate vocational needs of the school population.

(2) It seemed unwise, at least during the initial stages of the experiment, to burden the teachers with too complicated a mathematical program.

(3) The management of the school wished to follow a safe policy and decided to restrict the mathematics of the seventh grade to arithmetic, since rapid and accurate computation was regarded as of paramount importance to the children. Geometry could, therefore, not be introduced before the beginning of the eighth year. Algebra was assigned to the second semester of the eighth year, while at the same time geometry and arithmetic were continued.[34]

(4) The New York State Syllabus of elementary algebra imposed certain limits upon the correlated type of mathematics which was attempted during the ninth year.

It will be seen, then, that the plan actually followed is one of *progressive* correlation; in other words, each subject receives the major emphasis for a time, and is then kept growing, as occasion demands.

Recent changes.—The program described above needs considerable modification in order to reflect recent extensive changes in the Rochester plan as may be seen from the following note received from Mr. Betz just after this report was completed:

"Since that time (June, 1920) important changes have occurred in the program outline above. In January, 1921, intuitive geometry was introduced in three experimental seventh grade classes. The results were so satisfactory that in September it was decided definitely to make intuitive geometry a seventh grade requirement. It is now taught in all seventh grade classes at the rate of three periods per week of 30 minutes each throughout the year. This readjustment will affect the other two years very materially. Algebra will be begun earlier in the eighth year, more attention will be given to the geometry of solids and to trigonometric ratios, and the work of the ninth year will be enriched."

This plan is believed by the Rochester teachers to offer these advantages:

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[34] But see the next section on *Recent changes.*

1. It agrees more nearly with the historic development of mathematics.

2. It represents a good psychological and pedagogic sequence. Thus, geometry should precede algebra, because it is concrete and because it furnishes an admirable field of application for the algebraic symbolism.

3. It makes for economy of effort in presentation and assimilation.

4. It assigns a very definite new unit of work to each semester, which stimulates interest at critical points.

5. It has administrative advantages in connection with the grading and the reclassification of pupils.

The courses.—The topics listed below represent the usual achievement of the eighth grade during the last four years. Presumably the outline for all grades will be greatly modified in the direction toward general mathematics for grades seven, eight and nine, as may be noted in the quotation above. The outline of the seventh grade is omitted for the reason that although certain important changes are now being made, the course has to date been a conventional arithmetic course. The ninth grade is still being reconstructed experimentally.

Eighth Grade

First Semester

Geometry[35]

1.	Historical Introduction	5	recitations.
2.	Classification of forms	5	"
3.	Preliminary designs	3	"
4.	Straight Lines. Measurement of segments begun	5	"
5.	Circles	8	"
6.	Angles. Applications of angle measurement	12	"
7.	Triangles	10	"
8.	Polygons. Constructions and designs	8	"
9.	Parallels and applications	6	"
10.	Symmetry and related constructions	10	"
11.	Congruence	5	"
12.	Areas of geometric figures, both simple and composite	5	"

(This topic is continued throughout the year. Simple surveying exercises and scale drawing are introduced as often as possible.)

Arithmetic. Daily drill or review, as well as an extension of the work outlined for the seventh grade, continues throughout the semester.

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[35] See *Geometry for Junior High Schools*, by William Betz. It is a brief laboratory Manual published by the Board of Education, Rochester, N. Y.

Eighth Grade

Second Semester

Algebra.[36] Geometry forms the background for the work in algebra. Arithmetic is continued and is correlated to a large extent with algebra. The idea of estimating and approximating results is emphasized.

1. Historical introduction	5 recitations.
2. First principles	20 “
a. Use of letters	
b. Similar terms	
c. Simple formulas	
d. Evaluation (graphs begun)	
e. Rules of addition and subtraction. Similar and dissimilar terms	
3. The simple equation	25 “
4. Problems	15 “
5. Graphs, formulas	10 “
6. Multiplication (areas and volumes)	10 “
7. Square root 8. Pythagorean Theorem and applications	10 “

Ninth Grade

Correlated Mathematics. The course is still in the process of construction. The topics prescribed by the State are usually included. A beginning has been made in trigonometric work. It is rapidly becoming an integral part of the course. Exponents have been used for purposes of computation. The indications are that simple trigonometric computations can easily be incorporated.

Summary of results as seen by the Rochester teachers.— 1. The failures in mathematics varied from 10% to 11% in the period from June, 1916 to January, 1919. Unfortunately, statistics concerning withdrawal from mathematics classes were not included. It is safe to estimate that the number of failures is less than half the number of failures for the country generally. Here we need to keep in mind (1) that all pupils are required to take mathematics, (2) the unusual linguistic problem, and (3) the type of students.

2. The elimination statistics are equally encouraging. Not only do the majority of the 8th grade pupils go into the 9th grade, but the percentage of graduates entering the senior high school is extremely gratifying, ranging from 65% to 85%. It will be recalled that the

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[36] See *Introductory Algebra Exercises,* by William Betz. It is a laboratory manual published by the Board of Education of Rochester, N. Y.

figure before the organization of the Junior High School was 50 per cent. Naturally the credit for this gain can only be partially ascribed to mathematics or even to the organization of a Junior High School. During a recent semester two parallel classes were reorganized into two groups of average and slow pupils; the slower pupils receiving additional attention. The experiment was considered a very satisfactory one, but the department would like to go a step farther,—reorganizing into three groups, thereby giving the brightest pupil his chance. If a proper grading of pupils can be effected, it is hoped that superior classes will have a more extensive curriculum. Among possible optional topics are mentioned the study of logarithms, the use of the slide rule, simple trigonometric computation, more significant constructions, more complicated projects, and, perhaps, a first contact with demonstrative geometry.

3. Standard tests in algebra have yielded good results. From the data it appears that the school ranks with the upper third of four-year high schools that have applied these tests.

4. Pupils transferring to the senior high school experience no unusual difficulty in the mathematical work of this school. The mathematics teachers report good retention of the composite material taught in the junior high school. The longer "exposure" to geometry and algebra seems to develop a far greater degree of automatic response in many situations than is ever possible in the usual curriculum. Certain principles and processes had, in fact, become firmly implanted. This proved of advantage in many lessons.

5. *Attention to the appreciative aim of mathematics.* Finally, the teachers feel that the reorganized course gives attention to the cultural values of mathematics. The following statement prepared by Mr. Betz is included with the hope that it may help to make clear what is meant by the appreciative values of mathematics.

> What is mathematics? It is difficult if not impossible to give a brief and concise definition of this many-sided branch of learning. The mathematical sciences have ever maintained a close relation to philosophy. They have assisted in the continual endeavor to explain this world of a myriad ideas and of countless bewildering phenomena as a unified system controlled by law. The mathematician really tries to reduce all thinking to its ultimate bases and to comprehend the physical universe by means of a few fundamental concepts such as mass, energy, space, and time. In short, it is the passion for infinite harmony in a world of apparent chaos that has caused the mathematician to go on and on in relentless endeavor.

Every mathematical formula represents a storehouse of information, erected by ceaseless toil. Arithmetic, geometry, and algebra have made possible this storehouse; they furnish the key that unlocks its doors and releases the treasures of the past. To comprehend, even in a rudimentary way, that this universe is a world of law; to experience the economy and joy of ordered thinking; to sense, however slightly, the eternal symmetry implanted in the geometry of form; to see how a systematic body of universal principles is discovered and demonstrated—surely this is an ethical equipment, and a training in appreciation, of incalculable importance and beauty.

Needless to say, all the aims mentioned in this report are realized only very imperfectly in the ordinary classroom. In the standard four-year curriculum with its one year of algebra and one year of geometry, almost exclusive attention is given to formal technique. Rarely, if ever, does the pupil suspect the existence of a rich body of applications, not to mention the higher considerations referred to above. Lack of time is the invariable excuse given for this usual poverty of content. Hence, if the junior high school is to justify itself in the eyes of the scholar and of the general public, along mathematical lines, it must so administer its subject-matter and its methods, and must make such a time allowance, that the higher mathematical aims will also have a chance to come to light.

The Mary C. Wheeler School, Providence, R. I.

MATHEMATICS IN A GIRL'S SCHOOL

The Mary C. Wheeler School is unique of its type. Private tuition schools for girls (children of manufacturers, business, and professional men) are not expected to fight in the front trenches of an educational advance and never must they be caught wandering in the uncharted areas of a "No Man's Land" in education. It is then all the more gratifying to consider an exceptional girls' school. The school recognizes its clear obligation to prepare its girls for college, but it is unwilling to admit that college entrance requirements prevent the giving of worth while mathematics courses in the early years of the secondary school. To be sure a plane geometry[37] course is given in the eleventh school year and a formal review, designed especially to meet college entrance examination needs, in the twelfth year; but outside of this concession, the teachers try to forget college entrance examinations and put their minds on meeting life needs of pupils through the grades seven to eleven. It appears that the courses not only prepare pupils to get into college but to *stay in*. Besides a satisfactory college record, the mathematics classes have the credit of only a little over 3% failure and 3% withdrawal (1919-1920).

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[37] Discontinued however this year (1921-1922).

Junior high school courses.—The seventh and eighth grade courses are required of all girls. All are advised to take the ninth grade course. Beyond this the work is elective so far as the requirements of the school are concerned. The extent of the reorganization is shown in the outline of grades seven and eight which follows:—

THE SEVENTH GRADE

1. Review of fundamental operations in fractions (common and decimal).
2. Percentage relations and applications.
3. Practice in rapid calculations (drill cards).
4. Short processes.
5. Review of denominate numbers.
6. Mensuration.
7. Proportion.
8. Square root (Pythagorean theorem).
9. Simple graphing of statistics.
10. Simple concepts of congruence, similarity, and symmetry.
11. Informal discussion of angle, triangle, and circle.
12. Manipulation of geometric instruments.
13. Formulation of formulas from given facts.

THE EIGHTH GRADE

1. Review of intuitive geometry (already studied).
2. Intuitive geometry continued.
    - a. Simple geometric constructions and applications.
    - b. Indirect measurement.
    - c. Simple treatment of symmetry and loci.
    - d. Models in solid geometry made.
    - e. Planes discussed briefly.
3. Elementary ideas of trigonometry.
4. Demonstrative geometry.
    - a. Congruency theorems.
    - b. Parallel lines and related angles, angles of parallelogram and trapezoid.
    - c. Sum of angles in triangle and quadrilateral.
    - d. Isosceles triangle and equilateral triangle and problems based on them by drawing bisectors, altitudes, etc.
5. Algebra.
    - a. Positive and negative numbers.
        - i. Represented graphically.
        - ii. Fundamental operations (geometric and algebraic).

- b. Equations.
  - i. Linear in one unknown.
  - ii. Linear—two or more unknowns.
  - iii. Quadratic (by factoring and by completing the square).
  - iv. Ratio, proportion, and variation.
- c. Special products and simple factoring; types of factoring taught are monomial factors, binomial, difference of squares, trinomial square, and trinomial to be factored by trial method.
- d. Fractions.
  Simple fractions introduced in equations and dealt with by axioms.
- e. Graphs.
  - i. Statistical.
  - ii. Representing dependence.
  - iii. Solution of simultaneous linear equations.
- f. Formulas; meaning and use, and deriving one form from another.
- g. History of mathematics.

The ninth and tenth years (for three periods per week) are devoted to a correlated course of alegbra and geometry leading gradually to the formal treatment of plane geometry given in the eleventh school year.

From the preceding seventh grade outline it may be observed that at least six topics not ordinarily covered in seventh grade mathematics are included, namely:— (1) Graphing of statistics, (2) congruence, similarity and symmetry, (3) informal proofs of facts concerning the triangle and the circle, (4) geometric instruments, (5) construction of formulas from given facts and, (6) percentage problems solved by the use of formulas. These are added in the belief that they contribute toward making a child's reading and general contact with his surroundings more interesting and meaningful.

It would also be possible in similar fashion to enumerate significant departures in the eighth grade work from the conventional course. These are made in the belief that seven years can be made to cover the necessary arithmetic and the introduction to algebra. The teachers state that geometry and trigonometry in the eighth year stimulate the child's interest in mathematics and even though he leaves school early, these give him a chance to survey a fairly large part of the field of mathematics.

APPENDIX

References

The name of one person for each experimental course is given. Presumably the reader may gain additional information concerning any specific question from the person concerned.

1. Harry M. Keal, Detroit, Michigan.
   The Cass Technical High School.
2. Ernst R. Breslich, Chicago, Illinois.
   The University of Chicago High School.
3. William S. Schlauch, New York City.
   The High School of Commerce.
4. W. F. Downey, Boston, Mass.
   The English High School.
5. Matilda Auerbach, New York City.
   The Ethical Culture School.
6. Vevia Blair, New York City.
   The Horace Mann School.
7. Raleigh Schorling, New York City.
   The Lincoln School.
8. Tom Gladstone Rodgers, Las Vegas, N. M.
   The High School of the New Mexico Normal University.
9. John R. Clark, The Lincoln School, New York City.
   Formerly of The Parker School, Chicago.
10. William E. Breckenridge, New York City.
    The Stuyvesant High School.
11. Junius L. Meriam, Columbia, Missouri.
    The University High School.
12. William D. Reeve, Minneapolis, Minn.
    The University High School.
13. William A. Wetzel, Trenton, New Jersey
    The Trenton High School.
14. William Betz, Rochester, New York.
    The Washington Junior High School.
15. Mary Helena Dey, Providence, R. I.
    The Mary C. Wheeler School.

A Mathematics Laboratory Equipment

The older courses required little in the way of laboratory material. Perhaps therein lay their weakness, for there was consequently little stimulus to relate the principles of mathematics to science, industrial arts, recreational reading, and other phases of life. The older courses found supervised study difficult to incorporate, for one part of the recitation was very much like any other part.

An important characteristic of these experimental courses is the extent to which equipment is used to give concreteness and greater reality to mathematics.

To be sure, the equipment is simple (in contrast with General Science, Geography, and the like), but its importance as a device for relating mathematics to other school subjects can no longer be doubted.

Many of the materials found useful are listed in other parts of the report. It may be helpful to give special mention to the following room equipment: blackboard compasses, blackboard protractors, a yard stick (or metric), a slide rule for teaching, balances (perhaps made in the shop or at home), and a cross-section blackboard (easily painted on by the teacher). For the individual pupils, the following are practically essential: a pair of compasses, a protractor, a ruler graduated both to the inch and metric scale (metric largely to motivate decimals), and squared paper.

The mathematics set which has been especially prepared for use in grades seven, eight and nine by the Keuffel-Esser Company (Chicago-New York) has been found to be especially useful. It is an inexpensive[38] high grade set consisting of a pair of compasses, a brass protractor, enameled in a way to be easily read, an inch-metric ruler, and samples of cross section paper, one kind ruled to inches and tenths and the other ruled to centimeters with five divisions to the centimeter, both sets of graphic paper being especially helpful in motivating drill upon decimal fractions.

In no small measure is the success of these new courses dependent upon the degree to which the classrooms take on the physical appearance, spirit and attitudes of a well taught science laboratory in which development of new principles, the assignment, supervised study and recitation find their place and are economically inter-related.

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[38] The Junior Mathematical Set as described may be purchased for forty (40) cents per set in lots less than one hundred and for thirty-five (35) cents per set in lots greater than one hundred.

CHAPTER XIII

Standardized Tests in Mathematics for Secondary Schools

BY

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Introduction

The modern testing movement as applied to the field of secondary school mathematics is comparatively new and dates from the year 1914. Information concerning these tests is, in general, available only in scattered articles in educational periodicals, or in such technical form that often it is readable only by those who have had special training in statistics. Certain of the tests were worked out as dissertations for the degree of Doctor of Philosophy. Since the information concerning these tests is so inaccessible, it seemed desirable to the National Committee on Mathematical Requirements to have this movement brought to the attention of high-school teachers in popular form with the aim of describing the several tests in mathematics now available, and the methods of using them. The presentation herein given will be, therefore, as elementary as possible and will not assume any special training in statistics on the part of the reader. Thus it is hoped that this material will be of service to high-school teachers in general. This report will attempt mainly to summarize the present practice in reference to mathematical tests.

Typical tests for algebra and geometry have been included, since they particularly interest high-school teachers. Certain tests in arithmetic have also been included because arithmetic is taught in junior high schools as well as in a number of four-year high schools, and also because these arithmetic tests are especially helpful in understanding the two types of algebra tests.

PART I. TESTS IN ARITHMETIC

Certain Tasks in the Teaching of Arithmetic

The teaching of arithmetic involves two distinct tasks which are as follows:

(1) The first task is that of teaching the fundamental operations of addition, subtraction, multiplication, and division with whole numbers, common fractions, and decimals. These operations may be called the "machinery" of arithmetic. When a pupil has once acquired this machinery, his work with it is thereafter largely mechanical, and for life purposes it is very desirable that it should be so. The addition of a column of figures should be done quickly, automatically, accurately, and without any necessity on the part of the pupil for reasoning out the process. Similarly, skill in long multiplication means a complete mastery of the multiplication tables or facts (whether they be taught as tables or not), the automatic placing of the partial products, and the rapid addition of these partial products to obtain the final answer.

Psychologists have greatly assisted teachers of arithmetic by showing that the teaching of these fundamental operations is in reality the teaching of a large group of habits. It is possible, for example, to consider column addition as a group of many different habits, all of which need to be taught with full knowledge of the laws of habit formation. The first task in the teaching of arithmetic, therefore, is that of teaching this machinery of arithmetic, which is a group of many habits or skills. The teaching of this machinery is mainly the work of the primary grades. If this work is well done in the primary grades the work of the grammar grades can be devoted primarily to the second task in the teaching of arithmetic.

(2) The second task in the teaching of arithmetic is to show children how to use this machinery in solving problems and especially to make them familiar with a great series of life practices where the fundamental operations are used. To understand trade discount, for example, it is necessary for us to know those situations that occur in life which make a merchant interested to give a discount. Even the "January sales" advertised in the newspaper are a form of reduction in price with which we must make children familiar. Further, we must understand the practice of selling goods by catalogue and the

relation of discounts to such selling. We must know the extra inducement a merchant makes, by means of a discount, to one who buys in large quantities and also the further inducement he makes, through an extra discount, for cash payments. Throughout this subject of trade discount, even including chain discounts, there is no new arithmetic machinery. The same machinery is used in calculating these discounts which was learned in the primary grades. The teaching of trade discount, therefore, has been largely the process of making children familiar with the relations of the buyer to the seller. And if the child thoroughly understands these relations, most of his difficulties in "reasoning out" the solution of a problem in trade discount will disappear.

In the same way stocks and bonds, taxes, thrift, carpeting, paper-hanging, and the borrowing of money at a bank require for their successful teaching merely a lot of information as to how people in these modern days actually come in contact with and handle these things. The machinery of arithmetic is merely a servant to produce the answer in these life situations. The machinery of arithmetic is of no service whatever unless one knows enough of the life situation to know when to add, when to subtract, and when to multiply. Even percentage involves no new arithmetic processes; it is merely a new kind of language, a new way of talking about things, which new vocabulary people have found more concise and convenient than their previous methods of expression.

This second task in the teaching of arithmetic is largely the work of the grammar grades, although it is also going on as a matter of course in the primary grades, side by side with the teaching of the machinery of arithmetic. It is of interest to observe that the method of teaching suitable for making a child skillful in the machinery of arithmetic is often quite different from the method by which he learns the life uses of this machinery.

It has seemed best to go to some length in describing these two tasks in the teaching of arithmetic since they are intimately related to the modern tests in arithmetic. Practically all of the present tests in arithmetic are confined to a testing of the machinery of arithmetic and in certain instances they test only a small part of that. There are few tests which attempt to cover the second task in the teaching of arithmetic. The Courtis tests, for example, determine a child's skill only in the addition, subtraction, multiplication, and division of whole numbers. They do not include common fractions or decimals. The Woody test covers the fundamental operations with whole numbers and, to a limited extent, common fractions and decimals. These tests go no further than that. It is very important for teachers to get these points in mind because the present existing tests are in no way a comprehensive examination of all the instruction in arithmetic. If high scores are obtained in the Courtis tests in a given school, it does not follow at all that the arithmetic work in that school is thoroughly modern. On the side of the applications of arithmetic, that school may still be thirty years behind the times. Further, though the children may have come out well in the Courtis tests, they may still be particularly weak in handling the simplest common fractions or in the placing of the decimal point. Likewise, if a school makes a particularly poor showing in the Courtis tests, we have evidence only that the work of the school is poor so far as the teaching of the four fundamental operations is concerned.

The Courtis Arithmetic Tests

One of the vital questions which every teacher of arithmetic encounters is this: How skillful am I supposed to make my pupils in the fundamental operations of arithmetic? If we should give ,a problem in addition like the one here illustrated to a seventh-grade class in arithmetic, requesting the class to find the sum, we know that we should expect the child to find this sum accurately and we should expect that work to be done in a reasonable time. But what is a reasonable time for a seventh-grade child? Certainly 20 minutes is too long to allow a child to work a single problem like this; to that we would all agree. On the other hand, it does not seem reasonable for us to expect the child to obtain this sum in 3 seconds. Such skill would be called "lightning" addition, and only the most expert bookkeeper could do this problem in that time. The reasonable length of time for a seventh-grade child must lie, therefore, somewhere between 3 seconds and 20 minutes. If teachers could only know exactly what this time should be for a seventh-grade child, it would help materially to let them know when their task of teaching addition is finished.

$$\begin{array}{r}837\\882\\959\\603\\118\\781\\756\\222\\\underline{525}\end{array}$$

The Courtis tests in arithmetic are an attempt to answer this very question for each of the four fundamental operations with whole numbers, and naturally the answer to this question will vary with the school-year of the pupil. We will not expect from the third-grade pupil the same skill that we may reasonably expect of an eighth-grade pupil. In other words, Mr. Courtis has attempted to determine from the scores made by many children a goal or a standard of accomplishment for each grade which teachers will accept the country over and which, therefore, will enable them to compare with scientific definiteness the work of a child or the work of all children in the schools of one city with that in the schools of another. The moment we have such standardization, naturally the uses of such tests are many. It is of interest to note at this point that the standards set by Mr. Courtis fix 44 seconds as the time in which a seventh-grade pupil should be able to do correctly the addition problem given above.

The Courtis tests most used today, and more often recommended by Mr. Courtis himself, are the Standard Research Tests, Series B, which are an improvement upon an earlier group of Courtis tests, known as Series A.

The Series B tests consist of a pamphlet of four pages containing four tests in all, one for addition, one for subtraction, one for multiplication, and one for division. The complete test in addition, which occupies the first page of the pamphlet, is given on page 285.

It should be noted that all problems of this addition test are of equal difficulty—that is, each problem has the same number of items and the same number of columns. The problems have been so constructed that they contain different combinations of the same difficulty.

The subtraction test is constructed in a similar manner and contains twenty-four examples, each of the same difficulty. The multiplication and division tests are similarly constructed. Four alternate forms of the Courtis Research Tests, Series B, are issued, each of the same difficulty, in order to provide different problem material when the same group of children is tested at various intervals of time.

The practical details of giving the tests to children are given below as they appear in the *Manual of Instructions,* issued by Mr. Courtis.

> In giving any test, follow the standard instructions exactly, neither adding to, nor subtracting from, the directions.
>
> If on the day before the test is given, the significance of the test is discussed and the teacher makes a personal plea for maximum effort, the test

Courtis Standard Research Tests

ARITHMETIC. TEST NO. 1. ADDITION.

Series B—Form 1

Score: No. Attempted......; No. Right.......

You will be given eight minutes to find the answers to as many of these addition examples as possible. Write the answers on the paper directly underneath the examples. You are not expected to be able to do them all. You will be marked for both speed and accuracy, but it is more important to have your answers right than to try a great many examples.

927	297	136	486	384	176	277	837
379	925	340	765	477	783	445	882
756	473	988	524	881	697	682	959
837	983	386	140	266	200	594	603
924	315	353	812	679	366	481	118
110	661	904	466	241	851	778	781
854	794	547	355	796	535	849	756
965	177	192	834	850	323	157	222
344	124	439	567	733	229	953	525

537	664	634	572	226	351	428	862
695	278	168	253	880	788	975	159
471	345	717	948	663	705	450	383
913	921	142	529	819	174	194	451
564	787	449	936	779	426	666	938
932	646	453	223	123	649	742	433
559	433	924	358	338	755	295	599
106	464	659	676	996	140	187	172
228	449	432	122	303	246	281	152

677	223	186	275	432	634	547	588
464	878	478	521	876	327	197	256
234	682	927	854	571	327	685	719
718	399	516	939	917	394	678	524
838	904	923	582	749	807	456	969
293	353	553	566	495	169	393	761
423	419	216	936	250	491	525	113
955	756	669	472	833	885	240	449
519	314	409	264	318	403	152	122

may measure a very different set of abilities than if given without such an appeal. If the teacher, by the inflection of her voice, stresses speed, or quality, or other special features of the instructions, the results are not comparable with those obtained under other conditions.

The correct method of giving a test is to present it quietly but easily, vigorously, and *impersonally,* with as little departure from normal everyday conditions as possible.

Be very careful to say and do nothing which will create a tense, "nervous," apprehensive situation. Significant and reliable results will be secured when both the examiner and children are giving spontaneous and interested attention to a pleasurable "stunt." Enjoy giving the test. Above all things, don't make hard work of it and don't hurry. Poise, pleasure and "pep," or intelligent, efficient control of the situation are the qualities which make an examiner successful.

The purpose of a test is not to extract from each and every child the maximum of which he is capable. Inability to comprehend instructions adequate for the majority of the children is in itself significant. On the other hand, when practice material is provided for the instruction of the children, it should be used until all understand what is to be done.

In such cases, the examiner is not only free, but is expected, to supplement the printed instructions with others of his own. He must be careful, however, to use none of the test material proper and not to go further in his explanation than the model.

In giving a test, have the children read the printed instructions aloud with you. This collects and unifies the attention of the class. Immediately upon the completion of the reading, say, "That means, etc." repeating the instructions verbally.

Most of the children will have been so engaged in reading aloud that they did not pay attention to what was read. By listening quietly while the directions are repeated they have time to understand and adjust themselves to the idea of the action called for.

In giving any of the Courtis tests, proceed as follows: At the time of the test, hold up a copy of the test papers so all can see and say, "I am going to give you a standard test in arithmetic today. It is not an examination, but a kind of game, and I am sure you will enjoy it. This same test has been given in other schools all over the country, and I hope you will do your best for the honor of your school. I am going to give each of you a copy of the test. Please keep them face down until we are all ready. Clear your desks and make sure your pencils are sharp."

Put a package of test papers, face down, upon the first desk in each row, and have the children distribute them down the rows, putting the papers face down on each desk. Make sure the children do not look through the papers.

When all are ready, have the children read the instructions aloud, make the necessary explanations, and have the blanks, name, age, etc. filled out. Then say "In standard tests like this it is important that we all start together. We can do this easily if you will follow my instructions. Take hold of the lower right hand corner of your test paper, like this (suiting the action to the words), so that when I say 'Start' you will be ready to turn your paper over and begin work. Now, when I say 'Get ready' raise your pencil hand in the air (like this) as if you were going to ask a question. When I say 'Start' turn your paper over and begin to work."

Practice going through the motions a few times until sure that all understand.

Then say, "Get ready, eyes this way." See that every hand is raised. Then, when the second hand of your watch is exactly on the 60 mark, say "Start." Allow exactly the standard time allowance for your grade, then say, "Stop, hands up," and if the test is one that can be scored by the children, "Exchange papers."

Use a watch with a second hand for timing and take particular pains to keep exact time.

Every change in the time allowance will be reflected in the scores. Enough material is provided in most tests so that the children cannot finish in the standard time. Teachers frequently say, "If I had given more time, the results would have been better," forgetting that the same thing would be true of every other class tested. To give more time, and then to compare the results with the standard scores is therefore dishonest.

The purpose of the timed test is misunderstood by many. It has nothing to do with "speeding up" the children. The timing is to record the amount of work done per unit of time, for this is a significant symptom of development. The more perfectly a given habit has been automatized—the greater the degree of skill a person has—the less the time required to perform a given action. Many teachers do not realize that it takes time to make mistakes. Children who work slowly (barring those whose slowness is due to a naturally slow reaction) are those who work inaccurately, or those who have wrong habits of work. On the other hand, any one individual may vary his performance at will. That is, he may work slowly and very carefully, or rapidly and very inaccurately.

Therefore, the timing should be made an incidental feature of the testing. It should neither be concealed (for it is likely to be suspected and the suspicion to give rise to nervous apprehension) nor emphasized (for emphasis on speed, or effort, produces a nervous strain which makes for inaccuracy).

The desired effect is an impersonal presentation of the test situation, leaving it to each individual to determine for himself the rate and accuracy (or quality) at which he will work. The keeping of the time then drops into insignificance, as it is a mere means of determining the working interval.

The scoring of the papers may be done either by the pupils or by the teacher. It is customary in the Courtis tests to record the number of problems correctly worked and also the number attempted. When this is done we have for each pupil his particular record, and of course these records will vary considerably. A number of the students will have succeeded in doing many more problems than others. For many purposes, it is very desirable to have some means of recording the work of the entire class. One way of doing this is to find the average number of examples correctly worked. For many purposes, however, it is much easier and more desirable to find the class "median" instead of the average. The term "median" is explained in books on statistics, a list of a few such books being given in the bibliography on pages 427-428. For our purpose, however, it will serve to give a very crude idea of the median since this term is generally used in many of the tests for mathematics. If a group of 11 children should stand in a row, arranged according to their height, with the tallest child at one end and the shortest child at the other end of the row, the height of the middle child in the row is called the "median" height of the group. This median child, therefore, divides the group into two parts, all the children to the right of the median child being taller than the median while those to the left of the median child are shorter than the median. In other words, the median height is such that there are just as many children taller than it as there are shorter than it. If there had been 10 children in the row instead of 11 children there would be no middle child, though there are two middle children in this case. The median height here would be half way between the heights of the two middle children. Similarly we find the median score of a class, by arranging the scores of each child in order, the highest score at the top of the column and the lowest score at the bottom. The middle score is the median score.

The definition of the median just given is sufficient for our present purposes, but those who are interested in mathematical tests should become familiar with our more rapid and more refined methods of computing it. Two helpful references on this subject are Rugg's *Statistical Methods Applied in Education*, pp. 109-113, and an article by Dr. McCall in the *Teachers College Record*[1] for March, 1920, entitled "How to Compute the Median." A discussion of the median will be found, however, in any text on statistics.

The standard scores for each grade are given by Mr. Courtis in the pamphlets accompanying the tests. These standard scores are the answers to the question asked at the beginning of this chapter, "What skill in the fundamental operations is it reasonable for the teacher to expect of her pupils?" These standards are based on the results of giving these tests to thousands of school children all over the country. For example, according to the standard median scores given by Mr. Courtis, an eighth-grade child should finish 12 problems in the

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[1] Published by Teachers College, Columbia University, New York City.

Addition Test in the eight minutes allowed for that test. According to the ideal standard all 12 of these problems should be right. Few pupils actually achieve this, however. In that same test according to ideal standards, a seventh-grade pupil is expected to get 11 problems right while a fourth-grade pupil is expected to get only 6 problems right in the eight minutes. Naturally these standards are modified from time to time as greater experience with the tests is gained and as teachers themselves come to have more light upon all the factors involved in this work.

All arithmetic tests represent a great deal of experimental work and it is far from the purpose of the authors of any of them to claim that any standard has been fixed once for all. These standards, however, are far more reliable than the individual judgment of any teacher or any small group of teachers and therefore may be reasonably taken as representing a safe standard the country over for school work in the teaching of the fundamental operations of arithmetic with integers.

At the risk of repeating some of the facts already brought out, the author will give at this point a statement which Mr. Courtis prepared for him relative to his Research Tests, Series B. Certain facts in this statement apply equally well to all other tests:

> There are many different ways of measuring educational products, and as many different theories as to how educational products ought to be measured. The differences do not mean that one way is right and another wrong. They merely represent the attempts of various workers to solve the pressing problems in the ways best adapted to their needs and viewpoints. It often happens, however, that after a measuring instrument has been developed from one point of view and for a given purpose, it is used by later investigators from a different point of view and for a different purpose. It is not surprising that in such cases the test does not function well. Therefore, those who use standard tests should make sure they know the theory on which the tests are constructed and the purpose for which they were designed.
>
> The result of school training in a specific skill may be measured either by testing the individual with a series of problems which gradually increase in difficulty[2] in order to find out the most difficult type of problem he can solve correctly, or by presenting many problems of a given level of difficulty to find out the amount that can be completed in a given time and the accuracy with which the work is done. The Series B tests are of the latter type and should be called "rate or speed tests." They are in no sense arithmetic scales, but instruments designed wholly to determine children's ability to carry on given

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[2] The Woody Arithmetic Scales described on p. 292 are of this type, and are often called "power tests."

types of arithmetic computations. It should be noted particularly that the types of examples found in the tests were selected *arbitrarily* as representing objective definitions of the types of ability it was desired to develop by school training. Those who do not agree with the definitions adopted should not expect the tests to yield satisfactory results.

The Series B tests were issued in 1914 after six years of investigation with the Series A tests. During this interval the effect of various factors in work in the four operations with whole numbers had been noticed and the need for the construction of a measure of the "end products" of training realized. Accordingly an analysis of arithmetic work was made and examples of such size and character chosen that all the desired elements would be involved. The exact scheme of construction need not be repeated here as it has been given in full elsewhere. The objective character of this scheme, however, makes possible the construction of many editions of equal value, and at the present time four such editions are in common use. This makes possible repeated measurement of individuals or school systems, and objective definitions of the degree of skill to be attained at each grade level (i. e. standard scores).

At the time the tests were constructed the question of whether or not a test measures the ability it is supposed to measure had not been raised, nor the companion question of the reliability of the results obtained. Consequently no formal measure of either of these two important characteristics has been published. The ready acceptance of the tests at their face value by schoolmen everywhere and the high scores made by bookkeepers, accountants and other professionals constitute the only available evidence of the validity of the tests. Informal experiments show that the reliability of scores for rate of work is high, but for accuracy the reliability is low and extremely variable. Class averages are, of course, much more reliable than individual scores. Recent studies show that for scientific comparison scores must be reduced to some standard basis, as rate at a standard accuracy.

The Series B tests are of value in determining the degree of development attained by the individual, class, or school system. Standard scores are available, based upon the achievement of thousands of children in each grade. This standardization makes possible the use of the results for supervisory control, the standard scores being in effect an objective definition of the goals of the course of study in the fundamentals with whole numbers. The tests are of value also in all experiments designed to measure any phase of the products of training, as for instance, the determination of the relative value of drill material. The tests are also used for diagnostic purposes, but from this point of view they indicate merely the operations in which an individual is above or below standard. They do not indicate the cause of the trouble. Analysis of the mistakes made, however, will yield a more exact diagnosis but at considerable time cost.

Since the tests were issued in February, 1914, more than four million have been used in this country and in foreign lands. They may be obtained at cost from most university centers for the distribution of testing material, or direct from S. A. Courtis, 1807 East Grand Boulevard, Detroit, Michigan. As issued, folders of complete instructions for giving, scoring and tabulating the tests, and interpreting the results are sent with the test papers. They may be used, therefore, by teachers inexperienced in measurement work.

Courtis Supervisory Test in Arithmetic

Mr. Stuart A. Courtis, author of the Courtis tests described above, is a strong advocate of the principle that efficient teaching calls for the adjustment of school tasks to the individual powers and needs of the pupils. In order to determine these needs, he feels that in every school subject there should be (1) a test at the beginning of the semester to determine where the children are in relation to the grade standards for each subject, (2) a teaching period based on the results of this test in which special provision is made for drilling individual pupils in proportion to their needs, and (3) a second test at the end of the semester to determine the effects of this teaching effort. Mr. Courtis has devised his Standard Supervisory Tests to enable schools to carry out such a program of testing for spelling, arithmetic, geography, and writing. Later it is intended to add reading and composition tests to this group. This is a practical means of testing a school system on a large scale. The system is especially effective in those schools which have a full testing program for all subjects, since the system provides a uniform method of scoring for each of the subjects. The Supervisory Tests are not so well adapted for incidental testing in a single subject.

The Courtis Supervisory Test in Arithmetic is therefore a part of this general system of testing. The complete Supervisory Test in Arithmetic consists of two tests, known as Test A and Test B. Test A is a single sheet of simple problems covering the four operations with whole numbers, where no carrying is involved. This test is devised for Grades 4B, 4A, and 5B. The time allowance for the test varies with the grade, being 6¼ minutes for Grade 4B and 5¼ minutes for Grade 5B.

Test B is a single sheet covering the four operations with whole numbers, with carrying. The problems of Test B are a little more difficult than those of Test A. Test B is for Grades 5A to 8A. The time allowance varies with the grade, being 6¾ minutes for Grade 5A and 4½ minutes for Grade 8A.

The system of scoring for the Supervisory Test in Arithmetic is based on an ideal score of 1000 points. If every child in the class were up to standard, the class score would be 1000 points. The actual class score shows what part of the desired development has already been accomplished. Thus, if the score of a class is 600 points, the class has 60 per cent of the desired development. In other words, it is hoped that the teaching during the term should result in a gain of 400 points. This point system is of particular value in indicating the improvement which takes place during the year. An illustration will show how it is applied. Suppose a class at the beginning of the year had a score of 800. This would mean that the class had ahead of it a possible gain of 200 points, since 1000 points indicates standard accomplishment. If at the end of the year the final test showed a score of 900 points for the same group of children, the actual gain made by the class would be 100 points out of a possible gain of 200. The actual gain is therefore 50 per cent of the possible gain. Throughout the Courtis scoring system the final gain is found by determining what per cent the actual gain is of the possible gain. This scoring system is applied uniformly for the other subjects as well as for arithmetic.

The latest revision of the Courtis Supervisory Tests is that for 1922. The tests, together with a book of instructions for using them, may be had from Stuart A. Courtis, 1807 East Grand Boulevard, Detroit, Michigan.

The Woody Arithmetic Scales

We have already pointed out that the teaching of the fundamental operations of arithmetic is the teaching of a series of habits. For example, in learning to multiply, we may consider each fact of the multiplication tables as a habit. Thus, 2 × 3 = 6 is one habit, while 7 × 9 = 63 is still a different habit. If a pupil knows the multiplication tables, it does not follow that he can perform long multiplication, since that operation involves many other habits besides those represented by the multiplication tables themselves. To multiply 23 by 2 requires a special new habit, while to multiply 27 by 32 requires still other new habits. The successful teacher must ever be conscious of these new habits and provide for them in her instruction.

It is evident that an arithmetic test, like the Courtis Series B provides problems of a more complex rather than a more simple type. To multiply 8246 by 209 is representative of the difficulty of the problems in the Courtis multiplication test. Such a problem in multiplication includes practically all the habits necessary to perform this operation successfully with whole numbers. If a child fails to solve this problem correctly, it may be difficult for the teacher to determine in just what habits of multiplication he is weak. If a test in multiplication could be provided that would present problems of a number of different stages of difficulty, ranging from those so simple that they involve only the elementary facts of the multiplication tables up to those so difficult that they require the use of all the habits of multiplication, then such a test would enable the teacher to know just how difficult a problem in multiplication a child can solve and incidentally, in what particulars his skill in multiplication is weak. The Woody Arithmetic Scales are an attempt to fill this need for the teacher.

A specimen page of the Woody Multiplication Scale, Series A, is given on page 294.

Series A of the Woody Scales consists of three other sheets besides the one illustrated on p. 294, one of these sheets being for addition, one for subtraction, and one for division. Dr. Woody has also issued an abridged edition of this scale, known as Series B, which differs from Series A only in the omission of certain of the problems on the A sheet. A specimen of the Addition Sheet of Series B is given on page 295.

An examination of these scales will indicate that they differ radically in construction from the Courtis tests in that each problem is a little more difficult than the one preceding it, while in the Courtis tests all the problems are of equal difficulty.

It should be added that Dr. Woody has not only selected problems each more difficult than the preceding one, but he has actually measured the amount of difficulty in each problem and is able to express this difficulty numerically in terms of units of a "difficulty scale," just as we measure length in terms of a "length scale," such as a foot rule. The arrangement of the problems according to their difficulty has led specialists in educational measurements to call this type of test a "scale" to distinguish it from those tests where the problems

The Woody Arithmetic Scales

MULTIPLICATION SCALE A

(1)	(2)	(3)	(4)	(5)	(6)	(7)
3 × 7 =	5 × 1 =	2 × 3 =	4 × 8 =	23	310	7 × 9 =
				3	4	

(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)
50	254	623	1036	5096	8754	165	235
3	6	7	8	6	8	40	23

(16)	(17)	(18)	(19)	(20)	(21)	(22)
7898	145	24	9.6	287	24	8 × 5¾ =
9	206	234	4	.05	2½	

(23)	(24)	(25)	(26)	(27)	(28)	(29)
1¼ × 8 =	16	⅞ × ¾ =	9742	6.25	.0123	⅛ × 2 =
	2⅝		29	3.2	9.8	

(30)	(31)	(32)	(33)	(34)
2.49	$\frac{12}{25} \times \frac{15}{32} =$	6 dollars 49 cents	2½ × 3½ =	½ × ½ =
.36		8		

(35)	(36)	(37)	(38)	(39)
987¾	3 ft. 5 in.	2¼ × 4½ × 1½ =	.0963⅛	8 ft. 9½ in.
25	5		.084	9

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The Woody Arithmetic Scales

ADDITION SCALE B

(1)
2
3

(2)
2
4
3

(3)
17
2

(5)
72
26

(7)
3 + 1 =

(10)
21
33
35

(13)
23
25
16

(14)
25 + 42 =

(16)
9
24
12
15
19

(19)
$.75
1.25
.49

(20)
$12.50
16.75
15.75

(21)
$8.00
5.75
2.33
4.16
.94
6.32

(22)
547
197
685
678
456
393
525
240
152

(23)
1/3 + 1/3 =

(24)
4.0125
1.5907
4.10
8.673

(30)
2½
6⅜
3¾

(33)
.49
.28
.63
.95
1.69
.22
.33
.36
1.01
.56
.88
.75
.56
1.10
.18
.56

(36)
2 yr. 5 mo.
3 yr. 6 mo.
4 yr. 9 mo.
5 yr. 2 mo.
6 yr. 7 mo.

(38)
25.091 + 100.4 + 25 + 98.28 + 19.3614 =

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are all of the same or approximately the same difficulty. In these "scales" a child's ability is measured in terms of the "difficulty" of the problem he can work, rather than in terms of the number of problems done in a given time, as in the Courtis tests. For this reason, these "scales" are often designated as "difficulty" tests, "development" tests, or "power" tests, to distinguish them from tests of the Courtis type, which are known as "speed" or "rate" tests.

These Woody scales, as used at present, represent the final product gained from testing from 20,000 to 30,000 school children in Indiana, New Jersey, Connecticut, and New York. The preliminary tests, upon which the final tests were based, consisted of four sheets of problems, one for each of the fundamental operations. In devising different sheets, there was a definite attempt to select as great a variety of problems as the fundamental operations would permit, and to select them with various degrees of difficulties ranging from the easiest combinations that could be found to problems so difficult that they could be solved by only a very small percentage of the pupils in the eighth grade. In the first preliminary set of test sheets, little attention was paid to the social utility of the different exercises, but as experience was gained it became evident that each and every problem to be selected for the final scales must represent the type of problem frequently occurring in our complex social life. Consequently, after making a survey of the different investigations conducted to determine the actual needs for arithmetic in everyday transactions, i. e., the types of problems solved, the particular transactions used, etc., the original lists of problems were carefully scrutinized and all problems were rejected which did not represent common social needs. Whenever new problems were added, the social criterion was always taken into consideration. The task finally resolved itself into taking a series of problems representing a cross section of the minimal essentials for that operation, expressed in problems of as many types as possible, and arranging them in order of their degree of difficulty.

The final arrangement as given in the scales now in use was determined by giving the last revision of the preliminary tests to about 20,000 children.[3]

After this testing, the difficulty of each of the problems was determined by making an elaborate count to determine just how many children had solved each problem correctly. If 200 pupils, for example, solved problem M correctly, while 285 pupils solved problem N correctly, then problem N was regarded as easier than M because more pupils were able to work it correctly. By handling the facts thus obtained in accordance with modern statistical methods, the amount of difficulty of each problem was obtained in terms of a "difficulty" scale.

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[3] For further details, see Woody, *Measurements of Some Achievements in Arithmetic.* Revised edition. Bureau of Publications, Teachers College, New York City.

It should be noted that all the Woody scales provide for testing common fractions, decimals, and integers, all of which are part of the machinery of arithmetic. Nothing has been given to test the applied side of arithmetic.

The manner of giving the Woody scales to pupils is similar to that described for the Courtis tests. The method of scoring the papers most frequently used is likewise similar to that used in the Courtis tests, the teacher counting the number of problems correctly solved on each paper. In the Woody scales no attention is paid to the number of problems attempted, but incorrectly solved. No credit is given for a problem partially solved. The standard for marking a problem correct is absolute accuracy, and, wherever possible, reduction to its lowest terms. If the results of giving these tests are to be compared with the standards established by Dr. Woody, only those answers should be accepted as correct which he provides in the instruction manual entitled *The Woody Arithmetic Scales. How to Use Them.*[4] For purposes of obtaining the measure of an entire class or school, the median score is used, as in the Courtis tests. In the manual just mentioned, a second method of obtaining the class score is also given.

In giving the Woody scales the speed element is secondary. For each sheet or scale in Series A, twenty minutes is allowed; for each in Series B, ten minutes. In practice, most of the children finish before the expiration of these time limits. Those pupils who have not finished, have, in all probability, done all they can. The main object of the test, therefore, is to determine how difficult a problem in each scale the pupil can do.

The Woody scales claim as one of their chief merits that they are particularly suited for the diagnosis of arithmetic ills in the fundamental operations. The scales thus make it possible for one to study the mental or psychological process underlying errors in the solution of the problems. Knowing the psychological process causing the error, it is easy for the teacher to apply the proper remedy. By a careful study of the test sheets of a given group of pupils, it is possible to make a classification of all their errors and also to determine their relative frequency. Out of 1500 errors in a set of division papers, it was found, for example, that 18% of the errors were due to failure to invert the divisor, 12% to incorrect placing of the decimal point, 7% to addition or omission of a cipher in the quotient, etc.

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[4] For full information concerning each reference cited, such as the publisher, date, etc., see the bibliography at the end of this chapter.

It is unsafe, however, to pass final judgment on the achievement of a pupil or a class on success or failure in solving a single problem. To make certain of the diagnosis, it will be necessary to see what has been done with other problems involving that same operation or step. The safest diagnosis in a scale of the Woody type comes from giving to the pupil alternate or parallel forms of the same test. Such parallel forms have just been prepared by Dr. Woody. If a child fails on the same kind of problem on several alternate forms, we then have positive evidence that this failure represents a fundamental rather than an accidental error. Since Series A is the more complete edition of the Woody scales and offers a greater number and variety of problems, it is especially recommended for use when the maximum help in diagnosis is needed; while the abridged Series B is especially useful when the time for testing is limited, and where the purpose is mainly the measurement of the achievement of a class or of a school system.

The Woody Arithmetic Scales are especially adapted for grades three to eight. Standard scores for these grades are given in the pamphlet entitled *The Woody Arithmetic Scales. How to Use Them,* by Dr. Clifford Woody. The scales and instruction pamphlets may be obtained from the Bureau of Publications, Teachers College, New York City.

Objection is sometimes raised to scales like those of Woody in that the method of constructing them is based on a statistical record of the success pupils have in solving the problems in the tests, rather than on an analysis of the basic types of problems and habits found in each of the fundamental operations. By this statistical method, fundamental types of problems are often omitted, while the analytic method attempts to include those problems which incorporate every difficulty or new habit which could possibly arise. The Cleveland Survey Tests and the Monroe Diagnostic Tests, described later, represent an analysis of the several types of problems in the fundamental operations with integers and common fractions.

The Woody-McCall Arithmetic Scale

A special arrangement of the Woody Arithmetic Scales has been made by Dr. William A. McCall of Teachers College, Columbia University, which is known as the Woody-McCall Mixed Fundamentals. This test consists of a single sheet of problems selected from the addition, subtraction, multiplication, and division sheets of the Woody scales. It is intended to be given as a single test, a time allowance of twenty minutes being made for its completion. Four alternate forms of this test, each of the same difficulty, are available. Form I of this test is given on p. 300. Copies of this scale with standards and directions for its use may be obtained from the Bureau of Publications, Teachers College, Columbia University, New York City.

Monroe Diagnostic Tests in Arithmetic

In the Courtis Standard Research Tests, Series B, it was found, that each problem in addition was rather long and elaborate, consisting of three columns with nine figures in each column. To add such problems satisfactorily a pupil must possess the composite of all abilities necessary for adding. He must be familiar with his addition combinations, must know how to perform column addition, must be able to carry, and must have developed a certain "memory span." If he fails on an addition problem of the Courtis difficulty, the teacher has no way of knowing just what element of adding ability is deficient. We found that this objection to the Courtis test was remedied somewhat in the Woody test; but in that test only one problem of each stage of difficulty was provided, making it impossible for the teacher to determine whether a child's failure to solve a given problem was due to accident or to a fundamental deficiency in the principle involved. Had several problems been given involving the same element of adding ability, and of equal difficulty, it would have been possible for the teacher to have known whether the pupil's error was of the accidental or of the recurring type.

Professor Walter S. Monroe, of the University of Illinois, has devised a series of diagnostic tests which begin with simple exercises involving the addition of a single column of three digits and gradually increase in difficulty until all the adding abilities are incorporated in

Woody-McCall Mixed Fundamentals

Get the right answer to as many examples as you can in 20 minutes. Do all work on the front or back of this sheet.

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
Add			**Subtract**	**Multiply**	**Subtract**	**Add**		**Subtract**
2 3	2 × 3 =	$3\overline{)6}$	2 1	23 3	13 8	17 2	3 + 1 =	16 9

(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)
Multiply		**Add**	**Subtract**		**Add**	**Multiply**		**Add**
254 6	4 ÷ 2 =	23 25 16	393 178	$.2\overline{)13}$	9 24 12 15 19	5096 6	2¾ − 1 =	$12.50 16.75 15.75

(19)	(20)	(21)	(22)	(23)	(24)	(25)	(26)
Multiply		**Add**	**Multiply**		**Subtract**	**Add**	**Multiply**
7898 9	¼ of 128 =	547 197 685 678 456 393 525 240 152	287 .05	248 ÷ 7 =	27 12⅝	4.0125 1.5907 4.10 8.673	9742 59

(27)

⅞ of 624 =

(28)

Add

.49
.28
.63
.95
1.69
.22
.33
.36
1.01
.56
.88
.75
.56
1.10
.18
.56

(29)

⅛ × 2 =

(30)

Multiply

987¾
25

(31)

¾ ÷ 5 =

(32)

7.3 − 3.00081 =

(33)

$9\overline{)69\text{ lb. }9\text{ oz.}}$

(34)

Multiply

.0963⅛
.084

(35)

25.091 + 100.4 + 25 + 98.28 + 19.3614 =

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a single problem. The complete Diagnostic Test consists of twenty-one separate tests, each test containing a generous number of problems of the same type and difficulty, so that it can be determined if a pupil's error is of the kind which constantly recurs.

The complete Diagnostic Test is arranged in four parts. Part I contains Tests 1 to 6, and is made up of simple problems covering the fundamental operations with whole numbers. Part II contains Tests 7 to 11, which include more difficult problems covering the fundamental operations with whole numbers. Part III contains Tests 12 to 16 and covers the fundamental operations with common fractions. Part IV contains Tests 17 to 21 and covers the multiplication and division of decimal fractions. Each test is given a definite time limit. It is thus possible by this diagnostic test to obtain separate measures of each of the important abilities found in arithmetic

Specimens of each of the twenty-one tests are illustrated below, each of the tests containing a generous number of problems of the same type as those illustrated.

Monroe Diagnostic Tests in Arithmetic

ADDITION

Test 1

$$\begin{array}{r}4\\7\\\underline{2}\end{array}$$

Test 5

$$\begin{array}{r}7862\\5013\\1761\\5872\\\underline{3739}\end{array}$$

Test 7

$$\begin{array}{r}7\\6\\6\\5\\0\\5\\1\\8\\7\\3\\13\\1\\\underline{2}\end{array}$$

Test 12

$$\frac{1}{6}+\frac{1}{3}=$$

$$\frac{5}{6}+\frac{1}{2}=$$

$$\frac{3}{10}+\frac{3}{5}=$$

Test 15

$$\frac{1}{6}+\frac{3}{5}=$$

$$\frac{3}{12}+\frac{5}{8}=$$

SUBTRACTION

Test 2

$$\begin{array}{r}37\\\underline{5}\end{array}\qquad\begin{array}{r}94\\\underline{8}\end{array}$$

Test 9

$$\begin{array}{r}739\\\underline{367}\end{array}\qquad\begin{array}{r}1853\\\underline{948}\end{array}$$

Test 13

$$\frac{3}{4}-\frac{2}{5}$$

$$\frac{5}{6}-\frac{3}{4}$$

MULTIPLICATION

Test 3	Test 8	Test 10			
$\begin{array}{r}6572\\ 6\\ \hline\end{array}$	$\begin{array}{r}4857\\ 36\\ \hline\end{array}$	$\begin{array}{r}560\\ 37\\ \hline\end{array}$	$\begin{array}{r}807\\ 59\\ \hline\end{array}$	$\begin{array}{r}617\\ 508\\ \hline\end{array}$	$\begin{array}{r}840\\ 80\\ \hline\end{array}$

Test 14	Test 18		Test 20	
$\frac{2}{3} \times \frac{3}{4}$	$\begin{array}{r}657.2\\ .7\\ \hline 46004\end{array}$	$\begin{array}{r}67.50\\ .03\\ \hline 20250\end{array}$	$\begin{array}{r}487.5\\ .62\\ \hline 302250\end{array}$	$\begin{array}{r}57.28\\ 9.5\\ \hline 544160\end{array}$
$\frac{2}{5} \times \frac{3}{7}$				

DIVISION

Test 4	Test 6	Test 11	Test 16
$8\overline{)3840}$	$82\overline{)3854}$	$47\overline{)27589}$	$\frac{2}{5} \div \frac{1}{3}$
			$\frac{4}{7} \div \frac{2}{3}$
			$\frac{3}{8} \div \frac{2}{3}$

Test 17		Test 19		Test 21	
$.03\overline{)16.2}$	Ans. 54	$.4\overline{)148}$	Ans. 37	$.47\overline{)2758.9}$	Ans. 587
$.07\overline{)1.82}$	Ans. 26	$.9\overline{)65.7}$	Ans. 73	$8.2\overline{)38.54}$	Ans. 47
$.05\overline{).415}$	Ans. 83	$.6\overline{)1.68}$	Ans. 28	$79\overline{)36.893}$	Ans. 467
$.06\overline{)7.44}$	Ans. 124	$.7\overline{).301}$	Ans. 43		

In Tests 18 and 20 the pupil is simply to insert the decimal point in the product which is given. In the samples only the variations in the multiplier are given. Each multiplier is used with three types of multiplicands (657.2, 65.72, 6.572). Thus each test includes six types of examples.

In Tests 17, 19, and 21 the correct answer for each example, with the exception of the decimal point, is given at the right of the problem. The pupil is required to write the answer in its proper position and place the decimal point in its proper place and to place ciphers before or after the answer when they are necessary.

The Monroe Diagnostic Tests in Arithmetic, with directions for use, may be had from the Public School Publishing Company, Bloomington, Illinois.

The Cleveland Survey Arithmetic Tests

The Cleveland Survey Arithmetic Tests were devised in 1915 by Professor Charles H. Judd, of the University of Chicago, and Mr. Stuart A. Courtis, for the purpose of use in the survey made of the public schools of Cleveland, Ohio. The complete test consists of fifteen tests devised to analyze the arithmetic processes. In each test the problems are of approximately equal difficulty. In general, the tests are similar to the Monroe Diagnostic Tests and provide for common fractions as well as for the four fundamental operations with integers. There is no test for decimals. The Cleveland Survey Tests are fully described in the book entitled *Measuring the Work of the Public Schools,* by Professor Judd.

The Cleveland Survey Tests, with directions for use, may be had from Stuart A. Courtis, 1807 East Grand Boulevard, Detroit, Michigan.

Monroe General Survey Scale in Arithmetic

Professor Walter S. Monroe, of the University of Illinois, has devised a General Survey Scale in Arithmetic, the purpose of which is to provide a single score which will be a general measure of the pupil's ability to perform the operations of arithmetic. The purpose of this General Survey Scale is just the opposite of that of Professor Monroe's Diagnostic Tests, described above, which are designed to yield separate measurements of the detailed abilities necessary to perform successfully the fundamental operations of arithmetic.

The complete Survey Scale in Arithmetic consists of two parts. Part I is for Grades 3, 4, and 5. It is a pamphlet of four sheets, with eight tests containing simple problems covering the four fundamental operations with whole numbers. Part II is for Grades 6, 7, and 8. It is a pamphlet of four sheets, with seven tests containing more difficult problems covering the four fundamental operations with whole numbers; it also includes tests for the four fundamental operations with common fractions and a test for decimal fractions. In each individual test the problems are of approximately equal difficulty. There is a definite time allowance for each test. The Monroe General Survey Scale in Arithmetic, together with standards and directions for using it, may be had from the Public School Publishing Company, Bloomington, Illinois.

Progress Tests in Arithmetic

Professor Walter F. Dearborn, of Harvard University, and Miss Harriet E. Peet have constructed the Progress Tests in Arithmetic, which embrace a wider scope of testing than any of the tests already described. The Progress Tests in Arithmetic are intended to serve the following purposes:

(1) To measure the standing and progress of pupils in arithmetic from time to time.

(2) To differentiate the members of a class according to ability.

(3) To diagnose the difficulties of a class as a whole, and also those of individual pupils.

The tests are carefully planned to include the leading types of problems which a class needs to master, and to cover such difficulties as those met with in handling the zero, in placing the decimal point, and in computing with the most common business fractions. The problems in each test range in difficulty from very simple ones to those so difficult that only pupils of superior ability are able to solve them. In order to obtain a method of scoring which would be just, each problem has been given a value in proportion to its difficulty, statistically determined. A child's standing is not merely a place on a scale, but his record on the entire test. The standing of a class is found by comparing the class median, or standing of the child of middle rank, with a standard median. The work is so planned that a pupil may score his own or his neighbor's paper. This not only relieves the teacher of the work, but it gives the pupil an immediate benefit from the results. He is able to see wherein he has failed and in what ways his work needs improvement.

The tests run in two series: The Intermediate Series for Grades 4 and 5; the Upper-Grade Series, for Grades 6, 7, and 8. A Primary Series is in preparation. The tests may be given at the beginning, the middle, and the end of the year, or at any other time when it is desirable to get the relative standing of a class, to measure the progress made, or to find out with what types of work the pupils need most help.

Due to the length of each of these tests, it is not practical to duplicate them in full. A brief description will, however, serve to indicate their scope. The Intermediate Series for Grades 4 and 5 consists of five tests. Test 1 is made up of simple problems similar to those found in the reasoning tests described later. Test 2 covers addition, the problems ranging from simple ones of a single column to long problems of several columns; addition of decimals, mixed numbers, and denominate numbers is included. Test 3 for division, Test 4 for subtraction, and Test 5 for multiplication, each have problems ranging from simple to more elaborate ones and include mixed numbers, decimals, and denominate numbers, as well as integers.

The Upper-Grade Series for Grades 6, 7, and 8 consists of five tests. Test 1 covers problems similar to those in the reasoning tests described later, but more difficult than the problems in Test 1 of the Intermediate Series. Test 2 for addition, Test 3 for division, Test 4 for subtraction, and Test 5 for multiplication, contain problems more difficult than those found in the Intermediate Series and include decimals, mixed numbers, and denominate numbers, as well as integers.

The Peet and Dearborn Progress Tests in Arithmetic, with directions for use, may be had from the Houghton Mifflin Company, New York, Boston, or Chicago.

An Inventory Test in Arithmetic

Messrs. John R. Clark, Raleigh Schorling, and Harold Rugg, of the Lincoln School of Teachers College, Columbia University, found themselves handicapped in carrying on certain experimental work to improve the teaching of mathematics in the junior high school unless they could determine with a fair degree of certainty the knowledge which the children were supposed to have as a result of their study in the first six grades. Accordingly they devised an Inventory Test in Arithmetic to be given to children at the beginning of the seventh year as a measure of their knowledge of arithmetic obtained in the first six grades. The Inventory Test consists of two four-page pamphlets containing in all one hundred twenty-five questions. Through these questions the child's knowledge is tested concerning the simple tables of denominate numbers, the language and terms of arithmetic, the more important elementary principles of arithmetic, the understanding of decimals and percentage, the language of mensuration, simple graphs, and elementary business practice. The children are given ample time in which to answer these questions. The answers to certain of the exercises are numbers, while in others the answers are words or drawings.

For further information concerning this test, address the authors, The Lincoln School, 425 West 123rd Street, New York City.

Reasoning Tests in Arithmetic

The tests already described have been concerned with the measuring of a pupil's ability to perform the fundamental operations which constitute the machinery of arithmetic. One is not a master of arithmetic if he is skillful only with the machinery. A pupil must also know how to think through and solve problems in which various parts of this machinery may be used.

In other words, we also need to test a pupil's reasoning power in arithmetic. It has been far more difficult to get satisfactory tests for reasoning ability in arithmetic than for skill in the fundamental operations. Many factors account for this difficulty. No one seems to agree as yet as to just what is a satisfactory type of reasoning problem for each grade. Considerable work must still be done in the way of a detailed analysis of all the elements which go to make up the many types of applied problems found in arithmetic. Similarly, no satisfactory method has yet been devised to summarize the knowledge or acquaintance which a child should have with the various life situations in which the machinery of arithmetic is used. The possible applications of the machinery of arithmetic are so vast as to present many difficulties. Some of the attempts to determine the "minimum essentials" in arithmetic are partial movements in this direction, but they have only scratched the surface of an immense field ahead. The problem is further complicated by the aims in teaching arithmetic. Shall we teach only those topics which have practical everyday use in the bread and butter sense, or shall we also teach those topics in arithmetic which enable children better to appreciate the life that is going on around them, though they themselves may take no active part in it? The writer feels that the side of appreciation must not be neglected. We teach a number of topics in arithmetic as much for the purpose of enabling the pupil to appreciate their significance in world progress as for the purpose of enabling him to engage in actual transactions involving those topics. Stocks and bonds are an example of a type of topic taught as much for appreciative as for practical purposes. Many children will never buy a bond or a share of stock, yet they need to know the significance of these things in making possible our railroads, our trolley lines, our gas plants, and our Government participation in the War, or in the making of the Panama Canal.

Assuming a child to be familiar with the life situation or practice represented in a given problem, he still has further difficulties ahead in order to obtain a correct solution of the problem. In the first place, he must be able to read the problem with understanding. The vocabulary of the problem, which is made up of technical arithmetic terms as well as terms peculiar to some life situation, must be thoroughly understood. If the problem describes something which is not familiar to the child, correct reasoning is handicapped in advance. Next, the child must make a careful examination of the data given in the problem which he may use, and he must further decide exactly what he is required to find. The ability to select all the given data and to give all the factors their due consideration has an extremely important bearing upon the result. Then the pupil must recall the principles which apply to problems of this type, and with all of this in hand, he must outline a plan for solving the problem. After this is done, his machinery of arithmetic then comes into play, with all the possibilities of disaster due to mechanical mistakes. After a certain amount of adding, subtracting, and dividing, a number is obtained which is often called the "answer"; but this answer must be interpreted in the light of the conditions of the problem and given definite meaning. Finally, the pupil must review the whole procedure again to check his result, checking not only the mechanical processes, but also the reasoning by which he has arrived at his result.

It is at once evident that a reasoning test is attempting to measure a composite of activities along the lines sketched above. Naturally, the reasoning in certain problems will require only one or two steps, while other problems may be very complicated. There are therefore many stages of difficulty in reasoning problems. All the reasoning tests so far devised have taken fully into account this element of the complexity of the reasoning, and accordingly such tests start with simple problems and gradually proceed to those which are more difficult. The reasoning tests most in use today are those described below.

The Stone Reasoning Test *

The Reasoning Test, devised by Dr. C. W. Stone, is as follows:

1. If you buy 2 tablets at 7 cents each and a book for 65 cents, how much change should you receive from a two-dollar bill?

2. John sold 4 *Saturday Evening Posts* at 5 cents each. He kept ½ the money and with the other ½ he bought Sunday papers at 2 cents each. How many did he buy?

3. If James had 4 times as much money as George, he would have $16. How much money has George?

4. How many pencils can you buy for 50 cents at the rate of 2 for 5 cents?

5. The uniforms for a baseball nine cost $2.50 each. The shoes cost $2 a pair. What was the total cost of uniforms and shoes for the nine?

6. In the schools of a certain city there are 2,200 pupils; ½ are in the primary grades, ¼ in the grammar grades, ⅛ in the high school and the rest in the night school. How many pupils are there in the night school?

7. If 3½ tons of coal cost $21, what will 5½ tons cost?

8. A newsdealer bought some magazines for $1. He sold them for $1.20, gaining 5 cents on each magazine. How many magazines were there?

9. A girl spent ⅛ of her money for car fare, and three times as much for clothes. Half of what she had left was 80 cents. How much money did she have at first?

10. Two girls receive $2.10 for making button-holes. One makes 42, the other 28. How shall they divide the money?

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11. Mr Brown paid ⅓ of the cost of a building; Mr. Johnson paid ½ the cost. Mr. Johnson received $500 more annual rent than Mr. Brown. How much did each receive?

12. A freight train left Albany for New York at 6 o'clock. An express left on the same track at 8 o'clock. It went at the rate of 40 miles an hour. At what time of day will it overtake the freight train if the freight train stops after it has gone 56 miles?

An alternate form of this Reasoning Test has been issued. Dr. Stone reports, however, that his experience in using this alternate form has shown that it is not exactly of the same difficulty as the original test. Fifteen minutes are allowed for the completion of this Reasoning Test. A full description of the test, with standards and methods of scoring, is given in a pamphlet entitled *Standardized Reasoning Tests in Arithmetic and How to Utilize Them* by Dr. C. W. Stone, which may be had from the Bureau of Publications, Teachers College, New York City. This publisher also furnishes the tests.

Buckingham Scale for Problems in Arithmetic*

A reasoning test, entitled A Scale for Problems in Arithmetic, has been devised by Professor B. R. Buckingham, of the Ohio State University.

The test is in three parts. Part I is for Grades 3 and 4; Part II, for Grades 5 and 6; and Part III, for Grades 7 and 8. In each part each problem is a little more difficult than the one preceding. The number in parentheses at the end of each problem indicates its difficulty value. In general, a pupil's score is the difficulty (value) of the hardest problem which he answers correctly. The complete test is illustrated below.

PART I—GRADES 3 AND 4

1. We learn 2 words a day in our class. How many do we learn in 8 days? (Value 27)

2. 23 children belong to our class, but only 19 are present. How many children are absent? (Value 30)

3. James has 28 marbles. He gives half of them to Charles. How many has he left? (Value 33)

4. If you can get 3 ginger-bread dogs for 5 cents, how many can you get for 10 cents? (Value 36)

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5. A boy owned 3 kites, each of them having 150 feet of string. How many feet of string had he? (Value 39)

6. A baseball team took 12 players on a trip. The trip cost the team $36. How much was that for each player? (Value 42)

7. An automobile was run 30 miles every day for a week. How many miles did it go? (Value 44)

8. Henry gathered 5 quarts of nuts. He sold them at 8 cents a quart, and spent the money for oranges at 4 cents apiece. How many did he buy? (Value 48)

9. If an electric car runs 9 miles an hour, how many hours will it take to travel from one city to another, 117 miles away? (Value 51)

10. Ned sold his rabbit for 30 cents. This was 3/5 of what he paid. What did he pay for the rabbit? (Value 53)

PART II—GRADES 5 AND 6

1. A baseball team took 12 players on a trip. The trip cost the team $36. How much was that for each player? (Value 42)

2. An automobile was run 30 miles every day for a week. How many miles did it go? (Value 44)

3. Henry gathered 5 quarts of nuts. He sold them at 8 cents a quart, and spent the money for oranges at 4 cents apiece. How many did he buy? (Value 48)

4. If an electric car runs 9 miles an hour, how many hours will it take to travel from one city to another, 117 miles away? (Value 51)

5. Ned sold his rabbit for 30 cents. This was 3/5 of what he paid. What did he pay for the rabbit? (Value 53)

6. If a girl had two one-dollar bills, three five-cent pieces, two dimes and three quarters, how much money did she have? (Value 55)

7. How many pencils can you buy for 50 cents at the rate of 2 for 5 cents? (Value 57)

8. A boy had 210 marbles. He lost 1/3 of them. How many were left? (Value 59)

9. Two tubs of maple sugar weighed 42 lbs. One weighed 18 1/4 lbs. How many pounds did the other weigh? (Value 61)

10. A store takes in the following sums: $1,250, $300, $175, $16.25, $120.50, $32.75, $68.50. It pays out $600, $360, $166.67, $33.33, $240. How much remains after payments are made? (Value 63)

11. A man bought a house for $7,250. After spending $321.50 for repairs, he sold it for $9,125. How much did he gain? (Value 65)

12. How many weeks will it take Joseph to save 21 dollars for a bicycle if he saves 1½ dollars a week? (Value 67)

13. George walks at the rate of 2⅓ miles per hour. Henry starts with him and walks in the same direction at the rate of 3 miles an hour. How many miles apart will they be in 3 hours? (Value 69)

14. If 0.78 of the weight of potatoes is water, how many pounds of water are there in a bushel of potatoes, if a bushel of potatoes weighs 60 pounds? (Value 71)

15. Two boys agreed to cut a lawn for 90 cents. The first boy worked one hour and the second boy worked 4 hours. How much money should each receive? (Value 73)

PART III—GRADES 7 AND 8

1. A man bought a house for $7,250. After spending $321.50 for repairs, he sold it for $9,125. How much did he gain? (Value 65)

2. How many weeks will it take Joseph to save 21 dollars for a bicycle if he saves 1½ dollars a week? (Value 67)

3. George walks at the rate of 2⅓ miles per hour. Henry starts with him and walks in the same direction at the rate of 3 miles an hour. How many miles apart will they be in 3 hours? (Value 69)

4. If 0.78 of the weight of potatoes is water, how many pounds of water are there in a bushel of potatoes, if a bushel of potatoes weighs 60 pounds? (Value 71)

5. Two boys agreed to cut a lawn for 90 cents. The first boy worked one hour and the second boy worked 4 hours. How much money should each receive? (Value 73)

6. Find the total cost of the following purchases: 8½ yds. flannel at 96 cents, 4¾ yds. braid at 16 cents, 12 yds. embroidery at 22½ cents, 10 yds. lace at 27½ cents. (Value 75)

7. The distance from New York to Chicago is 998 miles. The running time, by the New York Central is 19 hours. Because of floods, an engine goes only 296 miles the first 7 hours. After that how many miles per hour must it go to reach Chicago on time? (Value 77)

8. A, B, and C owned all the stock in a certain store worth $4,800. A owned 16 shares, B, 3 shares, and C, 5 shares. What was the money value of each person's share of the stock? (Value 79)

9. A real estate dealer made a profit of $5,000 on a plot of ground. The rate of profit was 20%. Find the cost of the plot. (Value 81)

10. It is estimated that if the house fly were kept from food, deaths in cities in summer would be ⅓ less. If the deaths in the summer months are at the rate of 12 per 1,000 people, how many lives might thus be saved in one summer in a city of 1,000,000 inhabitants? (Value 83)

11. I bought a cask of molasses containing 84 gallons for $28. Nine gallons having leaked out, at what price per gallon must I sell the remainder to gain $4.25? (Value 85)

12. A contractor completed ⅖ of a job in 12½ days. How many days longer should it take to finish the job? (Value 87)

13. In the manufacturing of jute twine, the raw jute costs 8⅗ cents a pound, the spinning costs ⅞ of a cent a pound, and the twisting and finishing costs 13/16 of a cent a pound. Find the cost of 1,500 pounds of jute twine. (Value 89)

14. On a map, a line 2¾ inches long represents a distance of 132 miles. How many miles are represented by a line 8½ inches long? (Value 91)

15. Mr. Wood had 250 baseballs. He sold 20% of them the first day and 40% of the remainder the second day. What per cent of his original stock was left? (Value 94)

The Buckingham Scale for Problems in Arithmetic, with standards and directions for scoring, may be obtained from the Public School Publishing Company, Bloomington, Illinois.

Monroe Reasoning Test in Arithmetic

Professor Walter S. Monroe has devised a reasoning test in arithmetic in the scoring of which he takes into consideration two values for each problem, one a "principle value," which represents the credit to be given for correct reasoning in solving it, and the other a "correct answer value," which represents the credit given for making the calculations correctly when the problem has been worked according to the correct principle. At the end of each problem, illustrated below, there is given in parentheses its "principle value" (represented by P) and its "correct answer value" (represented by C). The complete test consists of three parts. Part I is for Grades 4 and 5; Part II, for Grades 6 and 7; and Part III, for Grade 8. The three parts are given on pages 314-316.

Monroe Reasoning Test in Arithmetic

Part I—Grades 4 and 5

1. Mr. Black received $2 a yd. for broadcloth. He sold 78 yds. How much did he receive? (P = 2, C = 2)

2. If a man has $275 in the bank and draws out $70, how much has he left in the bank? (P = 2, C = 2)

3. Oats weigh 32 lbs. to a bushel. How many bushels are there in a load weighing 1,344 lbs.? (P = 3, C = 2)

4. Find the cost of 864 bags of coffee of 130 lbs. each at 6 cents per pound. (P = 4, C = 3)

5. Mary worked 20 examples on Monday and 19 on Tuesday. How many did she work in the two days? (P = 2, C = 1)

6. After traveling 75 miles, how far must I go to complete a trip of 95 miles? (P = 2, C = 1)

7. A car contains 72,060 lbs. of wheat. How much is it worth at 87 cents a bushel? (P = 4, C = 3)

8. At 3 cents per foot, what is the cost of sufficient picture molding to go around a room 14 ft. by 14 ft.? (P = 4, C = 2)

9. How many pounds of hay are raised on 6 acres at 3,804 pounds to the acre? (P = 2, C = 2)

10. A Kansas farmer bought 80 acres of cheap land for $240. Oil being found on his farm, he sold his land for $60,000. What was his profit? (P = 3, C = 2)

11. Find the contents of a box 3 ft. long, 2 ft. wide, and 2 ft. high. (P = 4, C = 1)

12. Three boys bought a rowboat for $15.75, sharing the expense equally. Find how much each boy has to pay. (P = 3, C = 2)

13. By selling butter at 24 cents per pound a lady received enough money to buy 48 lbs. of coffee at 20 cents per pound. How many pounds of butter does she sell? (P = 4, C = 2)

14. A house rents for $35 a month. This is how much a year? (P = 2, C = 2)

15. Find the change from a two-dollar bill in paying the following amounts on packages to be sent by parcel post: 12 cents, 20 cents, 8 cents, 14 cents, 32 cents. (P = 3, C = 2)

Monroe Reasoning Test in Arithmetic

PART II—GRADES 6 AND 7

1. A girl having ¾ yd. of ribbon bought ⅛ yd. more. What part of a yard had she then? (P = 2, C = 1)

2. A piece of ribbon 4 ⅜ yds. long is cut from a bolt containing 10 yds. How many yards are left? (P = 2, C = 2)

3. There are 31.5 gallons in a barrel. How many gallons are there in 63 barrels? (P = 2, C = 2)

4. If a horse eats ⅜ bu. of oats a day, how long will 6 bu. last? (P = 3, C = 2)

5. When a 20-pound cheese is worth $1.90, how much will a 10-pound cheese cost? (P = 1, C = 1)

6. Four loads of hay are to be put into a barn. The first load weighs 1.125 tons; the second, 1.75 tons; the third, 1.8 tons; the fourth 1.9 tons. Find the weight of the four loads. (P = 1, C = 2)

7. A baker used ⅗ lb. of flour to a loaf of bread. How many loaves could he make from a barrel of (196 lbs.) of flour? (P = 3, C = 2)

8. My telephone bill is $12.85 a month. At that rate how much should I pay in 2 ¾ years? (P = 2, C = 2)

9. A man spends $6.50 for board, $12.25 for clothing, $5.20 for books, and had $12 left. How many dollars and cents had he at first? (P = 1, C = 2)

10. A boy saves 1 ¾ cents on a picture by doing his own developing and printing. This makes a saving of how much on each dozen pictures? (P = 2, C = 1)

11. Make out the following account for a day. Cash on hand, $174.30, Receipts, mdse. $12.50, $6.75, $0.42, $17.30, $9.50, $42.75; Expenses, Perry and Co., bill, $75.82. (P = 3, C = 3)

12. Muslin is to be bought for 12 new curtains each requiring 2 ⅞ yds. How much will the muslin cost at 12 ½ cents a yard? (P = 3, C = 3)

13. A market man has 7,850 pounds of ice put into his refrigerator at one time. How much does it cost at $3.90 a ton? (P = 2, C = 3)

14. A man bought two suits of clothes, one costing $35.75 and the other $28.50. How much more did the one cost than the other? (P = 1, C = 1)

15. A farmer raised 500 bushels of wheat on a field of 40 acres. What was the average yield per acre? (P = 2, C = 2)

Monroe Reasoning Test in Arithmetic

PART III—GRADE 8

1. A cow that cost $56 was sold at a gain of 12½%. What was the gain? (P = 2, C = 1)

2. A man invested $1,750 and lost $192.50. What per cent did he lose? (P = 2, C = 1)

3. The wages of the men in a certain factory are to be raised 20% from their present scale. How much will a man get after the raise who is getting $2.25 before the raise? (P = 2, C = 1)

4. A hardware dealer sold a furnace for $180 at a gain of 5%. What did the furnace cost him? (P = 3, C = 2)

5. If a man saves $18.75 out of his salary of $1,250, what per cent does he save? (P = 2, C = 1)

6. A boat carried 3,125 tons of iron ore. This ore will yield 43.8% of iron. How many tons of iron in the cargo? (P = 2, C = 3)

7. What is the tax on property assessed for $12,480 at $13.50 a thousand? (P = 2, C = 2)

8. If 33⅓% of the weight of meat is lost in shrinkage when cooked, what ought a ham weighing 12 lbs. when raw weigh when baked? (P = 2, C = 1)

9. Find the interest at 5% on $640 for 4 years. (P = 2, C = 1)

10. A salesman sold $75,000 worth of goods one year. His commission was 7½% of his sales. What did he earn? (P = 2, C = 2)

11. Ellen bought a pocketbook for $0.90 just after Christmas. She paid 40% less than was asked for it before Christmas. Find the price before Christmas. (P = 3, C = 1)

12. A fast train runs from Chicago to a station 356.4 miles distant in exactly 9 hours. What is the average rate of the train? (P = 1, C = 1)

13. Books that cost $1.50 wholesale were sold at a gain of 10%. What was the selling price? (P = 2, C = 1)

14. A dealer bought $167.40 worth of clocks and sold them at a profit of 33⅓%. How much did he gain? (P = 2, C = 1)

15. A clerk had his weekly wages increased $3, or 16⅔%. What were his wages before the increase? (P = 3, C = 1)

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These tests, with standards and directions for scoring, may be had from the Public School Publishing Company, Bloomington, Illinois.

Other Arithmetic Tests

The arithmetic tests already described include those most generally used in this country. There are still other tests for arithmetic, but it is beyond the purpose of this report to attempt to enumerate them all. The bibliographies given in such books as *Educational Tests and Measurements* by Monroe, DeVoss, and Kelly; *Measuring the Results of Teaching*, by Monroe; *How to Measure*, by Wilson and Hoke; *The Seventeenth Yearbook* of the National Society for the Study of Education; and the *Bibliography of Elementary School Tests* by Professor H. W. Holmes, of Harvard University, will in general provide the necessary information for those who care to pursue this matter further.

The Courtis Standard Practice Tests in Arithmetic

The Courtis Research Tests, Series B, the Woody Arithmetic Scales, and any other tests in arithmetic have as their purpose to determine whether or not a pupil or a class is up to the standard set for a given grade and what arithmetic difficulties, if any, exist among the pupils. Such tests also enable one to compare a given class with another class or a given school with another school. But all these tests stop at this point. If the tests have shown that the children are below standard, then the teacher is confronted with a new question: How am I to remedy this difficulty? And this question becomes all the more serious when the teacher finds that practically no two pupils in her class have exactly the same difficulties. Some of the pupils are not only up to standard, but even above it. Others are deficient in only a few of the habits essential for good computing; still others in many of these habits. The teacher realizes that class exercises intended to remedy one child's difficulties may be totally unsuited for the rest of the children. The fundamental question then becomes: How can I provide remedies for the difficulties of each of the children in the class, and at the same time economize the time of the arithmetic period? The Courtis Practice Tests in Arithmetic attempt to provide a solution to these problems. In the following description of these practice tests, it is very essential that the teacher should not confuse them with the Courtis Research Tests, Series B, already described.

The Practice Tests consist of 48 different cards called "lesson cards." Each card consists of a number of examples of a given type and of equal difficulty. The types of examples on the several cards range from the simplest examples in column addition up to the most difficult problem a child in the grades is usually called upon to solve. The mastery of the examples on each card insures the mastery of some one of the many component habits that enter into skill in the four operations. In addition, for instance, one of the early lessons is designed to teach the combinations. Another provides drill in the habits involved in bridging the tens; another, the habit of carrying; another, the habit of controlling the attention span. Specimens of three of these cards are given on pages 319-321.

It should be noted that all of the cards are confined to drill on the four operations with whole numbers. As yet no provision has been made for practice on common fractions or decimals. Three cards in the set are test cards, to be used in quickly diagnosing the practice a child needs and in measuring the efficiency of the child's efforts. Card 13, for example, is one of these test cards and covers the types of problems given on cards 1 to 12.

On the back of each of the cards the answers are given, in order to make possible the correction of the tests by the pupils themselves. In using these test cards, the pupils do not write directly upon the card but upon a piece of tissue paper under which the card has been placed and through which the child easily sees the example. After the child has worked on the tissue paper all of the examples of the card, the card is taken out and turned over so that the answer side is uppermost and is again placed under the tissue paper sheet. It is then easy for the child to compare his answers with the true answers and find his own mistakes. Each pupil is provided with a record card upon which he keeps a record of his own progress in the series of lessons. The lesson cards, being printed on strong cardboard, may be used year after year since no writing is done on them. It is the supply of tissue paper sheets that is renewed each year.

The examples in each lesson have been so chosen with respect to their number and difficulty that all the lessons require the same working time—that is, an eighth-grade child of standard ability will require three minutes to finish each and every lesson.

Courtis Standard Practice Tests

LESSON No. 1—ADDITION

Scores: Tried........; Right.........

Trial. No.................

9	4	7	8	8^{5}	2	6	9	4	5^{10}	1	3
3	0	8	6	1	6	7	3	5	3	3	9
6	4	8	5	2	6	7	0	4	9	8	4
4	2	7^{15}	6	1	5	3	3^{20}	9	4	1	5
7	1	5	0	6	4	8	4	2	0	6	6
4	9	7	9	1	7	9	8	3	3	7	8
8^{25}	9	7	8	3	6^{30}	3	4	6	8	5^{35}	4
8	1	9	9	3	1	9	3	9	6	1	7
7	5	0	8	1	4	9	4	7	0	2	8
8	8	9	5^{40}	2	5	7	7	9^{45}	4	2	7
2	4	7	1	2	2	5	5	7	8	2	2
9	0	5	8	5	5	8	6	0	8	6	4
9	8^{50}	7	0	7	8	6^{55}	9	7	9	6	6^{60}
5	6	6	3	9	5	1	8	6	7	0	4
9	3	2	5	6	5	4	4	2	6	7	2
1	7	7	7	5^{65}	1	6	6	2	8^{70}	8	4
4	0	8	4	2	3	7	4	6	0	9	6
6	7	6	5	8	7	9	4	9	5	5	4

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Courtis Standard Practice Tests

LESSON No. 13—TEST A (Lessons 1-12)

Scores: Tried........; Right.........

Trial No.................

Add

3	8	6	8	4	9
9	6	8	9	2	5
9	1	7	6	3	9
3	7	1	3	2	2
8	2	3	8	5	3
6	3	9	6	7	6
3	9	3	2	2	1

Subtract

152	118	165	94
56	68	97	23
120	152	90	57
97	78	71	29
71	121	125	123
37	46	57	38

Multiply

31	42	23	71	62	51
24	23	13	42	24	47

Divide

$21\overline{)399}$ $32\overline{)672}$ $43\overline{)559}$

$51\overline{)1887}$ $34\overline{)2788}$

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Courtis Standard Practice Tests

Lesson No. 18—Multiplication

Scores: Tried........; Right.........

Trial: No...................

582 37	704 48	593 59
691 62	374 17	268 45
670 69	348 83	768 9

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The program for the use of these cards so that each child secures the exact type of practice he needs without the necessity of all children doing the same thing, is given by Mr. Courtis as follows:

> At the beginning of the term, the teacher gives Lesson 13 (Test A) to determine which children need drill on Lessons 1 to 12. Those who have perfect scores on Test A are excused from drill work until the rest of the class has completed Lessons 1 to 12. Their drill time is spent upon other work. Experiments have proved that the children who are excused not only do not need drill, *but are likely to be injured by it,* so that they would have lower scores after taking the drill lessons.
>
> The next day the remaining children take Lesson 1. Make these children understand that the practice lessons are to develop in them the abilities which children who were excused already have. The third day those who are successful with Lesson 1 take Lesson 2, and so on. But those who failed on Lesson 1 spend the third drill period in practicing the examples in the lesson, and the following day try the lesson again, to test the effectiveness of their practice. Only when Lesson 1 has been completed successfully do they go on to Lesson 2.

In this way each child masters the simple work before attempting the more complex; each child practices only on the type of example in which he falls below the standard, and each child progresses as fast or as slowly as his native powers and personal efforts allow. At the end of two months each child in a class of 40 may be working on a different lesson.

To a teacher reading the foregoing description for the first time, it may seem that it would be very confusing to have children working on different lessons and progressing at different rates. It must be remembered, however, that as far as the teacher is concerned his work is the same whether the child is on Lesson 1 or on Lesson 40, and that at the start (the most confusing time) all the children work on the same lesson. By the time the number of lessons in use reaches four or five, the system will be running smoothly and will give no trouble.

The practice set provides two other test papers, Test B (Lessons 30 and 31) which is more difficult than Test A, and Test C (Lesson 44) which is still more difficult than Test B.

If half of the class or more have been excused from Lessons 1 to 12, give Test B (Lessons 30 and 31) to these children on the next day. If, again, half the class have perfect papers, give Test C (Lesson 44). In other words, children who fail on Test A should start on Lesson 1, those who succeed on Test A (Lesson 13) but fail on Test B should start on Lesson 14, while those who have perfect papers with the first two tests but fail on Test C should start on Lesson 32.

That is, Tests A, B, and C divide the series of lessons into three groups. Lessons 1-12, 14-29, 32-43, each more difficult than the one before it, so that by means of the tests the teacher is able to start each child at the exact point in the series where he needs drill.

Children who complete all the tests successfully do not need the slightest drill work in the four operations, as they already have more than average adult ability in these skills. Mr. Courtis gives emphatic warning that the drill lessons are designed only for children who need them. Failure to determine the needs of children and to adjust individual work accordingly is one of the greatest factors operating to decrease the effectiveness of almost all the drill work found in common practice.

Detailed instructions for giving these practice tests are given in the *Teacher's Manual for Courtis Standard Practice Tests in Arithmetic,* by Stuart A. Courtis, published by the World Book Company. Yonkers, New York. The tests themselves may also be had from the World Book Company.

Studebaker Practice Exercises in Arithmetic

Another series of practice exercises in arithmetic, designed to serve a purpose similar to that of the Courtis Practice Tests, has been prepared by J. W. Studebaker, Superintendent of Schools of Des Moines, Iowa. This series aims, as does the Courtis set, to give each child the particular drill on the four fundamental operations with whole numbers that he needs. No provision is made for common fractions or decimals.

The series consists of fifty different exercises, each exercise on a card in which holes are cut at the point in the card where the answer would ordinarily be written. The pupil writes the answer through this hole on a piece of ordinary paper placed under the card. Each exercise card has the answers printed on the back. When the card is placed, answer side up, in such a position that the holes in the card come directly over the answers the pupil has written on the paper, then the pupil can compare his answers with the correct ones given on the card. Thus the system is self-scoring. Since nothing is written on the card, the same set of cards can be used year after year.

Complete directions for giving these tests are to be found in the *Teachers' Manual for Studebaker Economy Practice Exercises in Arithmetic*, by John W. Studebaker, published by Scott, Foresman and Company, Chicago, who also supply the tests themselves.

There are still other practice exercises in arithmetic. It seems unnecessary to attempt to mention these here, since, so far as the author can determine, they are not as comprehensive as those already mentioned.

PART II. TESTS IN ALGEBRA

The Rugg-Clark Tests in First-Year Algebra

On page 281 we have pointed out that the teaching of arithmetic consists of two distinct problems: (1) the teaching of the machinery of arithmetic which is made up of the fundamental operations, and (2) the teaching of a number of life practices concerned with the problems of arithmetic. Similarly, the teaching of algebra has two distinct divisions: (1) the machinery of algebra, and (2) the problem or reasoning material in the working of which the machinery is used. Included in the machinery of algebra are such operations as: algebraic addition, subtraction, multiplication, division, factoring, the solution of equations of the several types, the removal of parentheses, exponents, the handling of radicals, etc.

After the initial explanation of these bits of machinery, it is believed by a number of investigators that they should become automatic habits, like the corresponding processes in arithmetic, so that the child can use them quickly and automatically. These investigators feel that it is not only desirable that the child should understand the principles involved in these fundamental algebraic processes, but also that he should attain such skill in the handling of this fundamental machinery that he will not have to think about it when he is using it in the solution of word or verbal problems.

In the first-year course in algebra we are perhaps much more conscious of this machinery than we were of this same element in our arithmetic classes, since it is customary in algebra to teach practically all this machinery during the first year, while in arithmetic the work of learning the fundamental operations is distributed over a period of four to five years. Because of the great task that has been assigned to the first year's study of algebra, the conventional courses in this subject have been predominantly a study of the fundamental operations. Verbal problem material and other applications have been relatively a minor offering.

It is not surprising, therefore, that the tests which have been devised for algebra should have followed to a great extent the pattern set by arithmetic tests, and should have devoted themselves in large measure to a test of the machinery of this subject.

The major part of the Rugg-Clark Standardized Tests for First-Year Algebra is a series of tests on different phases of the machinery of algebra. The Rugg-Clark tests, however, also provide means of testing the ability of pupils to solve word or verbal problems.[5] Those tests which are devoted to the machinery of algebra are included in two pamphlets, which together contain tests for each of the following operations:

---

[5] The original edition of the tests included a test for graphs and one for the solution of quadratic equations with irrational roots. These have been discontinued since the first publication of the Rugg-Clark tests. Dr. H. G. Hotz has published a superior graph test which is described on page 342.

1. Collecting Terms
2. Evaluation
3. Subtraction
4. Simple Equations
5. Parentheses
6. Special Products
7. Exponents
8. Factoring
9. Clearing of Fractions
10. Fractional Equations
11. Other Formulae
12. Quadratic Equations.
13. Simultaneous Equations
14. Radicals

Tests 1 to 9 are printed in one pamphlet, and tests 10 to 14, in a second pamphlet. In selecting these fourteen operations as the basis of the testing of the formal side of algebra, the authors chose these particular fourteen as those which they considered most commonly taught in the conventional first-year courses in algebra, the selection having been made from some thirty formal processes which they had found altogether. The fourteen tests given represent, therefore, operations which were shown to be most frequently taught, after a careful analysis of ten of the most commonly used textbooks in first-year algebra. The authors wish to emphasize that in their opinion these tests were constructed to fit the course *now taught in the public high schools*, it being assumed that the textbooks in question were representative outlines of these courses. The authors are equally emphatic in stating that they believe that these fourteen operations do not necessarily fit the content of an ideally organized course.

Specimens of Test No. 2 (Evaluation), Test No. 4 (Simple Equations), Test No. 5 (Parentheses), Test No. 7 (Exponents), Test No. 8 (Factoring), Test No. 11 (Formulæ), Test No. 13 (Simultaneous Equations), and Test No. 14 (Radicals) are given on pages 326-329.

The following characteristics of these tests are important:

1. Each test is made up of a series of examples, each of which is designed as a specific test for a definite, formal operation in algebraic manipulation.

2. The operations selected for testing are based upon a careful analysis of the mental processes involved in each operation, each problem aiming to test some ability essential for success in first-year algebra. The Rugg-Clark tests have been constructed, therefore, by the analytic method rather than by the statistical method used in the Woody Arithmetic Scales, or in the Hotz Algebra Scales described later.

Rugg-Clark Tests in First-Year Algebra

TEST NO. 2. EVALUATION

1) If $x = 4$ and $y = 2$ what does $2x^2 - 3xy$ equal? Answer......
2) If $a = 3$ and $b = 2$ what does $3ab + ab^2$ equal? Answer......
3) If $c = 2$ and $d = 5$ what does $cd^2 - 2cd$ equal? Answer......
4) If $p = 4$ and $q = 3$ what does $p^2 + 4pq$ equal? Answer......
5) If $x = 3$ and $y = 5$ what does $x^2 + 2x^2y$ equal? Answer......
6) If $a = 3$ and $b = 2$ what does $3a^2 - 2ab$ equal? Answer......
7) If $r = 4$ and $s = 2$ what does $2rs + rs^2$ equal? Answer......
8) If $u = 3$ and $v = 4$ what does $uv^2 - 3uv$ equal? Answer......
9) If $x = 5$ and $y = 2$ what does $x^2 + 3xy$ equal? Answer......
10) If $b = 3$ and $c = 2$ what does $b^2 + 3b^2c$ equal? Answer......

(Test No. 2 contains 10 more problems similar in form and arrangement to those above.)

TEST NO. 4. SIMPLE EQUATIONS

1) $2x + 3 = 11$ Answer.............
2) $4c = 6c + 12$ Answer.............
3) $5x - 3 = -20$ Answer.............
4) $13 = 2x - 8$ Answer.............
5) $12x - 7 - 15x = 10$ Answer.............
6) $3x + 4 = 16$ Answer.............
7) $8p = 15p + 14$ Answer.............
8) $7x - 6 = -29$ Answer.............
9) $17 = 3x - 8$ Answer.............
10) $9x - 8 - 13x = 7$ Answer.............
11) $5x + 2 = 27$ Answer.............
12) $6t = 9t + 21$ Answer.............
13) $8x - 7 = -35$ Answer.............

(Test No. 4 contains 12 more problems similar in form and arrangement to those above.)

Rugg-Clark Tests in First-Year Algebra

TEST NO. 5. PARENTHESES

1) $6(3x+8)$ Answer.............
2) $5(4x-2)$ Answer.............
3) $-3(8x+3)$ Answer.............
4) $-4(3x-4)$ Answer.............
5) $9(-7x-1)$ Answer.............
6) $-8(-4x-7)$ Answer.............
7) $8(5x+4)$ Answer.............
8) $7(4x-3)$ Answer.............
9) $-5(6x+7)$ Answer.............
10) $-6(4x-8)$ Answer.............
11) $7(-8x-3)$ Answer.............
12) $-9(-3x-6)$ Answer.............

(Test No. 5 contains 30 more problems similar to those above.)

TEST NO. 7. EXPONENTS

1) $a^3 \cdot a^5 =$ ....
2) $5x^7 \cdot 6x^5 =$ ....
3) $(n^2)^3 =$ ....
4) $\frac{c^3}{c^2} =$ ....
5) $(ba^3)^4 =$ ....
6) $x^7 \cdot x =$ ....
7) $e^2 \cdot e^4 =$ ....
8) $4y^6 \cdot 5y^3 =$ ....
9) $(x^3)^4 =$ ....
10) $\frac{m^5}{m^3} =$ ....

(Test No. 7 contains 26 more problems similar to those above.)

Rugg-Clark Tests in First-Year Algebra

TEST NO. 8. FACTORING

1) $5x^2 + 15x^3$ Answer.............
2) $a^2 - 64$ Answer.............
3) $y^2 - 6y + 9$ Answer.............
4) $b^2 + 11b + 28$ Answer.............
5) $5x^2 + 16x + 3$ Answer.............
6) $6a^2 + 9a^3$ Answer.............
7) $x^2 - 16$ Answer.............
8) $t^2 - 16t + 64$ Answer.............
9) $a^2 + 12a + 27$ Answer.............
10) $9x^2 + 36x + 32$ Answer.............

(*Test No. 8 contains 15 more problems similar to those above.*)

TEST NO. 11. FORMULAE

1) $P = ahw$ Solve for h Answer......
2) $c = \frac{E}{R}$ Solve for R Answer......
3) $E = \frac{PL}{K}$ Solve for P Answer......
4) $L = \frac{Mt - g}{t}$ Solve for M Answer......
5) $I = \frac{bd^3}{3}$ Solve for b Answer......
6) $E^2 = \frac{JWhr}{t}$ Solve for h Answer......
7) $i = prt$ Solve for r Answer......
8) $s = \frac{W}{L}$ Solve for L Answer......

(*Test No. 11 contains 13 more problems similar to those above.*)

Rugg-Clark Tests in First-Year Algebra

TEST NO. 13. SIMULTANEOUS EQUATIONS

1) $2x + y = 10$
$3x - 2y = 1$ Answer.............

2) $2x = 3y + 3$
$5x + 3y = 39$ Answer.............

3) $4n - 2r = 0$
$3n + 5r = 13$ Answer.............

4) $5b + c = 21$
$3b - 2c = 10$ Answer.............

5) $4p = 3s + 12$
$5p + 3s = 42$ Answer.............

6) $5x + 4y = 28$
$8x - 3y = 26$ Answer.............

(*Test No. 13 contains 9 more problems similar to those above.*)

TEST NO. 14. RADICALS

Leave answer in *simplest* radical form.

1) $\sqrt{8}$ Answer.............
2) $\sqrt{a^3b^4}$ Answer.............
3) $\sqrt{2/3}$ Answer.............
4) $\sqrt{27}$ Answer.............
5) $\sqrt{x^3y^6}$ Answer.............
6) $\sqrt{3/10}$ Answer.............
7) $\sqrt{12}$ Answer.............
8) $\sqrt{x^5y^8}$ Answer.............

(*Test No. 14 contains 13 more problems similar to those above.*)

3. All of these tests are "rate" or "time" tests. In this respect, therefore, they are like the Courtis Research Tests in Arithmetic. Test No. 2, for example, is allowed four minutes; Test No. 4, three minutes; and Test No. 7, two minutes. The time for each test has been so fixed that no pupil can quite finish the test in the time given; it is also fixed so that all pupils can do a considerable number of the problems in the test. Great care is taken to see that all pupils start and stop the test the same instant. The method of giving the test is similar to that described for the Courtis tests in arithmetic. All of Tests 1 to 9 (printed in one pamphlet) are to be given in one class period. Tests 10 to 14 (printed in the second pamphlet) are to be given in a second class period.

4. The examples in each of the tests are arranged in "groups" or "cycles," the examples of each cycle of a given test being of the same difficulty as those of each succeeding cycle. For example, in Test No. 4 printed on p. 326, note that examples 1, 6, 11 are of the same algebraic form and of the same difficulty. The same thing is true of examples 2, 7, 12 and examples 3, 8, 13, etc. By this arrangement examples 1 to 5 form one cycle, examples 6 to 10, a second cycle, etc., each cycle of a given test thus being of the same difficulty and containing the same types and variety of problems arranged in the same order. It should be noted, however, that the level of difficulty of the cycles of one test, like Test No. 4, is not the same as that of another test, like Test No. 10. The levels of difficulty vary from test to test.

This cycle arrangement has been rigidly followed throughout each of the fourteen tests covering the machinery of algebra, and is a distinguishing and very important feature of the Rugg-Clark tests. By means of this cycle arrangement, each test gives the pupil the opportunity to do the same kind of an example several times. In Test No. 4, for example, if the pupil fails on problem 3 because of lack of knowledge concerning the solution of that type of equation, he will also fail in problems 8, 13, etc. In other words, the pupil will make recurring errors, such errors being the important ones to discover for they always indicate fundamental weaknesses in learning on the part of the pupil and mistaken emphasis in teaching on the part of the instructor. It therefore follows that a test constructed on the cycle principle is of considerable use in diagnosing a pupil's fundamental errors, which is the type of error the teacher is much concerned to correct. This cycle arrangement is also of some assistance in locating accidental errors which may be due to lack of concentration, fatigue, or unfavorable conditions of work, but which do not recur with definite regularity.

The location of recurring errors shows one service of a standardized test of the Rugg-Clark type as a means of diagnosing difficulties of pupils and of improving instruction by indicating where the stress should be laid, and by stimulating the teacher to invent means of preventing such misunderstanding. Thus the test becomes a means of promoting economy of time in teaching.

Some of the types of errors which have been discovered by means of the Rugg-Clark tests are shown in the following illustrations:[6]

TEST 1. *Collecting Terms.* The error common to practically all algebraic work appears in the first test which is the mistake in signs in combining similar terms. The terms are recognized as similar, but before all are combined the pupil loses the sign, e. g., in $5x^2 + 3 - 3x^2 - 2 - 7x^2 - 5$, he combines $5x^2$ and $-3x^2$, but in combining this result with $-7x^2$ the sign of the $2x^2$ (which is positive) drops from the focus of his attention, and with the stimulus of the $-7x^2$, he combines $-2x^2$ and $-7x^2$, giving $-9x^2$.

The next most frequent error is that of adding the coefficients of similar terms without regard to their sign, e. g., in the foregoing exercise the pupil gets $15x^2$. Less frequent than these are: adding exponents of similar terms; failure to recognize similar terms at all (combines all terms); equating terms, evidently traceable to the influence of the equation; and accidental errors in omission of letters, exponents, signs, etc.

TEST 2. *Evaluation or Substitution.* Relatively few errors are made in this operation. Of the most frequent it will be noted that nearly one-third are due to squaring the product of literal factors instead of the one factor designated by the exponent. Thus the evaluation of ab^2 is many times more difficult than $2ab$. Here, as in most of the subsequent operations, the prevalence of a great amount of inaccuracy of a purely arithmetical nature is in evidence. Practically half of the errors in this operation are of this kind.

TEST 4. *Simple Equations.* By far the greatest difficulty in solving simple equations is that illustrated by problems of the type $4c = 6c + 12$. The pupil finds it very difficult to combine properly the similar terms $4c$ and $-6c$. He forgets the sign of $2c$, obtaining for his result $c = 6$. Evidently he knows how to "transpose" the $6c$ in the right member. He either does not think "$-2c$" at all, or, when his attention is immediately on the division of 12 by the coefficient of c, the sign drops from his mind. Following in frequency comes the ever-present arithmetical difficulty. Other errors of lesser difficulty are: error in signs in division, e. g., $-3x = 12$, $x = 4$; incomplete solution, e. g., leaving the work in the form $5x = 25$; the tendency to combine dissimilar terms, e. g., $3x - 4 = 14$, giving $x = 2$; and the inverted result in division, e. g., $5x = 13$, $x = \frac{5}{13}$.

---

[6] A complete tabulation showing the extent to which pupils commonly make each error will be found on pages 180-183 of Rugg and Clark, *Scientific Method in the Reconstruction of Ninth-Grade Mathematics.*

TEST 5. *Parentheses.* A study of the table shows that a rather high degree of efficiency has been secured in the teaching of this operation. As would be expected, about one-half of the mistakes were in the use of signs. Of these, twice as many were made when the minus sign was used outside the parentheses as when the minus sign preceded the first letter inside. The writers wish to explain here that no examples involving a complexity of symbols of aggregation have been included in this test. The importance of this kind of manipulation did not seem to warrant its inclusion.

TEST 7. *Exponents.* There are three prominent errors in the use of exponents. Reference to the table will indicate that the addition of exponents in involution involves the greatest difficulty. Following this is the error of squaring the exponent instead of multiplying as in $(x^3)^2 = x^9$, and the failure to raise all the factors of the product to the required power as in $(ab^2)^2 = ab^4$. A bit of psychological analysis at this point will show that specific knowledge of these errors will tend to increase markedly the teacher's effectiveness. Our experimentation proves this conclusively.

TEST 11. *Formulae.* The error most in evidence in this operation is that involved in selecting the coefficient of the unknown in the solution of fractional equations. For example, in $L = \frac{Mt - g}{t}$, it is common to find $M = \frac{Lt}{t - g}$ showing that the pupil considers $(t - g)$ as the coefficient of M. Then follows the tendency in division of each side of the equation to get the result inverted, for example, $V = LWH$, $W = \frac{HL}{V}$. The next most important error is interchanging factors on both sides of the equation, for example, $E = \frac{PL}{K}$, $P = \frac{EL}{K}$. The fact of the recurrence of this error certainly indicates a lack of clean-cut practice in habituation to this type of work.

The method of giving the tests to classes is similar to that used in the Courtis tests, each test being given under rigid time conditions. In scoring the papers of each pupil, a record is made of both the number of problems correctly solved and the number attempted. The class score is obtained by finding the median of the scores of all the pupils, as was done in the Courtis tests.

As a result of giving the Rugg-Clark tests in a great many first-year algebra classes, standard scores have been developed for each of the parts of the test. Some teachers have felt that the average of the scores of all these schools should be taken as the standard of attainment, but the authors believe that such an average sets up too low a standard. They have, therefore, taken the average of the upper third of these schools as the norm of attainment in first-year algebra. These standards, together with directions for giving and scoring the tests, and the tests themselves, may be had from the University of Chicago Book Store, 5802 Ellis Avenue, Chicago, Ill.

In addition to the tests already described which cover certain formal operations which make up the machinery of algebra, the Rugg-Clark series also provides tests for verbal problems, a specimen of which is given below:

25. Express the following verbal statement in algebraic form: the square of a side plus five.

30. If you represent a number by x, how will you represent 5 more than 5 times the number?

35. W and L are the width and length of a rectangle; write the equation for its area in terms of W and L.

40. 8 times a certain number equals 45 diminished by the number. State the equation by which you would find the number.

45. If the width of a rectangle is W increased by 10 and its length L increased by 20, write the equation for its perimeter.

50. What number has the property that when multiplied by ⅚, the result is greater by 1 than when multiplied by ⅘?

55. Two boys play at teeter. One weighs 100 pounds and sits 6 feet from the point of support. The other weighs 120 pounds. How far from the point of support must he sit in order to make the board balance?

60. A father 54 years old has a son aged 9 years. In how many years will the age of the father be just 4 times that of the son?

65. If a boy 4¾ feet tall casts a shadow 4½ feet long at the same time that a school building casts a shadow 57½ feet long, how high is the school building?

70. Twice the width of the Pennsylvania station in New York exceeds its length by 80 feet. 4 times the length exceeds the perimeter by 700 feet. Find dimensions.

75. Find two numbers whose sum is 51, such that if the greater is divided by their difference the quotient is 3½.

80. If a cistern can be filled by one pipe in x minutes and emptied by another in $(x-5)$ minutes, what part of the cisternful runs in one minute if both pipes are open?

85. A train running from Chicago to Denver at an average speed of 40 miles an hour takes 3 hours longer to make the run than one running at 45 miles an hour. What is the distance from Chicago to Denver?

The following characteristics of the verbal problem test should be noted:

The verbal tests are not sharply timed. Generally one class period is given to a list of problems like that in the sample test given above. The verbal problems in each list range from very easy to very difficult by approximately equal steps, each problem being scored on an absolute scale of difficulty from 0 to 100. No problem has been included in the lists which is harder than 85 on this scale or easier than 25. The number at the left of each problem in the above list indicates its *difficulty* and is not the serial number of the problem. Because of the gradation of the problems a teacher who uses these tests can distinguish between pupils of widely varying reasoning abilities. It should be emphasized that these tests are not tests of the rate at which pupils can do algebraic work. On the contrary, they are tests of pupils' reasoning and interpreting abilities, the training of which is the real goal of instruction.

The authors of the Rugg-Clark tests, in common with the authors of other tests, have found many practical difficulties in getting a satisfactory verbal problem or reasoning test for algebra. No one has yet had the time to make the detailed study and analysis of the applied problems of algebra which has been made on the mechanical side of that subject. And even if the time were available for such a study, the task is not so definite or so easy as for the more formal work. No teacher has ever been satisfied with the traditional stock of verbal problem material found in most of our algebras. It has always been felt and hoped that many more types of problems could be found which make use of the elementary tools of algebra, and more and more this hope is being realized. It is difficult to make a test when one doesn't know the range of material to be tested. Hence, any verbal problem or reasoning test is bound to be unsatisfactory to a number of teachers for some time to come, since they won't agree on the range or quality of the material included in the test. In other words, every algebra test is a definite reflection of the ideas of its author respecting the desirable aims or outcomes of a course in algebra.

In conclusion it should be stated that the authors of the Rugg-Clark tests realize that in their tests undue emphasis may seem to have been put upon the formal aspects of first-year algebra. They have prepared a monograph entitled *Scientific Method in the Reconstruction of Ninth-Grade Mathematics,* which is based on several years' experience with these tests and in which they have set forth completely their point of view with reference to the first-year algebra course and the position of formal work in such a course.

A quotation from that monograph will make clear the opinion of the authors regarding the real function of these tests and exercises:

> "The fundamental aim of instruction in first year algebra is to develop in the pupil the ability to use intelligently the most powerful instruments of quantitative thinking: the *equation*, the *formula*, the *graph*, and the *properties of the most common space forms*. Only in so far as habituation or automatic efficiency in the formal processes contributes to efficiency in the solution of problems primarily of the reasoning or interpretive type do we insist upon skill of habituation in them.
>
> "There is evidence, however, both of expert opinion and of experimentation, that ability in the solution of new and original problems is accompanied by, and varies roughly with, the degree to which the pupil has habituated these essential tool operations. We recognize, therefore, that this phase of the teaching process is subsidiary to, although a necessary correlate of, the more fundamental aim stated above."

The tests, together with directions for giving and scoring them, may be obtained from the University of Chicago Bookstore, Chicago, Ill.

The Rugg-Clark Practice Exercises in First-Year Algebra

As a result of giving standardized tests in algebra, teachers have come to ask two important questions: How can I remedy the weaknesses which the standardized algebra tests have shown my pupils to have? And, another year, how can I provide an economical means of giving the pupils through mass instruction the necessary skill in those formal operations in which they make the greatest number of errors?

The attempt to answer these questions is found in the Rugg-Clark Standardized Practice Exercises in First-Year Algebra, which have been widely used since 1917. These practice exercises in algebra correspond to the practice exercises in arithmetic prepared by Mr. Courtis.

The Rugg-Clark practice exercises consist of seven cards, each card containing two sets of exercises. These exercises include such operations as collecting terms, evaluation, the solution of simple equations, parentheses, subtraction, exponents, simultaneous equations, fractional equations, radicals, etc. They are thus seen to be similar to the 14 standardized tests mentioned on page 325. At the top of each test the standards for that particular test are indicated. The pupils do not write the answers upon the card but upon a sheet of paper laid under and at the side of the card. Answers are given on the back of the card so that the pupils can correct their own work. The set includes forms on which each pupil can keep a definite record of his progress. A specimen of one of these exercise cards is given on p. 337.

The purposes of these exercises are as follows:

1. In general, to make the pupil more efficient in his skill in manipulating the "tool" operations of first-year algebra.

2. To concentrate practice upon those operations that have been shown to be particularly difficult for the pupil, i. e., those types of problems in which the pupil makes the greatest number of mistakes.

3. To set up for the pupil and the teacher a definite, objective goal of achievement to be attained, after which further practice will probably not yield results commensurate with the time required.

4. To employ as a teaching device one of the most influential factors of learning, namely, short, spirited, frequent, "timed" practice periods.

5. To stimulate the pupil to compete against his own previous record, made definite and objective, and kept by himself as a measure of his own progress in the mastery of the subject matter.

6. To help reduce, as much as possible, the great amount of time now being given to class drills which careful measurement and individual experience show to be ineffective.

7. To provide a means by which individual differences may be practically recognized and met.

The following suggestions are made by the authors concerning the use of the practice exercises:

1. It is important to make clear that the practice exercises are in no wise intended to take the place of, nor to be given simultaneously with, the most thorough rationalization and explanation of the particular types of problems under consideration. They should be given after the class has clearly grasped the meaning and significance of the operation. The length of time devoted to getting this clear understanding of the various operations must necessarily vary with the teacher.

2. It is suggested that practice should be given on a particular exercise (e. g., the one on collecting terms, if that operation has just been explained and discussed thoroughly) each day until 75 per cent of the class attain, two days in succession, the standard number of problems set for that exercise, which is printed at the top of the card. That is, excuse any individual pupil from practice on a particular exercise as soon as he has made the standard score twice in succession.

Rugg-Clark Standardized Practice Exercises in First-Year Algebra

SET NO. 4. PARENTHESES.

Standard: 18 "rights" in 2 minutes.

1. $7(2a^2 - 3)$ ....
2. $-5(6b + 4)$ ....
3. $-3(-4c - 7)$ ....
4. $5(-7x^3 - 3)$ ....
5. $-(4a - 12)$ ....
6. $-(-60 + x)$ ....
7. $-7(-3q^4 - 6)$ ....
8. $-4(6p + 7)$ ....
9. $6(-7x - 8)$ ....
10. $-9(5x^3 - 3)$ ....
11. $-(-32 + y)$ ....
12. $-(4b - 17)$ ....
13. $-9(3y^2 + 4)$ ....
14. $8(-5p - 6)$ ....
15. $5(4x - 7)$ ....
16. $-(-27 + 8q^3)$ ....
17. $-6(7s - 5)$ ....
18. $-5(-4t - 9)$ ....

SET NO. 3. SUBSTITUTION.

Standard: 10 "rights" in 4 minutes.

1. If $c = 4$ and $f = 2$ what does $2c^2 - 3cf$ equal?
2. If $a = 3$ and $b = 2$ what does $3ab + ab^2$ equal?
3. If $x = 3$ and $y = 4$ what does $xy^2 - 2xy$ equal?
4. If $c = 5$ and $d = 2$ what does $\frac{c^2}{5} + \frac{4cd}{4}$ equal?
5. If $r = 2$ and $s = 4$ what does $r^2 + 3r^2s$ equal?
6. If $d = 3$ and $e = 4$ what does $4d^2 - 2de$ equal?
7. If $m = 2$ and $n = 3$ what does $2mn + mn^2$ equal?
8. If $a = 4$ and $b = 5$ what does $ab^2 - 4ab$ equal?
9. If $x = 4$ and $y = 3$ what does $\frac{x^2}{2} + \frac{4xy}{6}$ equal?
10. If $p = 3$ and $q = 5$ what does $p^2 + 2p^2q$ equal?
11. If $a = 3$ and $b = 2$ what does $3a^2 - 2ab$ equal?
12. If $r = 4$ and $s = 2$ what does $2rs + rs^2$ equal?
13. If $u = 3$ and $v = 4$ what does $uv^2 - 3uv$ equal?
14. If $x = 5$ and $y = 2$ what does $\frac{x^2}{5} + \frac{3xy}{6}$ equal?
15. If $b = 3$ and $c = 2$ what does $b^2 + 3b^2c$ equal?

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3. The presence of rapid, average, and slow pupils in the same class necessitates a decision concerning the best use to make of the time of the rapid pupil. Acting on the principle that we want the "optimum" amount of formal skill, not the extreme amount of which the pupil is capable, we suggest excusing the pupil from the class to work on problems of application or interpretation concerning those operations which have thus far been explained. That is, experience is proving that it is better to give the fast pupil extensive training with each operation, both "formal" and "applied," rather than to permit him to move on at once to the learning of other types of subject matter. In this way the members of the class are held together as regards formal skill, but are permitted to get as much facility as they are capable of in the use of algebraic devices in "thinking" situations.

In order to excuse the rapid pupil from the practice exercise the teacher must have "problem" work of some kind at hand from which to make assignments.

4. With the standard skill once attained on a given operation the drill time of the class goes largely to a new operation. The teacher, however, faces the problem of giving *review practices to hold the skill.* It is probable that it will be helpful to review a given exercise under time conditions as often as once a week, and not much oftener.

5. One of the most valuable outcomes of the use of the practice material is the part taken by the pupil in correcting the work, in noting his mistakes, in recording his own achievement, and in comparing it with the standard and with the achievements of other pupils in the same class. Thus systematize from the start the scoring, recording, and criticizing of the work done. If you feel that the pupils should not correct their own work, have them exchange papers.

One of the dangers against which all teachers must guard is that of giving these practice exercises too soon after the preliminary explanation of the new topic. Such a practice tends to make the pupil forget the principles upon which the various steps in a solution are based, and thus may make his work more mechanical and thoughtless than ever. It is very desirable, therefore, to have rationalization or explanation repeated by the pupil from time to time after the practice exercises are begun. There is little question, however, that the practice exercises, if properly and conservatively used, will promote real economy in the teaching of the "tool work" of algebra.

Copies of the tests may be obtained from the University of Chicago Bookstore, Chicago, Ill.

The Hotz First-Year Algebra Scales

The Hotz First-Year Algebra Scales were devised by Henry G. Hotz, Professor of Secondary Education in the University of Arkansas, and were first published in 1918.

These algebra scales are designed to measure the ability of pupils in first-year algebra, and consist of five different sheets or scales of algebraic exercises. These five scales are given the following titles: (1) Addition and Subtraction, (2) Multiplication and Division, (3) Equation and Formula, (4) Graphs, (5) Problems. The first two scales are designed to test the achievement of pupils in the fundamental operations involving integral, fractional, and radical expressions; the second two, to test the ability of pupils in handling the instruments of quantitative thinking; while the last is composed of verbal problems of the type usually stressed in first-year algebra.

There are two series of these scales, Series A and Series B. Series B is the long one, and contains from 11 to 25 exercises in each scale. Series A is an abridged edition of about half the length of Series B, containing from 8 to 12 exercises in each of the five scales. The exercises in each scale of Series A cover just as wide a range of difficulty as those of Series B.

Specimens of four scales of Series A are given on pages 340-347.

In the specimen scales given, it should be noted that the exercises in each scale are arranged in order of difficulty, that is, each scale begins with exercises so easy that they can be solved by practically every member of a class; each succeeding exercise, however, becomes increasingly more difficult, so that the last exercises in each scale can be solved by only a relatively small number of the pupils who try them. This arrangement of the exercises according to their difficulty is a distinctive feature of the Hotz Algebra Scales. In this respect, therefore, they are constructed on an entirely different principle from that found in the Rugg-Clark Algebra Tests, where each test has its exercises arranged in cycles of equal difficulty. It is thus seen that the Hotz Algebra Scales embody the same principle of construction as is used in the Woody Arithmetic Scales, and they thus serve for algebra the same purposes as are served for arithmetic by the Woody scales. Similarly, the Rugg-Clark tests may be compared to the Courtis tests in arithmetic.

As in the Woody scales, the difficulty of each exercise in the Hotz scales was determined by the percentage of pupils solving each exercise correctly, the exercises having been derived from data obtained from tests given to over 16,000 high-school students. By an elaborate statistical treatment of these results, Dr. Hotz has been able to make

Hotz First Year Algebra Scales

Series A

ADDITION AND SUBTRACTION

Carefully perform the operations as indicated.

1. $4r + 3r + 2r =$

3. $12b + 6b - 3b =$

5. $7x - x + 6 - 4 =$

9. $(4r - 5t) + (s - 3r) =$

10. $8c - (-6 + 3c) =$

12. $5x - [4x - (3x - 1)] =$

13. $\frac{3c}{4} - \frac{3c}{8} =$

15. $\frac{1}{a - x} - \frac{3x}{a^2 - x^2} =$

18. $3 - \frac{3 - 2x}{4} - 2x =$

22. $\frac{3 - 2x}{(x - 1)^3} + \frac{x + 1}{(x - 1)^2} - \frac{1}{(x - 1)} =$

23. $\sqrt{20} + \sqrt{45} + \sqrt{\tfrac{1}{5}}$

24. $\frac{a}{a - 2} - \frac{a - 2}{a + 2} + \frac{3}{4 - a^2} =$

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Hotz First Year Algebra Scales

Series A

EQUATION AND FORMULA

Solve the following equations and formulae:

1. $2x = 4$

2. $7m = 3m + 12$

4. $5a + 5 = 61 - 3a$

6. $10 - 11z = 4 - 8z$

8. $c - 2(3 - 4c) = 12$

11. The area of a triangle = ½ bh, in which
b = length of the base
and h = height of the triangle.
How many square feet are there in the area of a triangle whose base is 10 feet, and whose height is 8 feet?

14. $3m + 7n = 34$
$7m + 8n = 46$

18. $\dfrac{x+3}{x-2} = \dfrac{x+5}{x-4}$

19. $p^2 - 5p = 50$

23. $\dfrac{6x-2}{x+3} - 3 = \dfrac{3x^2+13}{x^2-9}$

24. $S = \frac{1}{2}gt^2$; solve for t

25. $\sqrt{x^2 - 1} - x = -1$

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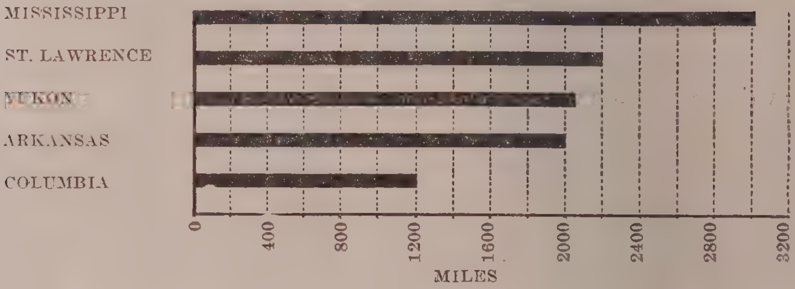
Hotz First Year Algebra Scales

Series A

GRAPHS

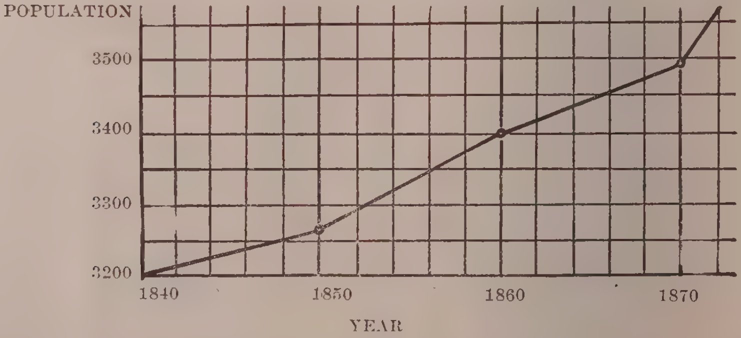
1. The following diagram represents the length of certain rivers:



How many miles long is the Arkansas river as represented in this diagram?

....

3. This graph indicates the population of a certain town for a period of years:



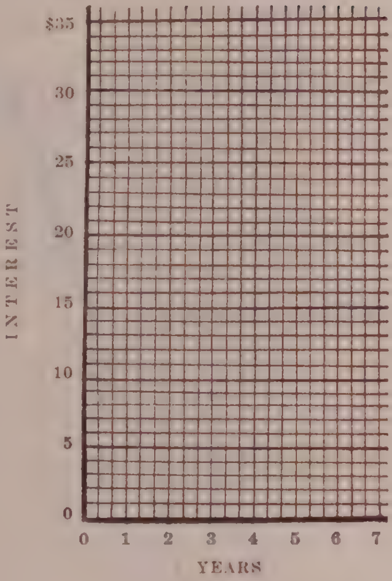
What was the approximate population of the town in 1856? ....

4. This table gives the interest at 6 per cent on $100 for a period of years.

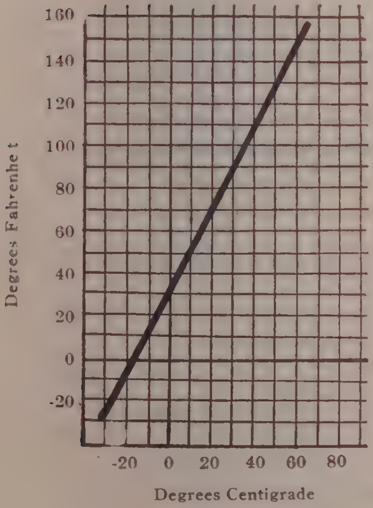
YEARS	INTEREST IN DOLLARS
1	6
2	12
3	18
4	24
5	30
6	36

Locate in the adjacent diagram the points corresponding to the above data.

Then connect these points by a line.

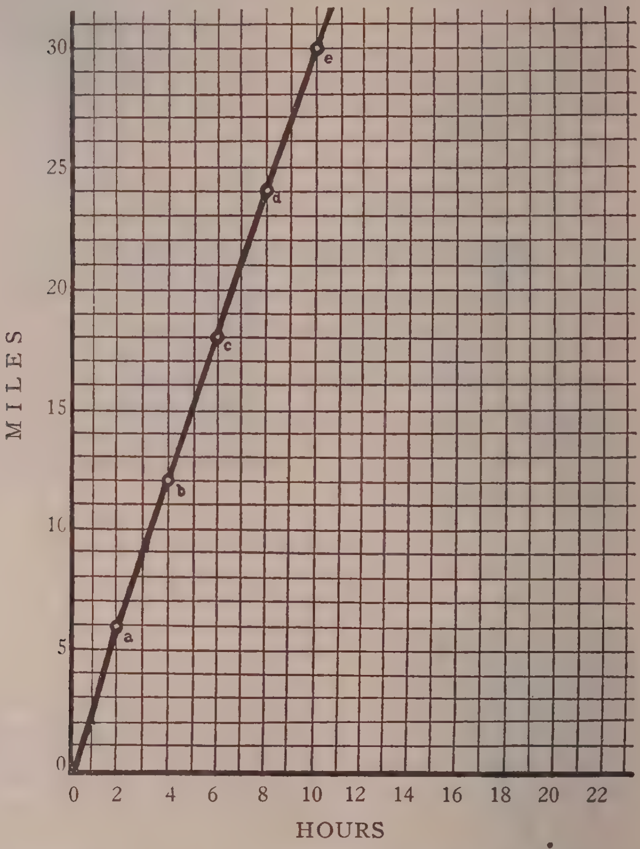


5. The adjacent graph is used to convert degrees of temperature from the Fahrenheit scale (F) to the Centigrade scale (C), and from the Centigrade scale to the Fahrenheit scale.



When it is + 20° on the F. scale, what is the temperature in degrees on the C. scale? ....

6. This graph represents the distance passed over by a man walking at the rate of 3 miles an hour for a number of hours:



By the conditions of the problem it may be said that the number of miles travelled equals three times the number of hours, or

$$m = 3h$$

That is— if $h = 4$, then $m = 12$;

and if $h = 6$, then $m = 18$, etc

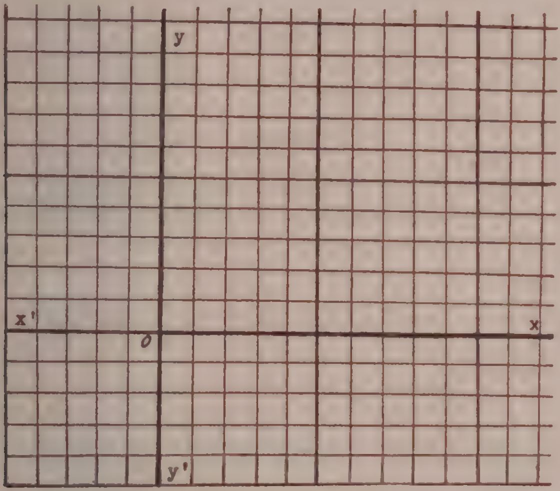
By locating these points, we have a, b, c, d, etc., of the graph.

In the same diagram draw a graph in which

$$m = 2h$$

7. Find three pairs of values for x and y in the following equation and then draw the graph of

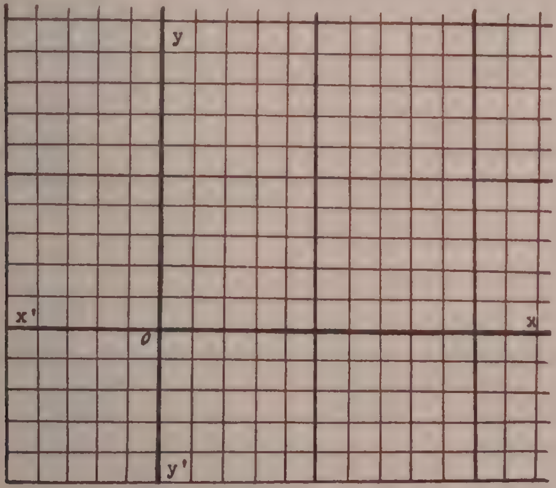
$$x + y = 5$$



1 Space = 1 Unit

10. Plot the following equations and find the values of x and y at the point of intersection of the graphs.

$$x + 4y = 11$$
$$2x - y = 4$$



1 Space = 1 Unit

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Hotz First Year Algebra Scales

Series A

Problems

Do not work out the answer to the problem—merely *indicate* the answer or *state* the equation in each case.

1. If one coat cost x dollars, how much will 3 coats cost?

Answer..

2. A man is m years old; how old was he r years ago?

Answer..

4. A gold watch is worth ten times as much as a silver watch, and both together are worth $132. How much is each worth?

Equation..

5. The distance from Chicago to New York by rail is 980 miles. If a train runs v miles an hour, what is the time required for the run?

Answer..

7. The total number of circus tickets sold was 836. The number of tickets sold to adults was 136 less than twice the number of children's tickets. How many were sold of each?

Equation..

8. A rectangular box is d inches deep, w inches wide, and contains r cubic inches. What is its length?

Answer. . . . .

9. The area of a square is equal to that of a rectangle. The base of the rectangle is 12 feet longer and its altitude 4 feet shorter than the side of the square. Find the dimensions of both figures.

Equation. . . . .

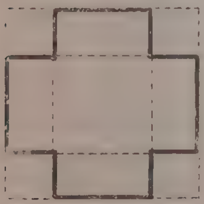
12. A train leaves a station and travels at the rate of 40 miles an hour. Two hours later a second train leaves the same station and travels in the same direction at the rate of 55 miles an hour. Where will the second train pass the first?

Equation. . . . .

13. A merchant has two kinds of tea, one kind costing 50 cents and the other 65 cents per pound. How many pounds of each must be mixed together to produce a mixture of 20 pounds that shall cost 60 cents per pound?

Equation. . . . .

14. An open box is made from a square piece of tin by cutting out a 5 inch square from each corner and turning up the sides. How large is the original square, if the box contains 180 cubic inches?



Equation . . . .

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an actual measurement of the amount of difficulty of each exercise in terms of a "difficulty scale," similar to that made by Dr. Woody in his arithmetic scale.

In the scales of Series A the exercises have been so arranged that the amount or degree of difficulty between successive exercises and problems is approximately equal; that is, the third exercise of a given scale is as much more difficult than the second exercise, as the second exercise is more difficult than the first exercise.

The schools which furnished the 16,000 pupils to whom the tests were first given, as a basis for standardization of the scales, varied all the way from the small rural high school to the large cosmopolitan high school. Classes were tested in all in 84 high schools, located in the states of Massachusetts, Connecticut, Rhode Island, New York, New Jersey, Ohio, Wisconsin, Missouri, Oklahoma, Colorado, and Washington.

The Hotz scales represent the first attempt to grade algebra problems on a scale according to their difficulty where the difficulty of each exercise was determined by the percentage of pupils solving it correctly. Professor E. L. Thorndike, of Teachers College, however, was the first to evaluate a series of algebra problems in terms of their difficulty. In 1914 he constructed such a scale for twenty-five problems in algebra, the difficulty of each problem being determined by the composite opinion of 200 teachers of high-school mathematics. Each teacher arranged the twenty-five problems in the order of their difficulty, according to his individual judgment. From the 200 arrangements thus obtained a single arrangement, representing the concensus of opinion of the 200 teachers was then determined. By applying modern statistical methods to these data it was possible to give each problem a numerical evaluation representing its difficulty. For a full description of Professor Thorndike's experiment, see *The Mathematics Teacher,* Volume VI, No. 3, March, 1914.

In using the Hotz scales to determine individual and class scores, the factor of primary importance is not so much how many exercises a pupil can solve correctly in a given time, but rather how far along on the scale of exercises, arranged in order of increasing difficulty, he can perform satisfactorily. In other words, the pupil is measured almost entirely by the point which he reaches on the scales. For this reason the Hotz tests are very properly called "scales," and are often known as "difficulty" tests or "power" tests to distinguish them from tests of the Rugg-Clark or Courtis type, which are known as "speed" or "rate" tests.

In the practical use of the scales, the question will arise as to which series to select. On the whole, Series A will be found to be more satisfactory than Series B, especially where the time available for testing purposes is limited. Series A is particularly suited for measuring the progress of a class for the purpose of comparing one class with another or one school with another. Series A, therefore, is particularly adapted for survey purposes.

On the other hand, if the purpose of the test is mainly diagnostic, that is, to discover difficulties which the pupils are encountering in their work, then Series B is preferable, since it contains a richer variety of exercises and a greater number of the type processes, which makes it possible for teachers to make a more complete analysis of the mistakes made by the pupils.

If the time for testing is so limited that only one scale out of a series can be given, it should be the Equation and Formula Scale, because it is more comprehensive and so tests a much wider function It is preferable, however, to use at least two scales, and in this case, the scale which undoubtedly comes second in importance is the Problem Scale. If these two scales are selected from Series A, there will be ample time to give both during a single class period. Whenever it is possible to do so, all five of the scales of a given series should be used, since the achievement on all of the tests gives a much more reliable indication of a pupil's ability than the results from one or two tests would give.

Teachers have found it most practicable to use the tests in rotation somewhat as follows:

1. At the end of three months—
    Addition and Subtraction Scale
    Equation and Formula Scale
2. At the end of six months—
    Multiplication and Division Scale
    Problem Scale
3. At the end of nine months—
    Equation and Formula Scale (repeated)
    Graph Scale

Whenever it is desired to use the same scale a second time, it is advisable to select it from a different series. It is feared that unless at least six months' time has elapsed since a given test was used some of the practice effect may survive.

Up to the present time no alternate or duplicate forms of the tests of Series A and Series B are available. In due time such alternate forms, in all probability, will be ready. The alternate form for any given scale will contain a different set of exercises of the same number and difficulty as the original exercises, thus making it possible to give an alternate form of any test very soon after the original form has been given, without the disadvantages which would naturally arise if exactly the same form were repeated in a short interval. Another advantage of the alternate forms will be that they will serve better for the purpose of making a diagnosis of the types of difficulties which pupils are having with their algebra work. If a pupil, on the alternate form of a test, makes the same type of mistake that he made on the original form, then that mistake is no doubt due to a fundamental misunderstanding or lack of knowledge on the pupil's part; but if only one form of a test has been given, it does not necessarily follow that the type of error made by a pupil is one which he would make a second time, if the opportunity were afforded. For all purposes of diagnosis, therefore, with the view of enabling the teacher to correct the difficulties of her pupils, or to improve her class instruction, alternate forms of tests of the Hotz or "power type" are desirable. For most effective work, five or six alternate forms for each test would be most helpful.

One of the most important questions that arises is when to give the tests. The tentative standards of achievement, based upon the 16,000 papers of the original study, were compiled for three-, six-, and nine-month intervals. It is, therefore, much more satisfactory for comparative purposes to submit these scales to algebra classes immediately after they have studied algebra for three, for six, or for nine months. However, data on the achievement at other intervals are being collected constantly, much of which is included in the pamphlet entitled *Teacher's Manual for First-Year Algebra Scales,* by H. G. Hotz. As time passes more and more information with regard to the achievement that may reasonably be expected at the other intervals will become available.

The scales may be used very profitably as early as the end of the third month of the school year.

The method of giving the tests to pupils is very simple and is similar to that used for the Rugg-Clark tests or for the arithmetic tests. Full directions for giving the tests are found in the manual referred to above. The directions given the pupils before beginning the tests include the following remarks: "Work directly on these sheets. Work the exercises in the order in which they are given. Work as many as you possibly can, and be sure you get them right. If you come to a problem you cannot do, leave it out and pass on to the next." In the graph scale, the pupils are permitted to use rulers.

The time limit for all the tests of Series A, except the Problem and Graph Scales, is twenty minutes. The Problem and Graph Scales require twenty-five minutes each. The total amount of time allowed for all five tests of Series A is 110 minutes. With classes that have studied algebra for less than nine months this time can usually be reduced quite considerably and reliable results still be obtained.

Forty minutes is allowed for each scale in Series B, or a total of 200 minutes for the five scales. The time allotment for this series may also be materially reduced with classes who have studied algebra for less than a full year.

In scoring the papers, all problems are marked either right or wrong, no credit being given for a problem that is partially right. Standards adopted in judging the correctness of an exercise were necessarily quite arbitrary. They were, nevertheless, determined after considerable experimentation and consultation with teachers of mathematics. The various answers which may be accepted as correct for each problem are given in the teacher's manual or in the original monograph concerning these scales entitled *First-Year Algebra Scales*, by H. G. Hotz.

On each test the pupil's score is the number of problems correctly solved. In order to determine the score of the entire class, the "median" of all of the individual scores is found in the same manner as for the Rugg-Clark tests or for the arithmetic tests.

Based upon the original study of 16,000 papers, tentative standards of achievement have been compiled for each of the tests of both series. These standards are given in the teacher's manual and also in the original monograph. Three sets of standards are given, providing for those who have studied algebra three months, six months, or nine months. The standards for the six-month group are higher than those for the three-month group. Naturally the standards for the nine-month group are the highest. All standards are given in terms of the median number of problems correctly solved.

Since the original standards were compiled, thousands of scores have been obtained from the results of using these tests in large cities, and in some cases, in state-wide investigations. Thus, new standards are included only in the teacher's manual. The manual and the tests themselves may be obtained from the Bureau of Publications, Teachers College, New York City.

At the risk of repeating some of the statements already made, a summary of the uses of the first-year algebra scales, taken from page 86 of the original monograph *First-Year Algebra Scales,* by Dr. Hotz, may be of service.

> These scales may be used by teachers for three distinct and very useful purposes. They may be used (a) to indicate attainment, (b) to measure progress, and (c) to diagnose difficulties.
>
> Scales which increase in difficulty by approximately equal steps furnish a most reliable objective means for determining the actual achievement of a student or a class of students. Any one of the scales may be used for this purpose, though the Equation and Formula Scale is perhaps to be preferred since, as previously stated, it is a more comprehensive test. It is well to keep in mind also, in this connection, that a low median class score is not always, nor even quite generally, due to poor instruction. Any one of a combination of several causes may be operating to keep a class score down. It is the duty of the teacher, however, to study these causes and to learn which ones are affecting the efficiency of the instruction, in order that proper remedial measures may be applied. This it is possible to accomplish with a much greater degree of certainty when the teacher knows the actual standard of achievement a class has attained. Such knowledge furnishes the teacher with a fact basis upon which to proceed and a motive with which to operate.
>
> The extent of progress made by a class can be quite scientifically measured by submitting the same scale to a class at intervals of about three months. Teachers should be cautioned very specifically, however, not to do any drill work upon the exercises or problems appearing in the scales. If it is feared that some of the practice effect may survive, it is suggested that another scale in the same series, or the same scale in a different series, be used for the second test. The most desirable method of measuring progress, very naturally, would be to have another parallel series of scales similar and equal in difficulty to those of Series A, and it is to be hoped that such a series will soon be constructed. For diagnostic purposes the scales of Series B have been found to be more serviceable.
>
> Finally, it must be stated emphatically that these are primarily power tests and as such should never be used for purposes of drill. Furthermore, with the time limits as now fixed, they are speed tests to a limited extent only;

i. e., the speed element is secondary. If a pure speed test is desired, the tests devised by Rugg and Clark could be used to supplement these tests. These would be particularly useful in determining whether a class has had sufficient drill upon the fundamentals.

Comparison of the Rugg-Clark and Hotz Algebra Tests

The descriptions already given of these two tests indicate their main differences and the purposes for which they are suited. It is not the intention to repeat those differences here. It will be helpful, however, to indicate a typical experience when both tests have been given to the same groups of pupils under carefully controlled conditions. Such an experience suggests problems which arise sooner or later as one progresses in his experience in giving tests.

In a Master's dissertation, written at the University of Chicago in 1919, Miss Eleanora Harris made a study of her findings in giving the Hotz and the Rugg-Clark tests to the same groups of pupils. In her study she used three of the Hotz Scales as follows: Equation and Formula Scale, the Graph Scale, and the Problem Scale; and seven of the Rugg-Clark Tests as follows: Tests 4, 10, 11, 12, and 13, the Verbal Problems Test, and the Graphs Test. The equation and formula exercises of both tests were given to 264 pupils, the problems exercises of both tests to 507 pupils, and the graphs exercises of both tests to 283 pupils. A summary of Miss Harris's conclusions is as follows:

1. No one of the tests was difficult to give.

2. The use of the term "equation" in Problems 3 and 5 of List A of the Rugg Clark Verbal Problems Test was confusing to the pupils and caused difficulty in scoring.

3. The use of the term "roots" in the Rugg-Clark test was probably the cause of failure to solve the problems on the part of many pupils tested.

4. In the Rugg Clark tests of the formal operations pupils are not instructed by the authors as to form in which to leave answers. This lowered the scores made by some pupils and caused difficulty in scoring.

5. The time and labor elements on the part of both examiner and pupils were approximately equal for the problem tests of both Rugg-Clark and Hotz.

6. In the equation and formula tests, the Rugg-Clark tests required approximately three times as much time of the examiner and two and one-half times as much time of the pupils as did the corresponding Hotz scale. This included giving, taking, and scoring.

7. Definite directions are given for scoring all the Hotz scales. No directions are given for scoring the Rugg-Clark Verbal Problems Test. This caused difficulty.

8. The Hotz Problem Scale and the Rugg-Clark Verbal Problems List A apparently measured, in quite a satisfactory manner, approximately the same ability.

9. The Hotz Equation and Formula Scale and the Rugg-Clark series of formal tests each partially measured the pupils tested, but apparently not for the same ability.

10. The Hotz Graph Scale measured the pupils tested and yielded some definite diagnostic evidence. The Rugg-Clark Graphs Test did not give satisfactory results.[7]

11. As judged by the pupils tested, the authors' tentative median standards are somewhat high.

12. In the three Hotz scales used in this study, the arrangements of problems in order of increasing difficulty as found by the pupils tested differed little from the order given by the author.

13. The exercises in the seven Rugg-Clark tests used in this study were found by the pupils tested to be of different degrees of difficulty. In the Verbal Problems Test List A the arrangement of problems in order of difficulty showed great irregularity for the last six problems. The arrangement of problems within the cycles in the formal tests was found to be irregular as to order of increasing difficulty.

14. The Rugg-Clark Formal Test No. 4 follows the cycle principle. The operations and symbols appear in rotations. Tests 10, 11, 12, and 13 do not follow the cycle principle in rigorous fashion. Therefore, the validity of the findings from these four tests is called in question.

The writer of this report would supplement the findings of Miss Harris by several other suggestions. At the present time we have no scientific basis for judging the relative merits of two tests such as the Rugg-Clark and Hotz tests, since in both cases we lack the correlations with a criterion of demonstrated ability in algebra.

In the form in which the Rugg-Clark and the Hotz tests are now issued, they should not in any sense be considered as competitors. In reality, these two tests in many respects supplement each other, one giving information of one type, while the other measures elements of another type. If a teacher has the time and can also afford the expense, it would be helpful to give both these tests to the same class. To economize time, perhaps a selection of certain individual tests from each of these tests would be most helpful. The following selection of tests might well be considered as affording a rather comprehensive measure of a pupil's ability: Rugg-Clark Tests No. 2 (evaluation), No. 4 (simple equations), No. 5 (parentheses), No. 7 (exponents), No. 8 (factoring), No. 11 (formulae), No. 13 (simultaneous equations), and No. 14 (radicals), together with the following three Hotz scales: Equation and Formula, Problems, and Graphs. This group of tests places emphasis upon those tools of algebra that are most essential for life purposes and, at the same time, gives an opportunity to test ability to use these tools in the solving of verbal problems. The above list may be abridged if time is limited by the omission of certain tests, especially those Rugg-Clark tests which may be fairly well represented in the Hotz scales.

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[7] Rugg-Clark no longer use their original graph test but recommend that it be replaced by the Hotz Graph Scale.

If time is very limited and only one test can be used certain of the Hotz scales (as indicated on page 349) will probably give the quickest measure of the actual attainment or general ability of the class in algebra and will perhaps serve as the quickest method of getting a fairly safe comparison of one class with another. The speed or skill side, however, will not be checked, and from a single testing no great assistance will be had in diagnosing the individual weaknesses or difficulties of each pupil.

If a diagnosis of the individual weaknesses of each pupil of the class is desired with the aim of improving the teaching in that class as efficiently as possible, then, under present conditions, the Rugg-Clark test will be the most effective single instrument, due to its cycle arrangement of the examples. At the same time, it will give a measure of the ability of the class as a basis for comparison with other classes.

In conclusion it should be stated that no test is perfect and probably never will be. And no one realizes this fact more clearly than the authors of the algebra and arithmetic tests already described. It was the spirit of improvement that originally induced the several authors to construct these tests, and that same spirit still prompts them to perfect their tests in every possible way, though improvement is necessarily a slow process, which can be made only after a long experience with the original test. The authors of the Rugg-Clark and the Hotz tests are each contemplating improvements in their original tests as soon as experiments now in progress have progressed reasonably far enough to make such changes advisable, which will probably be some time in the next three or four years.

The Thorndike Algebra Test

The Thorndike Algebra Test was prepared in 1921 by Professor E. L. Thorndike, of Teachers College, Columbia University, at the request of the College Entrance Examination Board. This test is a new type of examination in algebra for entrance to college and thus differs in purpose and in type from the algebra tests already described. The College Entrance Examination Board proposes to experiment with this new test and to compare the results of using it with the old form of college entrance examination in algebra to determine which form of test is superior. Possibly at some future examination conducted by the Board certain students may be given this new test, while other students will take the conventional test, thus affording an opportunity to compare the efficiency of the two tests.

This new test differs from the conventional college entrance examination in the following particulars:

(1) Each test is issued in a number of equivalent forms known as Forms A, B, C, and so forth. These forms have been so designed that they are all equal in difficulty, while the conventional algebra tests all vary greatly in difficulty.

(2) The new tests provide means for testing a pupil's mastery of the fundamentals of algebra rather than for finding out whether or not he has great mathematical talent. Almost all the fundamental abilities in algebra are tested in each of the tests.

(3) In general, these tests are in harmony with the recommendations of the National Committee on Mathematical Requirements as to the requirements in algebra. To this end, therefore, there is less emphasis on operations with polynomials, little emphasis on elaborate factoring, and no elaborate problems of a type that could not occur in real life. There is more emphasis than usual on the evaluation of algebraic expressions and on the testing of a pupil's understanding of the function concept.

(4) The new test can be graded by the examiner in much less time than the traditional college entrance examination.

The test is in two parts, the first part being devoted to algebra to quadratics, and the second part to quadratics and beyond. For each part ninety minutes is the time assigned. Thus, all the elementary algebra can be tested in three hours.

Forms A, B, C, and D, elementary algebra to quadratics, have already been widely circulated. Form A, which is printed in pamphlet form is reproduced on pages 358-361. The other forms, which are of the same difficulty as Forms A, B, C, and D will not be circulated in advance of the time that they may be used for examination purposes. Forms A and B of the examination, elementary algebra beyond quadratics, are also in circulation.

For further information concerning these tests, apply to the College Entrance Examination Board, 431 West 117th Street, New York City.

Other Algebra Tests

Space does not permit the detailed description of a number of other algebra tests which are less extensively used in the high schools than the tests already described. Among these other tests are the following:

Standard Research Test in Algebra. Prepared by Dr. Walter S. Monroe, of the University of Illinois.

There are six test sheets, dealing mainly with algebraic operations necessary in the solution of equations. This test has been revised and in its new form is known as the Illinois Standardized Algebra Test, described below. The Standard Research Test in Algebra was distributed by the Bureau of Educational Measurements, State Normal School, Emporia, Kansas.

Illinois Standardized Algebra Test. By Walter S. Monroe and Louis W. Williams.

This is a revision of the test described above, which it now replaces. The complete test consists of four sheets of equations. On each sheet the pupil is required to solve the equations. The Illinois Test, with directions for use, may be had from the Public School Publishing Company, Bloomington, Illinois.

Diagnostic Tests for First-Year Algebra. By Professor Harl Roy Douglass, of the University of Oregon.

The complete test consists of four sheets: One for addition and subtraction, one for multiplication, one for division, and one for the

The Thorndike Algebra Test

To Quadratics—Form A

1. $(5a^2 + 4 - 3a^2) + (a^2 - 2 - 7a^2 - 5)$

1. ....

2. $(-2a^3 - 10a - 4a^3) + (5a + 3a^3 - 4a)$

2. ....

3. From $3a + 4b$ subtract $5a - 9b - 3c$

3. ....

4. From $5a - b - 2c$ subtract $3c - 3a$

4. ....

5. $8a + 8b - (3a + 6b)$ 5. ....

6. $(5d - e) - (7e + 2f)$ 6. ....

7. $7d \times 2de^2$ 7. ....

8. $de^2 \times d^2e$ 8. ....

9. $8 - 5(d + 2)$ 9. ....

10. $4e^2 + e(-4e - 3)$ 10. ....

11. $5np - 3p(4n + 3p)$ 11. ....

12. $m + \frac{8m^2n^2}{mn}$ 12. ....

13. $4m + \frac{2m^2np^3}{mp}$ 13. ....

14. $\frac{m^2np}{mn} - \frac{m^2n - mp}{m}$ 14. ....

15. $(m^5n)(m^2n^3)$ 15. ....

16. $(2a - 7)^2$ 16. ....

17. If $a = 2$, and $b = 3$, what does $5a^2 - 2ab$ equal?

17. ....

18. If $a = .7$, and $b = 1.2$, what does $2a^2 - 5ab$ equal?

18. . . . .

19. If $a = 1$, $b = 2$, $c = .4$, and $d = 100$, what does $a^2 - b + cd$ equal?

19. . . . .

20. If $a = 12$, $b = 6$, $c = 5$, $d = 3$, and $e = 1$, what does $\frac{22}{7}[ab + c(d - e)]$ equal?

20. . . . .

21. If $d = 2$, $e = 3$, $f = 4$, what does $\dfrac{\frac{df}{d+e}}{ef}$ equal?

21. . . . .

22. $d = \frac{ef}{g}$. What does f equal?

22. . . . .

23. $\frac{e}{W} = \frac{r}{R}$. What does W equal?

23. . . . .

24. $\frac{PV}{T} = \frac{P_1V_1}{T_1}$. What does V equal?

24. . . . .

25. $4q = 7q + 5$. What does q equal?

25. . . . .

26. $15 = 7w - 4$. What does w equal?

26. . . . .

27. $\frac{V}{4} = a - 2$. What does V equal?

27. . . . .

28. $\frac{6}{2-u} - \frac{11}{3-u} = 0$. What does u equal?

28. . . . .

29. $P = \dfrac{b_1 b_2}{c_1 + c_2}$ b_1, b_2, c_1, and c_2 are all positive.

Write *"larger"* or *"smaller"* or *"you cannot know"* on the dotted lines.

If b_1 becomes larger, P will become ....
If b_2 becomes larger, P will become ....
If c_1 becomes larger, P will become ....
If c_2 becomes larger, P will become ....

30. $N = \dfrac{ab}{c} - d\,(e - f) + \dfrac{1.414}{g + \dfrac{1}{h}}$. a, b, c, d, e, f, g, and h are all positive.

Write *"larger"* or *"smaller"* or *"you cannot know"* on the dotted lines.

If a becomes larger, N will become ....
If b becomes larger, N will become ....
If c becomes larger, N will become ....
If d becomes larger, N will become ....
If e becomes larger, N will become ....
If f becomes larger, N will become ....
If g becomes larger, N will become ....
If h becomes larger, N will become ....

31. If $a = 1$, $b = 10$, and $c = 100$, express 216 in numbers and letters. Express 2.16 in numbers and letters.

32. To two times a certain number 2 is added. From 3 times the number 7 is subtracted. The two results are equal. What is the number? ....

Use cross section paper for Nos. 33, 34, 35.[8]

33. Make a little cross (x) at the point for which $x = 2$ and $y = 3$. Make a little circle (O) at the point for which $x = -4$ and $y = 1$. Make a little triangle (△) at the point for which $x = 1\frac{1}{2}$ and $y = -4$.

34. Draw a graph of $y = 1\frac{1}{3}x - 5$. Mark it No. 34.

35. Draw a graph of $y = \dfrac{1}{x} + 8$. Mark it No. 35. Draw enough of it to show clearly that you understand it.

---

[8] Co-ordinate graph paper is provided on which to give the answers to these questions.

36. Find two numbers such that:—
Twice the first, if added to three times the second, equals 2.
Six times the second, if added to ten times the first, equals 7.

36. ....

37. What is the length of a diagonal of a rectangle, if the sides of the rectangle are 8 ft. and 6 ft.?

37. ....

38. Multiply $\sqrt{\frac{a}{b}}$ by $\frac{b}{\sqrt{a}}$

38. ....

39. Write a plus sign (+) on the dotted line if the statement is true. Write a minus sign (−) on the dotted line if the statement is false.

$\sqrt{\frac{ab}{cd}} = \frac{\sqrt{abcd}}{cd}$ ....

$\sqrt{a}\ \sqrt{b}\ \sqrt{c} = \sqrt{a+b+c}$ ....

$a^{\frac{1}{2}} = \frac{1}{a}$ ....

$a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a\sqrt[3]{2}$ ....

$\sqrt{\frac{1}{5}} = \frac{\sqrt{5}}{5}$ ....

40. Write words and numbers on the dotted line, so as to make this a true statement.

If .... $a+b+c$ will equal abc.

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solution of simple equations. The test may be had from Professor Douglass, University of Oregon, Eugene, Oregon.

Scale for Testing Ability in Algebra. By W. H. Coleman.

This scale consists of a single sheet of twenty problems which include the essential elements of first-year algebra. The problems increase gradually in difficulty. This scale, with directions, is to be had from the Extension Division, University of Nebraska, Lincoln, Nebraska.

Indiana Algebra Tests. By H. G. Childs of the University of Indiana.

There are twelve tests in all, which are an elaboration of the Monroe Standard Research Test. These tests may be had from the Bureau of Research, Indiana University, Bloomington, Indiana.

Preliminary Algebra Tests. By C. E. Stromquist, of the University of Wyoming.

This is a series of seven timed tests, the problems of each test being of equal difficulty. The tests cover the fundamentals of algebra. This test is to be had from the author, University of Wyoming, Laramie, Wyoming.

The Relation of Tests to the Reorganization of Algebra Courses

Standardized tests, even in their present experimental form, are serving a number of valuable purposes in addition to those already pointed out. Concerning instruction in algebra, these tests are enabling teachers to know such things as the following:

1. That certain parts of algebra are much more difficult than other parts.
2. That a pupil's difficulty with a given topic is often due to neglect on the part of the teacher to provide proper instruction in certain small details of that topic.
3. That certain topics, traditionally thought to be difficult, are not so difficult, and in fact comparatively easy, if properly taught.
4. That teachers throughout the country have not been at all uniform in the amount of time or emphasis they have given to the various topics of first-year algebra.
5. That speed and accuracy have certain close relationships.

It is very clear that information upon such matters as those just mentioned is very important in the organization of a curriculum or of instruction to suit the needs of a child. The possibilities of the use of standard tests along the above lines are brought out in considerable detail in the monograph by Rugg and Clark entitled *Scientific Method in the Reconstruction of Ninth-Grade Mathematics*, which was based upon the experience of the authors in the use of their standardized tests in algebra. Since these matters are fully covered in that book, it will not be necessary to take them up here.

Two of the services performed by these standardized tests, however, deserve brief mention at this point.

First, the evidence obtained from the tests as to the difficulty of certain topics has shown that certain teachers are justified in eliminating from the ninth-year course in algebra certain material which has been included in the conventional text books, such as involved cases of factoring or fractions, tedious exercises in the long division of polynomials, the formal work in highest common factor, certain complicated fractional equations, certain involved work in exponents, and so forth. If these difficult topics are kept in the first-year course, and even if they are emphasized to any extent in the later courses, they require for their proper teaching an amount of time far out of proportion to the value of the topic.

Second, the tests clearly show that children can do almost anything in mathematics if it is clearly and simply presented. This suggests the possibility of our putting much new matter in a first-year course that we have hesitated to include, because we felt it was subject matter appropriate only for the higher courses in mathematics.

It is equally important to point out some of the things which the tests are doing that are not especially favorable to modern movements to reform the curriculum in algebra.

One of the fundamental weaknesses of all the algebra tests now available is that they contain certain types of problems which many teachers do not consider essential for a modern algebra course. The basis for the selection of the problems in the Hotz scales, for example, has been statistical, and therefore some types of problems most useful in algebra are not included. If the problems of the Hotz scales had been selected on a different basis, naturally the test would have assumed a much different character. Similarly, the problems of the Rugg-Clark tests were based upon an analysis of existing text books which is not the best source of material for reform purposes.

The main danger of the present algebra tests, therefore, is that they may tend to give teachers the impression that the topics that are being tested are the all-important ones, and thus they tend to direct emphasis upon types of problems which are relatively unimportant. All the "speed" tests of algebra, such as those of Monroe, Childs, Stromquist, and Rugg-Clark, also seem to put a premium upon algebraic manipulation, which thus tends to emphasize a type a mechanical curriculum in algebra from which the reform movements of the past few years have been trying to escape. The idea of twenty years ago that a first-year algebra course should be a series of exercises in the juggling of algebraic symbolism is, in a sense, kept up by the "speed" tests. The old-fashioned algebra course was all machinery, with little application of algebra to life situations. The "speed" tests tend to glorify this machinery. The present tendencies in the teaching of algebra, however, are to increase the number of uses of the subject and the amount of problem material, and to decrease to the minimum essentials the machinery taught in order to solve these problems.

Since one finds in all the algebra tests from ten to fifteen times as many exercises on the machinery of algebra as there are exercises on the uses of this machinery, these tests give the impression that the mechanical side of algebra is, after all, the principal thing in the subject. Thus again, the tests mislead those who are not fully familiar with modern reform tendencies.

Behind all the tests of the "speed" type there seems to be the assumption that it is desirable to raise all the fundamental operations of algebra to as high a plane of skill as is essential for the corresponding operations in arithmetic, on the ground that we have so much use for this algebraic machinery in the solving of life problems. Personally, I doubt that the fundamental operations of algebra should be brought to so high a plane of habit formation, since the great majority of people do not have anywhere near the opportunity to use algebraic machinery that they have for the fundamental operations of arithmetic. It is far from my purpose to question the desirability of developing certain skill in the use of these algebraic tools but the fundamental question is: What amount of skill do we need if the opportunities for displaying this skill are, compared with arithmetic, relatively infrequent?

While certain skill with algebraic machinery is desirable, it must not be obtained at the expense of understanding the principles involved in the fundamental operations since part of the value of an algebra course comes from the training it gives in logical thinking. In support of this viewpoint, Rugg and Clark state in their monograph[1] that the most important outcomes in the study of algebra are thought outcomes, that is, the ability to think logically, to use principles, to see relationships, to pick out essentials, to analyze, and to organize. In other words, they wish to develop in the pupil the problem-solving attitude, which implies ability to analyze a problem, to compare possible methods of solution, to generalize conclusions, and to recognize the field of application. Rugg and Clark give one the impression, however, that this problem-solving ability is to come mainly from problems of the verbal type. They desire to have the student make the operations of algebra automatic as soon as possible in order that he may have the maximum time for this problem-solving work.

It is my personal belief, however, that a great deal of this problem-solving attitude can also come from the right method of handling and thinking about the fundamental operations themselves.

To understand the principles by which an equation is cleared of fractions, by which a formula for the area of a circle may be changed to a formula for the radius of the circle in terms of the area, by which $\sqrt{1/3}$ is reduced to its simplest form, by which we know that $(A^2)^3$ is A^6 and not A^5, by which we relate the work in exponents to logarithms, by which we show the close relation between exponents and radicals, by which we know why we change signs when we "transpose" in an equation, or, better, by which we understand that we really never do "transpose" any term from one side of an equation to the other, and by which we know that $3x^4 - 2x^2 - 1$ can be put into the form of a quadratic expression,—to understand all these is to my mind as important an outcome in the teaching of algebra as the solution of verbal problems. The principles of equivalent equations, as a further example, and their use in explaining those mysterious "extraneous" roots, are to me quite as important and quite as thoughtprovoking, and often far more interesting to the pupil, than many verbal or "clothed" problems, as they are called abroad. Certainly, the opportunities for logical thinking in handling the above topics are as numerous as they are in any other part of the algebra course. There is a great danger, therefore, that all of these opportunities may be lost or subordinated in our endeavor to get the fundamental processes speeded up to the habit level as soon as possible.

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[1] *Scientific Method in the Reconstruction of Ninth-Grade Mathematics.*

It is far from my purpose to discount the value of having a certain amount of skill in the use of the essential machinery of algebra. But all good teachers know that when the fundamental operations are so taught that the child emphasizes the speed element, he may be tempted to work by rule rather than by principle, and soon he finds himself greatly embarrassed when a problem presents itself that has the slightest variation from the type he has been taught to handle. The traditional error that a childr makes in trying to simplify $\frac{a+b}{a+c}$ by cancelling the *a's*, is a pretty good example of the force of a habit as opposed to a principle. With many children cancellation has come through this force of habit merely to signify the privilege of drawing a line through like letters wherever they are found. And most of these children do not realize that in certain instances cancellation may represent the operation of division, while in other places it means subtraction.

Speed drills may tend to put a premium on such practices by making the child think that it is far more important for him to do something quickly than it is for him to reflect whether or not he is doing the thing the right way. If he stops and reflects about a problem, it does not improve his record in a speed contest.

Naturally, all of the above considerations are but parts of a more fundamental question, namely: What is the aim of the first-year algebra course? Any test which has as yet been devised for algebra is but a reflection of the ideas of its author as to what an algebra course should be. Any algebra test is a good one, therefore, to the extent to which it tests the kind of an algebra course the teacher believes to be the proper one. As stated above, the algebra tests now on the market are testing a number of things which many teachers regard as relatively unimportant. Likewise, they are failing to test certain material which many teachers believe to be very fundamental. Rugg and Clark have very frankly said that their test is not representative of their present notion of an algebra course, but that it is merely a test of the kind of a course they found after an examination of a number of text books. To evaluate an algebra test, therefore, the teacher must first ask himself this question: Does this test cover the kind of an algebra course that I believe is in harmony with modern demands? In this I am ignoring, of course, any arbitrary standards that may have been set up for college entrance.

Perhaps the greatest service that these standard tests have performed is to make teachers more and more conscious of the necessity of thoughtful study concerning the content of the ideal algebra course and the aims of instruction in this work. Year after year as new tests in algebra have appeared, teachers came to look upon them as official standards or patterns to guide their teaching. Immediately there was a revolt and an emphatic objection to the acceptance of any such dictation as these tests implied, especially since many of these teachers did not understand the real purpose of these tests. Destructive criticism, which naturally arose after such a feeling, has gradually given place to constructive effort to find the right kind of an algebra course for American boys and girls.

It is perhaps not out of place at this point to state some of the elements that must be considered in formulating this ideal algebra course. In the first place, I doubt that the topics now found in the traditional first-year algebra course will find application in applied problems of a materially different type or of a much greater variety than those we now have. The movement to find "real problems" in algebra began in this country about twenty years ago and has been promoted more or less since that time, with the result that we have found some "real problems" not before given in the text books; but, on the whole, I believe we would all admit that in algebra the yield has hardly been commensurate with the effort, and altogether, it has been somewhat disappointing. The greatest promise in the direction of a good supply of "real problems" has come not so much in searching applications for the machinery generally taught, as in finding new types of applications, involving types of machinery we had previously not been teaching in the first year of high school. Graphic representation and statistics are one example of a new source of apparently useful material that seems to have come to stay. The teaching of the alegbra of thrift, investment, and insurance, is a field not yet worked out for the first-year course, but which has already proved to have more than passing possibilities in the more advanced high-school courses.[10] The gradual introduction of the slide rule in the algebra class has put new energy into the chapter on exponents, and I have no doubt that sooner or later the alluring speculations of the simpler theory of probability will come within the realm of high-school teaching. Throughout this new algebra course there will be a constant development and exercise of the function concept, that great notion of how a change in one quantity influences or causes a change in another quantity. So far, we have done practically nothing with this idea in American schools. Since the National Committee on Mathematical Requirements has issued a separate report on this subject, it is unnecessary for me to dwell on it at greater length here. Likewise, there will be included as an application of the function concept the important notion of a mathematical law and the great service algebra has performed in making it possible to discover and to use effectively such laws. As an outcome of this study of laws, we shall find that our algebra is the universal language of all the sister sciences, such as geometry, physics, astronomy, and finance, when they need to express their quantitative relationships in precise and usable form. And the use of algebra as an aid to appreciation, interpretation and explanation will not be ignored. But such a course will be largely an opportunity for pleasurable and stimulating thinking, rather than wholly a speed contest. That it will need its mechanical side, I will not deny, but I doubt that any one type of mechanical operation will be so often repeated in these applications as to put it in the class of such staple operations as the addition or subtraction of integers in arithmetic.

The algebra course of tomorrow, therefore, will probably have a much wider range of topics than that of today, including many that were formerly regarded as appropriate only for the more advanced courses in mathematics. There will also be more emphasis throughout on straight thinking, so that in this latter respect it will contribute at least a small share of some of the better things that should come in the way of good thinking from the study of geometry. I am hopeful that this new algebra course, *if properly taught,* is going to have genuine disciplinary values in the sense in which the modern psychologist uses this term. I mean by this that I believe our new subject matter can be so presented that certain habits of work, or of thinking, or of procedure can be developed very economically in the algebra course which will "carry over" in many ordinary life situations and demands. I shall give a single illustration. One often sees the type of person who, if given an article that contains detailed descriptions or directions compactly and precisely written, merely skims over it and gets little or nothing out of it. Especially is this true if the description gives a sequence of events which are related to each other. An income tax blank, an accident insurance policy, the text of a bond, the interpretation of a new law, or the deed to a house are illustrations of what I have in mind. To read such papers successfully requires, among other things, such accomplishments as the following:

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[10] For suggestions along these lines see article entitled "The Secret of Thrift" by Clifford B. Upton in the *Teachers College Record* for November, 1918, published by Teachers College, New York City.

1. The development of the patience and willingness to go through a mass of details, keeping each item in mind.
2. The ability to do close and detailed thinking.
3. The ability to give sustained attention (concentration).
4. The ability to weigh the significance of each word in a sentence.
5. The ability to know whether a sentence has meaning or not.
6. The ability to think rigorously and soundly rather than carelessly.
7. The inquiring or questioning attitude of mind.
8. The ability to subject a statement to a severe test of its truth or validity.
9. The ability to discover whether sufficient data have been given.
10. The ability to analyze.
11. The ability to draw conclusions.

These are exactly the abilities the successful study of mathematics encourages and develops. To work through an algebraic explanation or demonstration, or to apply an equation to a verbal problem, one must always go through this patient step-by-step analysis, reflection, and criticism, weighing all statements, questioning their validity, and sticking to the task until all the steps have been taken. No step can be ignored lest the whole chain break. I am confident that mathematics, if properly taught, has the opportunity more than any other subject in the high school to develop this homely but extremely vital and valuable accomplishment, and I believe that tests properly conducted by a brilliant psychologist will prove that mathematical training does "carry over" and "carry on" into such situations as those illustrated above. Naturally, it is too optimistic to hope that all of this will come from first-year algebra alone, but, *if properly taught*, a definite start can be made in the first-year algebra class which can be further developed in the succeeding mathematics classes of the high school.

It goes without saying that no existing algebra scale tests the kind of an algebra course that has been outlined. But this type of course will not finally be possible without the generous aid of mathematical and psychological tests in the development of appropriate subject matter and effective ways of teaching it. After we have found this ideal course, then it, in turn, will furnish the basis for the final standardized test of its subject matter.

The Kelley Mathematical Values Test

During the year 1918-19, in the tenth year of the Horace Mann School of Teachers College, an experiment was conducted in the teaching of algebra to girls by offering to one class a year's study of algebra of a type quite different from that which prepares for the usual college entrance examinations. This new course placed less emphasis upon drill and the mechanical features of algebra, and more upon the inspirational and cultural elements. It also included several topics, such as thrift and investment, not usually given in algebra courses, which were assumed to be of practical value to girls. Certain features of the algebra course, outlined on pages 367-369, were included in this experiment. A full account of this experimental course appears in the *Teachers College Record* for November, 1921. This course is also described in this report in the chapter on Experimental Schools. See page 226.

Before beginning this experiment, it seemed desirable to have some means of measuring its success. To do this, a second class of girls was used as a check, this second class being taught the conventional college entrance topics in algebra. At the end of the year a special test was given to both classes in order to determine, if possible, which type of curriculum was the more desirable. The test given was the Kelley Mathematical Values Test, which was especially devised by Dr. Truman L. Kelley for the purpose of comparing these two classes. In devising this test, Dr. Kelley was obliged to break away entirely from the type of algebra test previously constructed, and to ask the question, What are the desirable values that should come from the study of high-school algebra? To answer this question, a questionnaire was mailed to a large number of prominent men and women, asking what they considered to be the chief values to be derived from the study of high-school mathematics as a whole, and also from high-school algebra. Based upon the replies to this questionnaire, a number of values of high-school mathematics were obtained. The Kelley Mathematical Values Test, given below, was devised as an attempt to measure these values.

Kelley Mathematical Values Test Alpha*

1. How would you find the value one year hence of a W.S.S. for which you now pay $4.16?

2. It has been claimed that there is an algebraic type of thinking. What does this mean to you?

3. A certain professor gave a lecture in a town some distance away, for which he was to receive $100 and his expenses. The note-book in which he kept track of his expenses read as follows:

Feb. 2,	Ticket	$19.10
	Dinner	1.00
	Cab	1.50
	Hotel Bill	18.20
	Liberty Bond	50.00 paid first installment
Feb. 5,	Couple dollars of tips and Pullman 2.00	
	Return fare 17.10	

One morning the professor tossed this note-book to his wife and asked her to make out a bill covering all moneys due him. She did her best.

(a) Express your opinion with reference to each item in the note-book as to whether or not it is complete and satisfactory.

(b) Make out the best statement or bill that you can from the note-book.

In your mind how is algebra related to

4. Religion?
5. Life in the home?

**Copyright, 1920, by Teachers College.*

6. Life out of doors?

7. Other school subjects (*a*) home economics? (*b*) sciences? (*c*) history? (*d*) any other school subjects?

8. Write down just what you have been told higher mathematics deals with. Who told you this?

   Now think of mathematics that is still more advanced than this that you have just described. Make a good guess and describe what you think it deals with.

9. Where did our present numerals originate and about when were they first used in Europe?

   What system of numerals did people in Europe generally use before they used our present one, and how did they perform such operations as addition, subtraction, multiplication, and division?

10. The area of a circle is πa^2. What does π stand for? a^2?

11. Suggest three or four problems other than those discussed in your algebra class that could be made clear by means of a graph.

12. The algebraic method is one of supposing the unknown quantity known, making a statement (an equation) which relates this to the known quantities, and then determining the unknown in terms of the known (solving the equation). Can you think of some problems suggested by your other school work, or your life outside of school, in which this method applies? Explain.

13. In the solution of what kind of problems is it desirable to use logarithms?

14. The expression of a physical law by means of a mathematical formula is probably the most powerful tool in modern science for the interpreting of scientific facts. For example $S = \frac{1}{2} gt^2$; in which S = the space passed over by a falling body, g = the force of gravity, and t = the length of time that the body is falling. The simple formula $S = \frac{1}{2} gt^2$ tells more about the law of gravity than a whole volume could tell without it.

    Write down any other formulas that you know.

Write down for each of the following fields two relationships which you think have been, or are capable of being, expressed by means of formulas:

15. Electricity (Sample answer: There is probably a formula which gives the relation between the amount of current which can flow through a wire and the size of the wire.)

16. Air (Sample answer: There is probably a formula which gives the relation between the temperature of the air and the rapidity with which sound travels through it.)

17. Mention some recreational activities in a person's life that algebra can make more enjoyable.

18. Do you know any game in which algebra is used? If so, describe it.

19. Two men meet and the one says to the other "My father is your father's son." What kin were they?

20. What do any of the following names suggest to you in connection with mathematics?

    Pythagoras; Newton; Euclid; Pascal; Archimedes; Leibnitz; Leonardo of Pisa; Descartes.

    Add the names of any others whom you know who have made important mathematical contributions, and tell what they did.

21. Certain great concepts or ideas have been discovered as humanity has evolved. Some of these are listed below. Number them 1, 2, 3, etc., in the order of their momentousness or importance. (*Consider that one which has revolutionized procedure most to be the most important;* for example, (*a*) below is of tremendous importance, for the present number system displaced the old cumbersome Roman system composed of X's, L's, C's, V's, etc., and it is extremely difficult to do so simple a thing as to multiply in the Roman system.

    a. The present number system.

    b. The idea or concept of the letters x and y as unknowns in an equation and of a and b as knowns.

c. The concept of 0, zero.

d. The concept that $a^2a^5 = a^7$.

e. The concept that $x^2 - y^2 = (x + y)\ (x - y)$.

f. The concept that an equation can express the path of a comet or planet around the sun.

g. The concept that an equation of this type, $4y - x = 7$, is a straight line.

h. The concept that \$1.00 at 5% interest compounded annually = \$1.10¼ at the end of 2 years.

i. The concept that if $2x^2 - 10x = 28$, then $x = 7$.

j. The concept that in many problems the unknown quantity can be dealt with as though known, in building up equations, the solution of which gives the value of the unknown.

k. The concept that $ax^2 + bx + c = 0$ is the same as

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

22. If a man 5 ft. tall weighs 110 lbs., a man 6 ft. tall should weigh how much?
If a dwarf 3 ft. tall weighs 30 lbs., a man 6 ft. tall should weigh how much?
If a new born baby 20 inches tall weighs 8 lbs., a man 6 ft. tall should weigh how much?

23. The area of a triangle equals one-half the product of the base and altitude. If the area of a triangle = ab, what does a stand for, and what does b stand for?

24. If F represents *force* of attraction, G is a constant, M represents the *mass* of one body, m the *mass* of a second body, and D the *distance* between the two bodies, what does $F = \frac{G\,M\,m}{D^2}$ mean?

Is there any mathematical symbol or operation that makes you think of, or seems in some way similar to:

25. God's all pervading power—that is, his presence in nature around us?

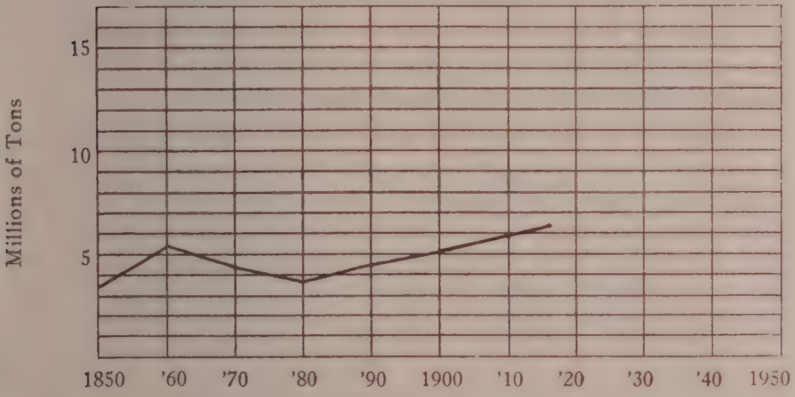
26. The intimacy of God's power—that is, his presence in our own natures?

27. Living creatures can be divided into two fundamental groups, male and female. Mention all the mathematical symbols or operations that show an equally fundamental division into two aspects. There are many illustrations, so give more than one.

28. $R = \dfrac{nr}{1 + (n - 1)r}$ $\qquad r = .45 \qquad R = .90$

What is the unknown in this equation? Solve for it.

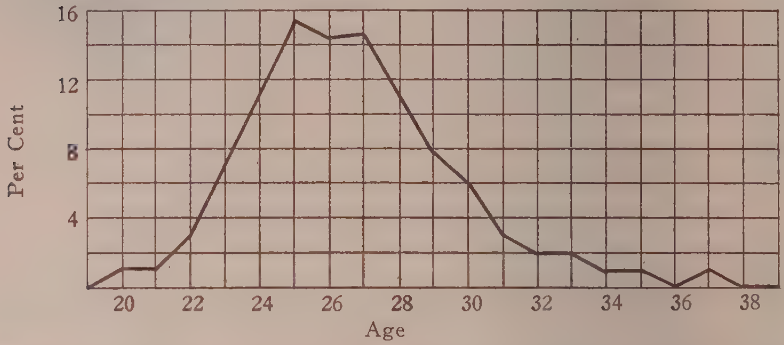
29. The accompanying graph gives the amount of U. S. tonnage for a number of years. How do you account for the low spot in the curve?



30. In January, 1918, the U. S. expected to build about 3 million tons during 1918 and 6 million in 1919. It was feared that submarines, storms, etc., would sink $\frac{1}{6}$ as much as was built. Assuming these estimates to be correct, draw the curve from 1918 on to 1920. Make a guess and draw it from 1920 on to 1950.

What effects have the important wars of the U.S. had upon tonnage?

31. The accompanying graph gives the percentages of married graduates of a certain girls' college who married at different ages. At what age did the most marriages occur? At what ages were there just 10% of marriages?



32. Mrs. A. has a skirt pattern cut for a 40 inch hip measure. Her own hip measure is 44. The pattern is in two parts, one measuring 25 inches at the bottom and the other 30 inches at the bottom. What should be the bottom measures for the two parts in order to conform to Mrs. A's hip measure?

Since Dr. Kelley's test and the results of giving it are described very fully in non-technical form in the *Teachers College Record* for May, 1920, it seems unnecessary to devote further space to it here. This article shows clearly how the test is to be used. Copies of the test, together with very full instructions for giving and scoring it, may be obtained from the Bureau of Publications, Teachers College, 525 West 120th Street, New York City.

PART III. TESTS IN GEOMETRY

Standardized Tests in Geometry

The problem of constructing standardized tests in geometry is identical with that in algebra or in arithmetic, and depends upon a clean-cut answer to the question, What are the aims of geometric instruction in our secondary schools? But on this question there is at present a greater divergence of views, perhaps, than there is for any other high-school subject. There are some who go so far as to recommend that all instruction in geometry be dropped from the high school on the ground that it serves no useful purpose, while there are others who believe this subject so valuable that they demand much more instruction in it than has usually been offered, requesting that it be spread over more school years and that it include intuitive geometry as a preparation for deductive geometry.

Another element that has complicated the problem of making a geometry test is the fact that geometry has no machinery that stands out in the same emphatic way in which the fundamental operations of arithmetic or algebra do. A still further complication arises from the practice now found in a number of American schools of teaching two kinds of geometry. One of these is intuitive geometry where conclusions are based upon observation or upon repeated measurement made with instruments of no high degree of accuracy, such as the ordinary ruler or protractor. The other type of geometry is the more formal demonstrative one, which is usually taught in the second or third year of the four-year high school. Which of these types shall we test?

The other difficulties that confront one in making a satisfactory geometry test may be realized by noting some of the things that are being emphasized today in the geometry classes of various schools:

1. A number of the better schools are giving considerable time to the practical applications of geometry, in which the pupils are required to do field work with simple measuring instruments, such as finding the distance across a river, determining the height of a building, surveying a field, or drawing to scale the floor plan of a building. This involves a certain amount of practical computation, the use of simple numerical tables, the ability to apply the theorems concerning congruent and similar triangles, some skill in the use of measuring instruments, and the ability to draw to scale. This work is appropriate for classes in both intuitive and demonstrative geometry.

2. Another type of work, emphasized in many classes in formal geometry, but not requiring field work, is computation of a practical nature, based upon certain fundamental theorems of geometry, such as finding the length of the hypotenuse of a right triangle, the area of a field, or the volume of a cistern.

3. Still another application of more recent development is the relation of the work in geometry to that in mechanical drawing and the use of geometrical theorems and instruments in the construction of all types of artistic geometric designs, which play so great a part in the industrial and architectural design of the world.

4. A new element in geometric instruction is the application of theorems to explain and illuminate many of the things that are going on about us or that we frequently see. In this respect geometry becomes a distinct aid to the interpretation or appreciation of the life of today.

5. There is also found the use of geometric theorems as working tools for the applied sciences, such as wood-working, metal-working, mechanics, physics, astronomy, automobile designing, civil engineering, and many others.

6. Geometry is still considered in some schools as fundamentally an exercise in memory training, this training being supposed to come from the pupil's repetition of the details of a proof as he finds it in the text. Fortunately this kind of geometric work is becoming more and more unpopular.

7. In the great majority of schools the most prominent work in the class in deductive geometry is the proving of theorems and exercises. This is the place where clear thinking and systematic methods of work have the opportunity to be predominant. After some experience in the proof of theorems, there is usually much opportunity to prove original exercises and in some schools systematic methods of attacking such exercises are given.

8. There is now beginning to be a small group of teachers who are carrying the work of (7) much further by making pupils still more conscious of the details of the methods of proof found in geometry, making clear that this is one of the most convincing kinds of proof available and seeking to extend the methods thus learned to reasoning and proof in life situations in general. This means a careful study of definitions and the assumed elements in any proof (axioms and postulates in geometry) and also a full command of both direct and indirect methods of demonstration. By such a treatment of geometry these teachers are trying to secure from it genuine disciplinary values of a kind which a modern psychologist would approve and which will carry over into life situations in general.[11] The disciplinary values are not unlike some of those hoped for in the ideal algebra course outlined on pages 368-369. Such disciplinary ends will be realized, however, only if the teacher consciously guides his instruction in this direction.

These eight divisions by no means cover all the types of work given in geometry classes. They are helpful, however, when one is thinking of the problem of testing, since they suggest the question, Which of these elements shall we test? The answer to this depends upon another question, Which of these elements are the most important and what should be the relative amount of time assigned to each? These are very difficult questions to answer and teachers are by no means agreed upon them. Not until there can be some substantial concensus of opinion can we hope to approach a satisfactory solution of a geometry test, or better, of a series of geometry tests, because no one test can provide adequately for all these elements.

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[11] This treatment of geometry has for several years been given considerable attention by the author of this report in his classes on the methods of teaching geometry.

It is of interest at this time to point out that many good teachers have supposed that element (7) mentioned above, which treats of the formal proof of theorems and exercises, is the one most emphasized in the usual formal demonstrative geometry course as it is given in most high schools today. It would naturally be considered safe, therefore, to draw the conclusion, that pupils whom teachers had marked high in geometry would be able to handle skillfully some of the simple elements of a formal proof, such as drawing a figure to satisfy the conditions of an exercise, or determining from a proposition what is given and what is to be proved, or having ability to select from previous theorems or facts those essential to the completion of the proof. Certainly these are essential abilities in proving a theorem. It is surprising, however, that Dr. J. H. Minnick found, as a result of giving his geometry tests which are described on page 381, that there seemed to be no very close correlation or relationship between the presence of these abilities in pupils and the marks they had been given in geometry by their teachers. In other words, many pupils who had been marked high in geometry were not very proficient in these elementary steps in proving an exercise. The question then arises, Upon what accomplishments in geometry had the teacher been basing his marking of the pupils if these abilities had not been developed?

Evidently the teacher was marking on the basis of excellence in some of the other elements enumerated on pages 377-378 or upon still other considerations not mentioned there. Possibly the teacher was marking only upon faithful effort on the part of the pupil. So long as such conditions exist the country over so far as instruction in geometry is concerned, it is very clear that we can hope for no immediate agreement as to the aims of instruction in geometry, and consequently for no immediate acceptance of any geometry test as meeting all requirements.

In spite of all these difficulties there have been some beginnings in the making of geometry tests. In 1916 Stockard and Bell made "A Preliminary Study of the Measurement of Abilities in Geometry,"[12] and devised a test which is a kind of examination in geometry. The test includes problems demanding a variety of abilities, but it does not attempt to provide for examining a pupil's ability in geometry with the same detail found in the algebra tests. No standards have been published for the Stockard and Bell test. Professor W. H. Metzler has also made a study[13] of some of the fundamental activities involved in geometric work.

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[12] An article in the *Journal of Educational Psychology,* Vol. VII, pp. 567-580.

The writer knows of several recent efforts to construct new geometry tests which are now in the process of development, but which are not yet sufficiently perfected to justify their authors in announcing them to the public.

The most satisfactory contribution yet made in the way of a geometry test is that of Professor J. H. Minnick, Dean of the School of Education of the University of Pennsylvania. This test limits itself to the measurement of certain fundamental abilities needed in the formal demonstration of a theorem. The test is intended to be given after a pupil has studied the first two books of the usual high-school course in formal or deductive geometry. Dr. Minnick's test is based upon an analysis of certain of the important elements of a demonstration and has already been fairly well standardized. This test is fully described on pages 381-389.

A more recent geometry test, issued in preliminary form in 1920, is that of Mr. Raleigh Schorling, which is intended to be given at the end of the school-year, after the pupil has completed all the work in plane geometry. This test covers a wide field of geometric information, including applications or knowledge of practically all the more important theorems of plane geometry. The test constitutes a new type of final examination in this subject. Another feature of this test is its very economical scoring scheme, which makes it possible to grade each paper in a few minutes. The test is described in detail on pages 390-395.

Another distinct contribution to the field of geometry tests is that of Dr. Agnes L. Rogers, which is described fully on pages 395-414. While the purpose of Dr. Rogers' study was not to make a geometry test as such, but to solve the broader problem of measuring the innate mathematical ability of pupils, still she found it necessary to devise a test for certain geometric abilities as one unit of measurement contributory to her larger task. In addition to a provision for testing some of the elements that are necessary in order to make a geometric proof, Dr. Rogers also provided a test for one group of capacities, which greatly influence success in elementary geometry, namely the various functions included under intuitive grasp of spatial relations. The proof that two triangles are similar depends upon the power to apprehend immediately the spatial relations underlying the application of the principle of superposition. Individuals differ by native endowment and also by training in the dexterity with which they can accomplish this. A means of measuring the degree to which it has been developed is of value in throwing light upon the causes of geometrical weakness or talent in any student. Dr. Rogers also includes a test for space intuition, which is necessary for successful work in solid geometry.

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[13] "The Problems in the Experimental Pedagogy of Geometry," *Journal of Educational Psychology*, Vol. III, pp. 345-360.

In *The School Review* for October, November, and December, 1918, Mr. Irwin presents a "Preliminary Attempt to Devise a Test of the Ability of High-School Pupils in the Mental Manipulation of Space Relations." Three hundred thirty-seven pupils in three high-school classes were examined in one test and three hundred fifty-two in another. The material was not strictly geometrical, but the classification of errors is valuable for teachers of geometry.

The Minnick Geometry Tests

The Minnick Geometry Tests are five in number and since their publication in 1918 they have been used somewhat extensively throughout the United States. A complete description of the origin and development of these tests is given in a thesis by Dr. J. H. Minnick, entitled *An Investigation of Certain Abilities Fundamental to the Study of Geometry.*

In the introduction to this thesis, Dr. Minnick states the purpose of his investigation as follows:

> High-school geometry should include both the formal and the practical phases of the subject. The most important parts of formal geometry are the demonstration of theorems, the construction of figures under given conditions, and the solution of numerical problems. This study is limited to an investigation of certain fundamental abilities involved in the demonstration of theorems.
>
> An examination of the steps in a demonstration will reveal these abilities. The first step in a demonstration is to draw the figure described in the theorem. As a second step, the pupil should state the hypothesis and conclusion accurately in terms of his figure. The third step is to recall additional known facts concerning the figure. Only in the simplest cases will the conclusion follow directly

and solely from the facts stated in the hypothesis. Hence it is necessary for the pupil to have the important properties of a geometrical figure so definitely associated that he can recall them at will. Finally, as a fourth step, it is necessary to select from all the available facts those essential to the proof and to arrange them in the order necessary to arrive at the desired conclusion. *Here there are really two steps involved*,—the selection and the arrangement of facts. However, these steps are so closely related that it seems impossible to separate them for the purposes of this investigation. The selection of facts to be used will depend upon the arrangement or the method of proof. On the other hand the method of proof must depend upon the facts which can be recalled. Hence in this study these two elements are considered as a single step. Corresponding to these four steps are the four fundamental abilities with which this study is concerned; namely,

1. The ability to draw a figure for a theorem.

2. The ability to state the hypothesis and conclusion accurately in terms of the figure.

3. The ability to recall additional known facts concerning the figure.

4. The ability to select from all the available facts those necessary for a proof and to arrange them so as to arrive at the desired conclusion.

Dr. Minnick has constructed four tests, known as Tests A, B. C, and D, each of which measures one of these abilities. The purpose of these tests is diagnostic; that is, to determine whether or not the weakness of a class is due to the lack of development of one or more of the above abilities. Test A consists of a series of geometrical exercises for which the pupil is to draw the figure. In Test B the theorem is stated and the figure is drawn; the pupil is to state the hypothesis and conclusion. In Test C the pupil is asked to state as many facts as possible about given figures. Test D consists of a series of exercises in which the figures are drawn, the hypotheses stated, and lists of known facts given; the pupil is asked to prove the exercises. In giving these tests, the time limit is thirty minutes for each. This gives time for practically every pupil to complete each test if he is at all able to do so, the purpose being to test ability to solve the exercises when speed is eliminated.

Specimens of Tests A and D are given on pages 383-386.

Minnick Geometry Test A

The purpose of this test is to measure the pupil's ability to draw accurate figures for theorems.

The exercises follow in the order in which they are given:

1. Draw the figure for the following proposition:
   If two radii of a circle are perpendicular, and a tangent to the circle cuts these radii produced at points A and B, the other tangents drawn from A and B are parallel.
2. Draw the figure for the following proposition:
   If two lines which are on opposite sides of a third line meet at a point of the third line, making the non-adjacent angles equal, the lines form one and the same line.
3. Draw the figure for the following proposition:
   The perpendicular drawn from the point of intersection of the medians of a triangle to a line without the triangle is equal to one-third the sum of the perpendiculars from the vertices of the triangle to the line.
4. Draw the figure for the following proposition:
   The bisectors of the interior and the exterior vertical angles of a triangle meet the circumscribed circumference in the mid-points of the arcs into which the base divides the circumference, and the line joining those points is the diameter which bisects the base.
5. Draw the figure for the following proposition:
   The bisectors of the angles included between the opposite sides (produced) of an inscribed quadrilateral intersect at right angles.

DIRECTIONS FOR GIVING TEST A

The directions for giving Test A are as follows:

1. See that each pupil is supplied with pencil, ruler, and a pair of compasses.
2. *The following is to be read to the pupils:*

I am going to give you some geometry exercises. In order that all of you may have the same chance I want you to start at the same time. Do not open the set of questions which you are about to receive until I give the signal to begin work by tapping on the desk.

3. Distribute the questions.
4. Have pupils fill out blanks on the cover sheet of the questions.
5. *The following is to be read to the pupils:*

At the top of each sheet which you have received there is an exercise from geometry. When the signal is given to begin work fold back the cover sheet, read one of the exercises carefully, and then in the space below draw the figure for the exercise. Then read another exercise and draw the figure. Continue in this manner until you have drawn the figures for all the exercises. Draw the figures as accurately as possible, but you need not make actual constructions. Do not attempt to prove the exercises. All I want to know is whether you can draw the figure. You may do the exercises in any order you care to. You will have thirty minutes in which to complete your work.

6. Give the pupils a chance to ask questions concerning the instructions but do not reveal the content of the questions by your answers.

7. Note the time and then give the signal, thus: "Ready," and then tap on the desk with your pencil.

8. In the case of any irregularity on the part of any pupil during the test make a note on the cover sheet of his questions indicating the exact nature of the irregularity.

9. Collect all papers promptly at the close of thirty minutes of actual work.

Minnick Geometry Test D

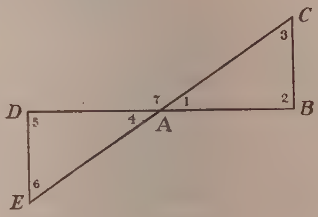
The purpose of this test is to determine the pupil's ability to select and organize facts to produce a proof. At the top of each sheet there is a figure. Below is a statement of what is given and what is to be proved. To eliminate the factor tested for by Test C a list of "Other known facts" is given at the left hand side of the lower half of the sheet. To the right of this list is ample space for the pupil's proof. The list of "Other known facts" contains those facts essential to the proof and also facts not needed in the proof. The pupil is free to select facts from this list if the figure does not suggest them. After selecting the facts needed, the pupil is to arrange them so as to arrive at the desired conclusion.

The questions in the order in which they are given follow:

1.

Given AB = AD, ED is perpendicular to DB, CB is perpendicular to BD, and EC passes through A.

To prove that triangle ABC is congruent to triangle EAD.



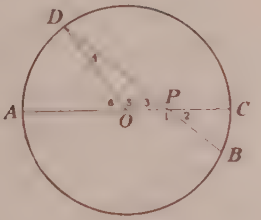
Other known facts:

$\angle 6 = \angle 3$
$\angle 1 = \angle 4$
$\angle 4 + \angle 7 = 180°$
$\angle 2 = \angle 5$
ED is parallel to BC
$AC - CB < AB$

Proof:

2.

Given: P is any point within the circle O, AC is a diameter through P, BD is any other chord through P, OD is a radius.



To prove that AP > DP

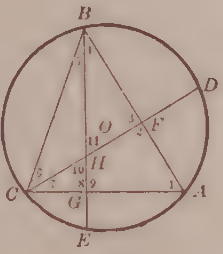
Other known facts:

$\angle 6 = \angle 3 + \angle 4$
$PD - OD < OP$
$OD + OP > DP$
$\angle 2 = \angle 3$
$AO = OD$
$\angle 5 + \angle 6 = 180°$
$AO + OP = AP$

Proof:

3.

Given: The triangle ABC inscribed in the circle whose center is O, CD is perpendicular to AB, BE is perpendicular to AC.



To prove that arc AE = arc AD.

Other known facts:

$\angle 8 = 90°$
$\angle 6$ is measured by half of arc BD
$\angle 7$ is measured by half of arc AD
$\angle 5$ is measured by half of arc EC
$\angle 4$ is measured by half of arc AE
$\angle 2 = 90°$
$\angle 9 = \angle 5 + \angle 6 + \angle 7$
$AC < AF + FC$
$\angle 9 = 90°$
$AC - AF < FC$
$\angle 2 = \angle 4 + \angle 5 + \angle 6$
$\angle 2 = \angle 8$

Proof:

Directions for Giving Test D

The directions for giving Test D are as follows:

The first 4 steps are the same as for Test A.

The following is to be read to the pupils: At the top of each sheet which you have received there is a geometrical figure. Below this figure there is a statement of what is given and what is to be proved. Below this and on the left-hand side of the sheet there is a number of other facts about the figure, some of which may guide you in proving the exercise. When the signal to begin work is given fold back the cover-sheet, read carefully what is given and what is to be proved in one of the exercises, and then in the space at the bottom and to the right of the sheet give a complete proof of the exercise, but you need not give reasons or authorities for the different steps in your proof. Refer to the facts stated on the left-hand side of the sheet as much as you care to and use any of them that will help in your proof. When you have completed the proof of this exercise proceed in a similar way to prove the other exercises. You may do the exercises in any order you care to. You will have thirty minutes in which to do your work.

The most difficult element in administering the tests is the scoring of the papers. Two scores are used for each of Tests A, B, C, and D. One is known as the *positive* score and is based upon the per cent of the correct elements of the answer given; the other is known as the *negative* score and is the total number of incorrect and unnecessary elements given. The value of the two scores is evident when we remember that the purpose is diagnostic and it is therefore important that the scores shall indicate both the amount of work done correctly and the extent to which error has been introduced. The diagnostic value of these tests is greatly increased if each teacher scores the papers of her own classes. She then sees not only the abstract measures of the abilities of her class as a whole but she also discovers the nature of the errors which each pupil makes. Rules for scoring each test have been carefully prepared and may be obtained, together with the tests, from Dr. J. H. Minnick, Dean of the School of Education, University of Pennsylvania, Philadelphia.

Although Tests A, B, C, and D are for the purpose of diagnosis, it is desirable for each teacher to compare the results of her classes with those obtained throughout the country. For this purpose standards based on the results of tests given to several thousand children throughout the country have been established. These standards accompany the directions for scoring each test. In establishing the standards sufficient cases were not included to make them absolutely reliable. Dr.

Minnick will therefore appreciate it if those giving the tests will send copies of their results to him. Extra score sheets are furnished for this purpose.

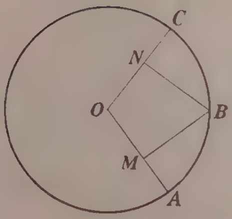
In addition to Tests A, B, C, and D, Dr. Minnick has more recently developed a fifth test known as Test W which is a perfected form of Test E which was described in his original study entitled *An Investigation of Certain Abilities Fundamental to the Study of Geometry*. The method of developing Test W is described in *The School Review*, Vol. XXVII, pp. 101-109, February, 1919.

Test W is for the purpose of comparing one class with another and thus differs from Tests A, B, C, and D which are primarily for the diagnosis of class difficulties. Test W consists of a series of original exercises in which the figure is drawn and the hypothesis stated. The pupil is then asked to make any additional drawings necessary to prove the exercise. The test is built on the assumption that if a pupil draws the correct auxiliary lines he can prove the exercise. This is not a safe assumption unless it is carefully checked. Many pupils are willing to take a chance on such a test and will guess at the correct drawings. Hence pupils frequently draw lines making possible proofs which they cannot complete. It was therefore necessary to develop a key for scoring which would, as nearly as possible, eliminate such guessing. After an extensive investigation such a key has been devised,[14] which is fairly accurate as long as the test is used for group testing. It should never be used, however, for testing individuals.

A specimen of Test W is given below.

Minnick Geometry Test W

1.



Given: The circle whose center is O, arc AB = arc BC. BM is perpendicular to AO, and BN is perpendicular to OC.

Make any additional drawings that are necessary to prove that BM=BN.

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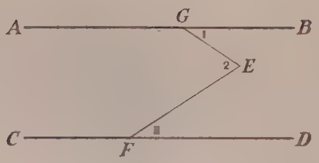
[14] The method of devising this key is described in *Educational Administration and Supervision*, Vol. VI, No. 9, December, 1920.

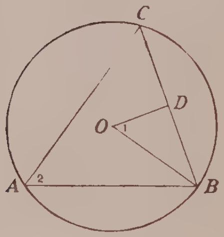
2.



Given: AB is parallel to CD, and lines GE and EF meet in E.

Make any additional drawings that are necessary to prove that ∠2 = ∠1 + ∠3.

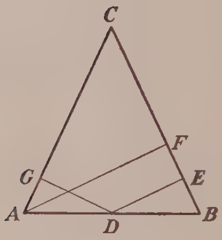
3.



Given: Triangle ABC is inscribed in a circle whose center is O, and OD is perpendicular to CB.

Make any additional drawings that are necessary to prove that ∠1 = ∠2.

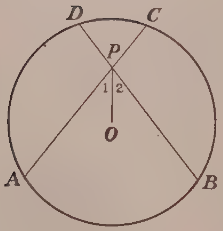
4.



Given: The triangle ABC, AC = BC, D is any point on AB, DE is perpendicular to BC, DG is perpendicular to AC, AF is perpendicular to BC.

Make any additional drawings that are necessary to prove that DG + DE = AF.

5.



Given: The circle whose center is O, chord AC = chord BD, AC and BD intersect in P and the line OP is drawn.

Make any additional drawings that are necessary to prove that ∠1 = ∠2.

DIRECTIONS FOR GIVING TEST W

The directions for giving Test W are as follows:

The first 4 steps of the directions are the same as for Test A, given on page 383.

The following is to be read to the pupils: At the top of each sheet which you have received there is a geometrical figure. Below this figure there is a statement of what is given and what is to be done. When the signal to begin work is given, fold back the cover sheet, read carefully what is given and what is to be proved in one of the exercises, and then make any additional drawings that are necessary to prove the exercise. Thus, if in the triangle ABC (place drawing on the board) AC = BC and we are to prove that ∠A = ∠B we may draw CD bisecting ∠C. When you have completed this exercise, do the other exercises in a similar way. Do not write any explanation on your papers and do not prove the exercises. All I want to know is whether you can make the correct drawings. Do not draw any unnecessary lines. Do the exercises in the order in which they are given. If you come to one which you cannot do, leave it and pass to the next one. You will have twenty minutes in which to do your work.

The directions for scoring Test W are as follows:

1. Mark each exercise right or wrong.
2. All drawings that may be accepted as correct are given in a sheet furnished by Dr. Minnick. All other drawings should be marked wrong.
3. Assign to each exercise the following values:

Exercise I.	10
Exercise II.	17
Exercise III.	24
Exercise IV.	31
Exercise V.	40

4. Remember that the score is not in per cents. The highest possible score is 122.
5. The standard median score made by all pupils tested by the author is 61.2.

For Test W only a *positive* score is used, there being no supplementary negative score such as was used for Tests A, B, C, and D. The standard score for Test W is more important than the standard scores for the other tests, since Test W is used wholly for purposes of comparison. Dr. Minnick will be glad to have teachers who use Test W send their scores to him in order that he may have as great a variety of data as possible upon which to base later modifications of the tentative median score of 61.2 given above.

A New Type of Geometry Examination

The Plane Geometry Test, devised by Mr. Raleigh Schorling, is a new type of final examination in geometry. It represents an effort to obtain a test which will be free from the most serious objections which characterize the typical plane geometry examination. Among the desirable characteristics of an examination in plane geometry, which Mr. Schorling has endeavored to incorporate in his new test, are the following:

(1) It is desirable that a test should cover as large a number and as wide a range of the principles of geometry as possible. The usual three-hour examination given by the College Entrance Board covers only a limited number of propositions or principles. In taking the usual examination, so much time is spent on the mechanics, such as rewriting the proposition, drawing the figure, stating what is given and to be proved, etc., that a wide sampling of propositions and exercises is not possible. As a result, we have on the one hand frequent failures, or at any rate, very low grades being made by pupils who are believed by their teachers to be good students. On the other hand, a very poor student, if properly tutored, may make an acceptable grade. Hence, there is reason to doubt whether the college, through the present examination system, gets the best selection of entering students. The new geometry test rests on the assumption that if a student knows plane geometry, he can express that information. That is, if a strong student learns the mechanics of plane geometry in the early weeks or in the early months of his course, there is no need to test him on the mechanics. The test may well consist of functional parts only. Whether or not this assumption is true, remains to be proved by further study of the new test. This assumption makes a wide sampling possible.

It is also desirable that this sampling should be sufficiently wide in range to discourage the tutoring of students with a low grade of intelligence. This point has, of course, been implied in the discussion above, but there are some supplementary facts that need to be stated. The new test covers all but three of the theorems of plane geometry that appear in the list recommended by the National Committee on Mathematical Requirements. (See p. 55). This means both the fundamental and the subsidiary list of theorems. Moreover, it covers the important constructions. The test would seem to be sufficiently comprehensive; so that if any student, through tutoring, succeeds in passing the test, he would probably deserve to pass. In the event, however, that undesirable tutoring is possible, tests of equivalent difficulty can easily be constructed.

(2) It is desirable to have a test that can be graded economically. The College Entrance Board in its June examinations uses the energy of approximately twenty-five to thirty well-trained mathematicians, many of them college professors, for a period of about two weeks, in grading the papers of the plane geometry examination. It seems that this energy could serve society better than to be used in grading papers of doubtful value. The new test can be graded with a stencil by a person untrained in plane geometry, for example, an eighth-grade pupil or a satisfactory clerk.

(3) It is desirable that a test should be given in less time than the three hours now required for the college entrance examination. This new geometry test can be given in 44 minutes.

The Plane Geometry Test consists of a pamphlet of sixteen pages, containing seven different tests in all. This pamphlet represents a total of 78 different questions. The entire pamphlet is intended to be given in a single class period, with a total time assignment of 44 minutes. There is a preliminary four-page test or Practice Sheet to be given the day before the regular test in order to acquaint the pupils with a test of this kind.

Specimen elements of each of the seven tests which constitute the pamphlet are given below:

TEST 1. COMPLETION TEST

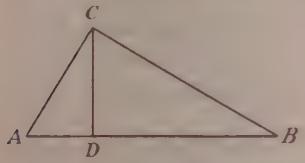
This test consists of a number of sentences which are not complete, as shown below. The student is required to insert a word in each blank which will make a statement of a geometric fact.

7. The.....................of two.....................polygons have the same ....as any two corresponding.....................

9. π is a Greek letter which represents the.............................. of the.........................of a circle to its............................

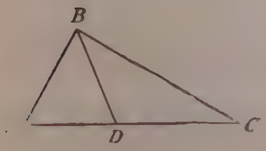
TEST 2. TRUE-FALSE TEST

In this test will be found some geometric figures. After each figure is given one or more facts relating to the figure. Then follows a conclusion based on these facts If the conclusion is true, place a cross in the square under "True." If it is false, place a cross in the square under "False."



11. Given $\angle ACB = 90^\circ$ and $CD \perp AB$
Conclusion: $\triangle ACD \sim \triangle ABC$

True False
☐ ☐



17. Given: BD, a bisector of the angle ABC.
Conclusion: $\frac{AD}{DC} = \frac{BC}{AB}$

True False
☐ ☐

Test 3. True-False Test of the Converse of a Theorem

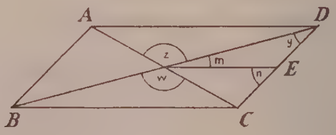
You will find below a list of propositions. Think about the converse of each proposition. If the converse is true, put a cross in the square under "True." If the converse is false, put a cross in the square under "False".

28. Proposition: If a straight line is drawn through two sides of a triangle parallel to the third side, it divides the sides proportionately. True ☐ False ☐

29. Proposition: If a straight line is drawn through the midpoints of two sides of a triangle, it is parallel to the third side. True ☐ False ☐

Test 4. The Matching Test

The test below consists of both a left and a right-hand page. On the left page is printed a series of statements. When you begin, read the first sentence on the left-hand page. Then find the reason for it in the list of geometric facts on the right-hand page. Write the number of the geometric fact in the square after the sentence on the left-hand page.



Now try these:

The left-hand page looks like this:

Statements

In the parallelogram:

(a) Angle z = Angle w ☐

(b) AB = CD ☐

(c) AB || CD ☐

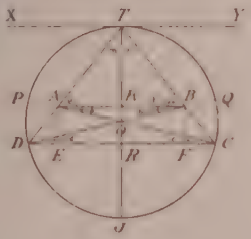
The right-hand page looks like this:

Geometric Facts

1. The opposite sides of a parallelogram are parallel.
2. Vertical angles are equal.
3. The opposite sides of a parallelogram are equal.
4. A parallelogram is divided into two congruent triangles by either diagonal.

TEST 5. DRAWING CORRECT CONCLUSIONS

You will find below a number of statements relating to the given geometric figure. Each problem will consider two or three of these statements as given, and from them you will form some conclusion. Write your conclusion in the space inside the rectangle.



Sample:

Given: Statements 1, 4, 12.

Conclusion: △ TAK is congruent to △ TKB

The conclusion is that △ TAK and △ TKB are congruent and this fact is to be written in the blank rectangle provided.

Be sure to use these symbols: for similar use ~, for equality use =, and for congruence use ≅.

Statements to be used:

1. AT = TB
2. ∠AOB = ∠DOC
3. AB ∥ EF
4. AK = KB
5. ATB is a △
6. TD = TC

.
.
.
.
.

12. TK ⊥ AB

Now try these:

1. Given statements 1 and 5.

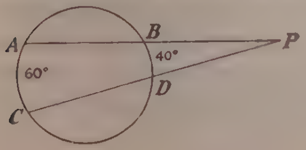
Conclusion.

2. Given statements 1 and 6.

Conclusion.

TEST 6. COMPUTATION

This test consists of ten exercises in computation with a numerical answer, which answer is to be placed in the circle after each problem.



54. The arc AC = 60°, and the arc BD = 40°

How large an angle is formed by the two secants at P?

TEST 7. CONSTRUCTIONS

This test consists of eighteen problems requiring the analysis of constructions. In each problem you are asked to study a construction and to answer certain questions about it.

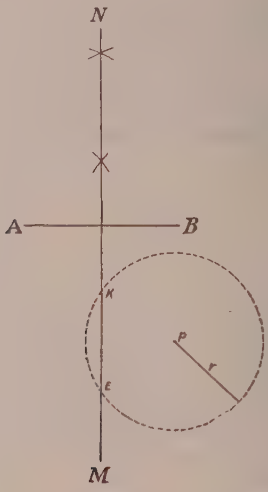


Fig. D.

Study Fig. D and answer the next three questions:

65. Study the figure and complete the following: The line MN is the locus of all points ....

....

66. The circle is the locus of all points r units distant ....

....

67. The points E and K constitute the locus of points which are ....

The test is to be given after the pupil has completed his entire course in plane geometry. A good student is permitted to go through the booklet without being stopped on each individual test. Slower students who have not finished each test by a given time are stopped and required to go on to the next test.

Teachers will ask if this test indicates a pupil's ability to solve original exercises. Preliminary evidence seems to show that success with the test correlates closely with ability to solve original exercises.

During the year 1921-1922 each of the above tests was extended so as to cover all the important theorems of plane geometry. These tests were given in a number of schools with the following purposes: (1) to determine the difficulty of individual elements, (2) to determine the value of the different types of tests, and (3) to secure tests of equivalent difficulty. Based on the results of this experimentation, Miss Vera Sanford, instructor in mathematics in the Lincoln School, is constructing a number of different forms of the complete test, all of which will be of equal difficulty.

Since the test is still in the experimental stage, it is available at present only for those teachers who wish to co-operate in its further development. Correspondence in reference to the test should be addressed to Mr. Raleigh Schorling, The Lincoln School, 415 West 123rd Street, New York City.

PART IV. TESTS OF MATHEMATICAL ABILITY

The Rogers Test of Mathematical Ability

All the tests in algebra, arithmetic, or geometry so far described have tested the results of instruction in these respective subjects. The Rugg-Clark or the Hotz algebra tests, for example, have shown whether or not the pupil has the ability to solve an equation, to add fractions, to handle exponents, to simplify a radical, to evaluate a formula, to solve a quadratic, or to graph a given algebraic expression. When these tests were given to the pupil, it was assumed that he had had thorough instruction in these and other algebraic topics. These tests, therefore, were attempting to find out whether the pupil could do certain essential things in algebra which he had been taught, and if he could not do them, in what particulars he was deficient.

In contrast to all these tests, the Rogers Prognostic Test of Mathematical Ability is attempting to solve a much more difficult problem, which is that of enabling the teacher to know *in advance* the innate mathematical ability of a pupil. By the term "mathematical ability" we mean that composite of specific abilities that are necessary for success in a wide range of mathematical studies starting with arithmetic, algebra, and intuitive geometry, and continuing through deductive geometry, trigonometry, and the subsequent college mathematics courses. Further, the problem before us is not the measure of a student's mathematical ability *after* he has studied all these subjects, or even *after* he has studied only those offered in the high school, but our whole concern is to determine the pupil's mathematical ability *before* he commences the intensive study of mathematics beyond the ninth school-year in American schools.

The Rogers test does not attempt, therefore, to be a comprehensive test of those mathematical topics studied by the pupil up to and including the ninth year. In fact, as will be explained more fully on page 400, the greater part of the test is on material that the child has not previously studied. The test is primarily a means of determining the inborn power of the pupil along mathematical lines. The Rogers test is of such a nature that a child of native mathematical gifts may obtain a very good score in it, even though he may not have secured good marks in his previous mathematical work, due to lack of application, carelessness, or failure to co-operate with the teacher.

All teachers will undoubtedly agree that such a type of test can be of great service in the solution of three pressing school problems. The first of these problems is that of advising pupils at the end of the ninth school-year concerning their further studies in mathematics. Is the pupil one who will profit by taking the mathematics courses offered in the remaining three years of the high school, or is his ability along mathematical lines so weak that he will in all probability never succeed in any field requiring mathematical gifts, and should therefore be excused from the further study of mathematics in order to devote himself more fully to other subjects? The answer to this question is particularly important, because first-year algebra consists to a considerable extent of the formation of a narrow range of specific habits, and success in this direction gives comparatively little ground for expecting assured success in more advanced study. Consequently, there is need for a means of measuring in advance capacities relatively unexercised in algebra or arithmetic but essential to success in later mathematical work. A typical illustration of the lack of proper advice for pupils at the end of the ninth year is the following: I well recall a high-school boy who had been weak in mathematics during the ninth year—in fact he never had had a good record in this subject. This boy wanted to be an engineer and so was encouraged at the beginning of the tenth year to take the usual course in plane geometry which was followed in subsequent years by more algebra, solid geometry, and trigonometry. After several trials and much tutoring he finally succeeded in passing the college entrance examinations in mathematics with a low grade and was admitted to the Engineering School. He spent his freshman year in this school and was in trouble most of the time, due to his inability to do well in any of his studies, most of which were either mathematical or dependent upon mathematics, such as physics and mechanics. At the end of the year, he was dropped from the rolls of the school, when he came to realize vividly for the first time that he did not have the innate mathematical qualifications for an engineer. This boy then set about to find himself. He had always succeeded in English; he had had considerable success in writing, and had a love for affairs in general. Further, he had a most attractive personality. He commenced to think that journalism most closely matched his abilities. He therefore entered the School of Journalism and finished the course with full credit to himself and the school, and is now successfully practicing that vocation. What a pity that this boy could not have been told with definite assurance, certainly not later than the tenth school-year, that the calling of an engineer was not suited to one of his weak mathematical attainments! At least two years of wasted time could thus have been saved. It is the aim of the Rogers test to discover such boys in advance and to help them to know definitely that they are not fitted for mathematical callings.

The second problem arising in every school is that of knowing whether or not the pupil's previous school marks in mathematics can be depended upon, and especially is this a vital question when there seems to have been a difference of opinion among teachers as to the pupil's ability. The teacher of the eighth year may have found the pupil very good, while the teacher of the ninth year ranks him among the poorer pupils in the class. Is this difference due wholly to personal matters or to the varying individualities of the teachers, or is it a matter of definite mathematical inability on the part of this particular pupil? If the pupil's promotion depends upon the mark of the last teacher, it is very desirable that we have some means of making an independent check on the teacher's estimate of this pupil's capacity. The Rogers test aims to be of service in such situations.

The third problem which teachers are now beginning fully to realize is the necessity of adapting instruction in mathematics to the capacities of the pupils, since careful investigations have shown that great individual differences in mathematical skill and insight exist among pupils. It is more and more being found that this adaptation can be made most effectively by having pupils of high mathematical ability together in one class, those of average ability in a second class, and those of below average attainments in a third class, rather than to permit pupils of all grades of ability to be together in the same class. To be most effective this classification should be made at the beginning of the school-year. Formerly, the only means available for such classification were the school marks of the preceding year, but, since these are not always dependable, an independent or supplementary means of classifying in advance on the basis of mathematical ability becomes very desirable. Here again, the Rogers test lends assistance. In this connection it should be noted that Dr. Rogers' studies have pointed out in a striking manner "the ability of the pupils to attack much harder tasks than we commonly assign to them. We seem prejudiced against the child's ability. The extreme instances of sheer incapacity influence us unduly. Again the need for classification according to inborn power shows itself to be urgent. When the question of reconstructing the content of the elementary courses in mathematics is before us, it will be necessary for many of us to take a less depreciatory view of the capacity of the pupil and to ask ourselves whether we have a correct conception of what he can do, rather than to doubt the advisability of introducing more advanced material earlier to the student. I incline to agree with Thorndike that there is no kind of reasoning a child of eleven cannot do, provided the material lies within the realm of his experience. Indeed, it would be easy to find a child of eleven who could do more efficient mathematical reasoning than many adults. Opportunities for the gifted should not be lacking, nor should we take so low a view of the ability of the average child."[15] But effective work with these groups can be done only by having them classified according to their native abilities.

The Rogers test was deliberately planned to determine the mathematical intelligence of pupils whose training in mathematics had included about five months of algebra and five months of intuitive geometry, and it was assumed that the test would be given at the end of the ninth school-year. The modern junior high-school course in mathematics, which covers the seventh, eighth, and ninth school-years, offers that variety of introductory treatment of the several mathematical subjects which Dr. Rogers would like the pupils to have before taking her test. Such a training in mathematics seems only fair to presuppose on the part of the pupil, since "it would seem right in a democracy that no child of normal intelligence should be deprived of the opportunity of becoming acquainted with a realm of thought, which has meant so much for the advancement of science and of civilization as mathematics. Nor should he be deprived of a training in the use of the tools of quantitative thinking, except where he lacks the ability of profiting by it. Both from a cultural and a practical standpoint knowledge of the fundamentals of mathematics should not be withheld, since some degree of mathematical mastery is necessary for advanced work in so many spheres. Nor should the child be considered a fit judge of what will be best for his future development, nor indeed should the parents in most cases. We need to safeguard the rights of the child, so that he is not steered away from the high types of work, of which he is capable."[16]

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[15] See *The Mathematics Teacher,* Vol. XI, No. 4, June, 1919, p. 160.

It is also possible to give the Rogers test at the end of the first year of a four-year high-school course where the previous mathematical instruction has been organized along the conventional rather than the more modern junior high-school lines. All standards at present available for the Rogers test assume the test given at the end of the ninth school-year, but it may be given in the eighth year or in the tenth year, and it is expected, in the near future, that standards for these particular years will also be available.

The Rogers Test of Mathematical Ability in practical use today consists of six separate tests, sometimes called the Rogers sextet, which are bound together in a single booklet. These six tests have the following titles:

Test 1. Algebraic Computation
Test 2. Interpolation
Test 3. Geometry
Test 4. Superposition
Test 5. Mixed Relations
Test 6. Trabue Language Scales

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[16] See *The Mathematics Teacher*, Vol. XI, No. 4, June, 1919, pp. 156-157.

The first two of these tests are algebraic, the first one measuring the ability to grasp and manipulate numerical and algebraic symbols, while the second tests ability to analyze numerical symbols, to perceive a general rule implicit in them, and to apply the principle so derived. The third and fourth tests are geometric, the third testing ability to prove simple geometric exercises deductively, and the fourth the ability to grasp intuitively spatial relations. The fifth and sixth tests are of language ability, that is, the ability to apprehend and manipulate words. The inclusion of the language ability tests may at first consideration appear strange; but as Dr. Rogers' preliminary work showed, the ability to conceive clearly the meaning of the words of any problem is as important an element in its solution as any connection it has with the abstract symbols of algebra.

Attention should be called to the fact that Test 1 and possibly Test 3 are the only ones which cover material which the pupil may have studied in a junior high-school course, or in the first year of a four-year high-school course. Test 3 covers work in simple demonstrative geometry, which in many schools will not have been a subject of study previous to the time this test is given, although intuitive geometry probably will have been studied. Tests 2, 4, 5, and 6 (and for some pupils also Test 3) are, therefore, practically new material for all pupils. Thus we see that the Rogers test measures abilities untested by ordinary examinations, but important for success in the future study of mathematics.

It requires a longer time than the usual recitation period to give the Rogers test. When it is inconvenient to give the entire test at one sitting, the test should be given in two sittings, care being taken to collect the examination booklets immediately to prevent pupils becoming acquainted with the remaining tests. At the second period of testing, the same examination booklets may be distributed again, and the examination completed without vitiating results. Test 1 (Algebraic Computation) and Test 3 (Geometry) can be conveniently given in one period, and the remaining four tests in a second period.

The actual working time for the six tests is sixty minutes. The preliminary explanations by the teacher take approximately twenty-five minutes. Thus, the time for testing, explanation, distribution, and collection of examination booklets amounts to an hour and a half.

Descriptions of Tests 1 to 6 are given on pages 401-403.

TEST 1. ALGEBRAIC COMPUTATION

This test consists of two pages of algebraic examples which include the evaluation of algebraic expressions, the addition of similar terms, the solution of simple equations, the solution of fractional equations, and the solution of simultaneous linear equations in two unknowns. One quarter of a minute is allowed for the teacher's explanation, and ten minutes for the test itself. It seems unnecessary to show a specimen of this test.

TEST 2. INTERPOLATION

This test occupies two pages, one page of which is shown on page 408. In explaining this test to the pupils, the teacher writes on the board two series of numbers such as the following:

2	4	..	8	10	..	14	..	..	..	..
5	10	..	20	25	..	..	40	..	..	..

The children are asked to supply the missing numbers in the first series, it being pointed out how each number in the series is obtained from the preceding number. The second series is similarly handled. The pupils are then given the test itself, the blank spaces of which are to be filled in with the missing numbers. Five minutes are allowed for the teacher's explanation, and thirteen minutes for the pupils to write the test.

TEST 3. GEOMETRY

The Geometry Test consists of three sheets in all, one sheet, illustrated on pp. 404-405, which gives a list of five geometric facts to be used by the pupil in his proofs, and two sheets of exercises (one sheet of which is illustrated on p. 406), which constitute the test to be taken by the pupil. In working each of the exercises the pupils are cautioned to refer after each step of their proof to such of the five given facts as may have been used. It is thus seen that this is a simple test in demonstrative geometry. This test is given, however, to all pupils, whether they have studied any demonstrative geometry or not in their junior high-school course. Most students will have had considerable work in intuitive geometry, and a few may have had a little introductory work in demonstrative geometry. The test has even been taken successfully by pupils who have studied neither intuitive nor demonstrative geometry, that is, by those pupils who have had only arithmetic in the seventh and eighth years and only algebra in the ninth year. In giving the Geometry Test the teacher takes eight minutes for explaining the test, and allows twenty-two minutes for the pupils to do it.

Test 4. Superposition

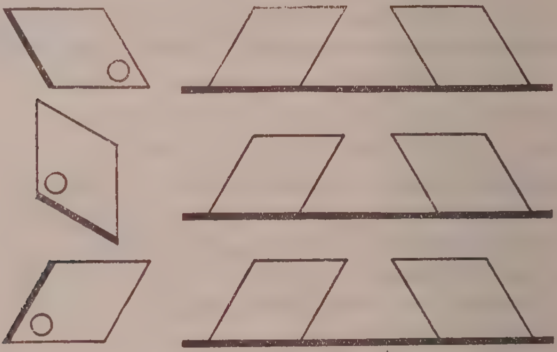
The children are first requested to read the following directions concerning this test, which are printed on the test itself:

Suppose that the figure with a circle in it is a small card with one of its edges painted black and with a hole in one corner.

If this card is moved around so that its black edge lies upon the long heavy black line, it will fit one of those two figures shown.

Decide which it fits and then with your pencil *draw a circle where the hole would be.*

Try the three following examples.



In order to make the directions still clearer, the teacher prepares three pieces of cardboard similar to the three figures with circles which appear at the left in the above illustration. These cards should be 10 inches on the side with one edge of the card black. Holes should be cut as indicated by the circles. The teacher will then reproduce on the blackboard the other three figures appearing above at the right. The teacher now takes the cardboard representing the first of the three examples above, slides it around on the surface of the board over the first figure drawn on the board, which we will call Figure A (being careful NOT to turn the card over so that the under side comes on top). This sliding should so be done that the black edge of the card lies upon the heavy black line of Figure A drawn on the board. The card will fit either the left parallelogram or the right parallelogram of Figure A. Having determined that it fits the left parallelogram, the teacher will then make a circle through the hole with chalk. While doing this, the teacher will repeat orally the following:

> "You are to imagine that the figure with a circle in it is a small card with one of its edges painted black and with a hole in one corner Then this card is slid around so that its black edge lies upon the long heavy black line to the right. It will fit one of the two figures on that line. Decide which it fits and then with your pencil draw a circle where the hole would be."

The children are then given the Superposition Test which consists of two pages, one of which is illustrated on p. 407. Five minutes are allowed for the explanation and two minutes for the test itself.

TEST 5. MIXED RELATIONS

A specimen of this test is shown on page 409. Before giving the test the teacher writes on the blackboard the following analogies:

color — red   name — John
page — book   handle— ?
fire — burns   soldiers— ?

The teacher says: "The first two words on the first line on the blackboard express a certain relation. You have to find a fourth word, which along with the third word will give the same relation. Thus COLOR is related to RED as NAME is related to JOHN, that is, RED is a COLOR and JOHN is a NAME.

The teacher then points to the second line and asks: "PAGE is related to BOOK as HANDLE is related to WHAT?"

The teacher explains that PAGE is part of a BOOK and HANDLE is part of a KNIFE (DOOR, SUIT-CASE, PAIL, etc.).

Similarly, the teacher points to the third line and asks: "FIRE is related to BURNS as SOLDIERS is related to WHAT?" The teacher thus brings out that words like FIGHT or MARCH have the desired relation.

If not sure that the test is fully understood by every pupil, give further illustrations. The teacher then asks the pupils to turn to the test and says: "On this page similar relations must be found. You must always find a fourth word that is related to the third as the second is related to the first. Your score depends on the number of correct answers."

Three minutes are allowed for the explanation, and three minutes for the test itself.

TEST 6. TRABUE LANGUAGE SCALES

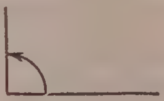
This test consists of two pages of incomplete sentences, one page of which is shown on page 410. The pupils are told to write *one* word in each blank space, selecting the word which will make the most sensible statement. Two minutes are allowed for the teacher's explanation, and 10 minutes for the pupils to write the test.

Rogers Test of Mathematical Ability*

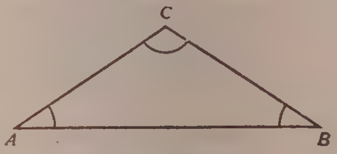
Facts To Be Used in the Geometry Test

This sheet is for reference only. The following facts are given as true.

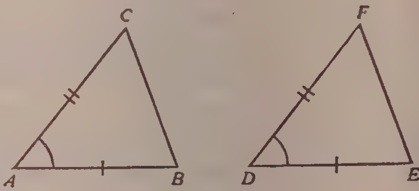
1. *There are 90 degrees in every right angle.*
   This is a right angle.



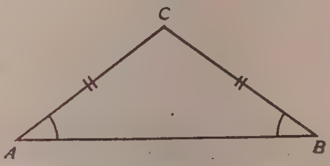
2. *All the angles of a triangle added together equal 180 degrees.*
   Thus in triangle ABC, angle A, angle B, and angle C added together equal 180 degrees.



3. *Two triangles are equal if two sides and the included angle of one triangle are equal respectively to two sides and the included angle of the other.*
   Thus the triangle ABC equals triangle DEF, since side AC equals side DF and side AB equals side DE, and angle A equals angle D.



4. *An isosceles triangle has two sides equal and the angles opposite them equal.*
   Thus ABC is an isosceles triangle, since AC equals BC and angle A equals angle B.



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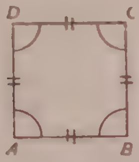
5. *The sides of a square are equal and all its angles are right angles.*

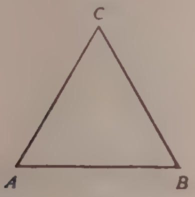
Thus $ABCD$ is a square, since AB equals BC equals CD equals DA, and angles A, B, C and D are right angles.



Directions

Taking the above facts (1), (2), (3), (4), (5) as true, do the work required on the next page and write your answer in the space reserved. Do not take anything for granted not given above in (1), (2), etc., or under *Given* on the next page. Do the exercises on the next page in the order in which you find them. In every case when you use any of the facts above, (1), (2), etc., in your work, write the number to show what fact it is you use.

Example:



Given: Angle $A = 60$ degrees and that the triangle is isosceles.

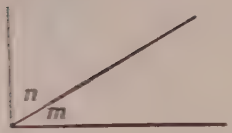
How many degrees are there in Angle B? State the reasons.

Answer:

The triangle is isosceles (Given)
Therefore angle A equals angle B by fact (4)
Angle A equals 60 degrees (Given)
Therefore angle B equals 60 degrees.

GEOMETRY TEST

1.

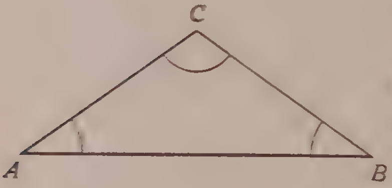


Given: Angle $m = 30$ degrees, and the sum of the angles m and n is a right angle.

How many degrees are there in angle n?
State the reason.

Answer:

2.

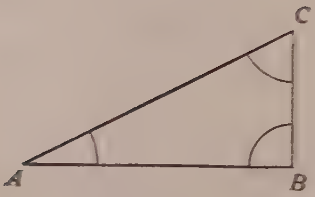


Given: The triangle is isosceles and angle $A = 30$ degrees.

How many degrees are there in angle B?
State the reasons.

Answer:

3.



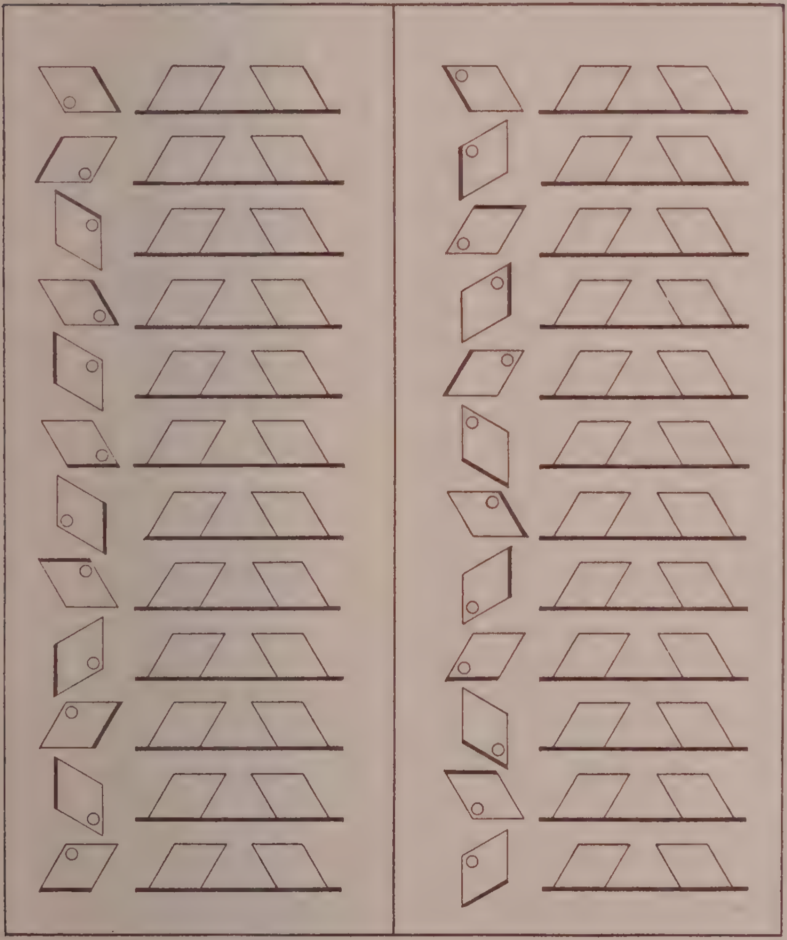
Given: Angle B is a right angle and angle $A = 30$ degrees.

How many degrees are there in angle C?
State the reasons.

Answer:

Rogers Test of Mathematical Ability

SUPERPOSITION TEST



Rogers Test of Mathematical Ability

Interpolation Test

A.	1	8	15	22	....	36	43	50	....	64	71
B.	3	7	11	15	....	23	27	31	....	39	43
C.	4	10	16	22	....	34	40	46	....	58	64
D.	6	10	14	....	22	26	30	....	38	42	46
E.	9	20	31	....	53	64	75	....	97	108	119
F.	5	17	29	....	53	65	77	....	101	113	125
G.	8	17	26	....	44	53	....	71	80	89	98
H.	0	....	16	....	32	....	48	....	64	....	80
I.	7	....	13	....	19	....	25	....	31	....	37
J.	2	....	16	....	30	....	44	....	58	....	72
K.	1	....	....	7	....	....	13	....	....	....	21
L.	4	....	....	31	....	....	58	....	....	....	94
M.	7	....	....	....	27	....	....	....	47	....	....
N.	9	....	....	....	57	....	....	....	105	....	....
O.	...	....	13	....	....	....	29	....	....	....	45
P.	...	....	16	....	....	....	44	....	....	....	72
Q.	3	....	....	....	....	33	....	....	....	....	63
R.	2	....	....	....	....	47	....	....	....	....	92
S.	6	....	....	....	....	....	72	....	....	....	116
T.	5	....	....	....	....	....	83	....	....	....	135

Rogers Test of Mathematical Ability

Mixed Relations Test

Examples:
color—red name—John
page—book handle—knife
fire—burns soldiers—fight

1. eye—see ear—
2. Monday—Tuesday April—
3. do—did see—
4. bird—sings dog—
5. hour—minute minute—
6. straw—hat leather—
7. cloud—rain sun—
8. hammer—tool dictionary—
9. uncle—aunt brother—
10. dog—puppy cat—
11. little—less much—
12. wash—face sweep—
13. house—room book—
14. sky—blue grass—
15. swim—water fly—
16. once—one twice—
17. cat—fur bird—
18. pan—tin table—
19. buy—sell come—
20. oyster—shell banana—
21. past—present present—
22. come—came go—
23. North—South far—
24. mend—clothes bake—
25. lily—flower oak—
26. ton—pound pound—
27. elbow—arm chin—
28. pea—pod nut—
29. bell—rings clock—
30. deep—valley high—
31. growls—dog roars—
32. brick—wall page—
33. lathe—machine hammer—
34. pencil—lead book—
35. high—low up—
36. sheep—lamb dog—
37. kettle—brass cup—
38. Thursday—Friday June—
39. build—house paint—
40. one—single two—
41. mice—cat worms—
42. London—England Paris—
43. church organ—banjo Hamlet—

Rogers Test of Mathematical Ability

Trabue Language Scale L

Write only one word on each blank

1. Children .... are rude ....not easily win friends.

2. Plenty .... exercise and .... air ....healthy .... and girls.

3. In .... to maintain ....health, one should have nourishing ....

4. .... happiness can not be .... with money.

5. One's .... do ....always express his thoughts.

6. To .... to wait, after having ....to go .... very annoying.

7. It is sometimes .... to .... between two .... of action.

8. One can .... do his .... at one .... while .... of another.

Much more detailed directions for giving the Rogers test are to be found in a pamphlet entitled *Directions for Giving the Rogers Test of Mathematical Ability*, by Agnes L. Rogers, which together with the tests themselves may be obtained from the Bureau of Publications, Teachers College, Columbia University, New York City. This pamphlet also describes in detail the method of scoring the test.

The pupil's total score on the entire six tests is expressed as a single number. Pupils obtaining a total score of over 600 have mathematical capacity superior to the average high-school pupil and sufficient to warrant their being made to cover ground more rapidly. Pupils with a total score less than 400 have inferior mathematical capacity and unless unusually industrious will fail to master the same amount of mathematics as the average student. They can probably cover two years' work in three years. Pupils obtaining a score above 700 have very superior mathematical ability and should be allowed to progress at their own rate, while those obtaining a score below 300 have very inferior mathematical ability and other things being equal should be released from further study of the subject. The above facts are presented below in tabular form:

Score on Test	*Mathematical Ability*
Above 700	Very Superior
Between 600 and 700	Superior
Between 400 and 600	Average
Between 300 and 400	Inferior
Below 300	Very Inferior

While the reliability of the Rogers test is satisfactorily high, nevertheless in all group examinations there is a possibility of a particular pupil failing to make a score which is representative of his actual capacity owing to temporary indisposition or excitement or worry.

Whenever a case of this kind occurs, in which a pupil's score in the test is at variance with his ordinary achievement in the mathematics class, it is desirable that he should be examined a second time. A duplicate sextet of tests will be issued later for this purpose. Meantime it is recommended that the test should not be used independently of school marks and teachers' judgments, but only to confirm or challenge them.

The standards which are given above are merely tentative. It is desirable that the test should be more extensively applied and to groups other than those for which they were originally determined. Dr. Rogers, the author of the test, will assist in their interpretation, if results are forwarded to her at Goucher College, Baltimore, Md.

It is the single score number, described above, that enables the teacher to make use of this test as a basis for advising students in reference to their future mathematical studies. It can also be used as a check upon a teacher's mark or judgment of a pupil, and likewise to classify pupils in advance on the basis of mathematical ability. This score is also serviceable as an additional check where it is necessary to select those pupils who should be limited in their study of mathematics, for example, those whose study of geometry should be confined to intuitive geometry, or those who should take the same work as the average student of high-school mathematics, but should spend an additional term or terms upon it.

The total score obtained as a result of giving the Rogers sextet of tests to a group of 61 pupils in the Horace Mann School of Teachers College, New York City, showed a very close relation to the future mathematical accomplishment of the students. For those acquainted with statistics it will be of interest that the coefficient of correlation was .82, which indicates a very close relationship between the scores obtained in the test and the pupils' future mathematical accomplishment.

It is thus seen that in an hour and a half the Rogers test may enable a stranger to learn as much about a pupil's native mathematical ability as it has taken the regular teacher two or more years to determine.

"Much more scientific work remains to be done along the line of the Rogers test. We need tests to measure innate algebraic intelligence, before the child has had any algebraic training whatever. We should have a test or a group of tests of such a character applied the first day in the ninth-grade algebra course, so that we could from the outset group pupils more appropriately, in consonance with their powers and their promise as students of the subject. The same holds good of geometry. For it, too, we need a team of tests or a single test of high diagnostic power, which presupposes no previous knowledge of the subject, and which could provide us with an adequate measure of innate ability in that field."[17]

A statement concerning the origin and development of the Rogers Test of Mathematical Ability may be of interest. The six tests comprising the sextet were selected by Dr. Rogers as jointly providing a reliable test of mathematical ability after a very comprehensive study[18] and experimentation with seventeen earlier tests, five of which had covered phases of algebraic ability, five others providing for the several geometrical capacities, and four being tests of language ability in order to discover how far inferiority in mathematics is due to lack of ability in the use of words. There were also three other tests. By rigidly controlling the conditions of giving these seventeen tests the various elements that go to make up mathematical ability were obtained, and the particular value of each of the tests in measuring this composite ability was determined. "It was found that mathematical ability is not a single function but complex in character, involving a number of mental processes which are loosely related to each other but all equally indispensable for successful mathematical accomplishment. While some of these abilities have more importance than others, all had some part to play in determining the presence of mathematical capacity."[19] After this analysis "it was clear that no single test could afford a satisfactory measure of mathematical proficiency. It was possible, however, to discern three dominating types of ability, each essential to success with the subject. These were algebraic abilities, geometrical abilities, and verbal or language abilities, and these three proved of equal influence in determining the presence or absence of mathematical efficiency."[20] Dr. Rogers' study also showed that the abilities dominant in high-school algebra are not those dominant in geometry. This only bears out the experience of many teachers that pupils who have been successful in algebra will not necessarily be successful in geometry. It was found that the relation between ability in the language tests and ability in geometry is just as close as the relation between ability in algebra and ability in geometry. Thus it would seem that algebra and geometry are in no special degree closely related, which is of particular interest these days in connection with the current discussions which in some quarters are favoring a close fusion or correlation of algebra and geometry, so that they may become practically a single subject.

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[17] *The Mathematics Teacher,* Vol. XI, No. 4, June, 1919, p. 161.

[18] See Rogers: *Tests of Mathematical Ability and Their Prognostic Value.*

[19] From *Teachers College Record,* Vol. XIX, No. 4, September, 1918, p. 415.

[20] *Ibid.,* p. 416.

"It is a noteworthy fact that verbal ability tests enable us to predict efficiency in algebra with as great a chance of success as geometry tests would and similarly that they are as reliable indices of the characteristics which make a successful student of geometry as algebra tests are.

"If there is any one factor common to these diverse types of ability involved in mathematical functioning, it appears to be the ability to respond to partial elements in a complex total situation, in other words, to analyze and think selectively. Therefore, this, if anything, is the crucial quality determining success. Tests possessing it in high degree would best serve to indicate the presence or absence of mathematical capacity. Besides this common factor, however, which is found in different amounts in all the tests, there are specific factors of varying value which are also necessary for efficiency."[21]

Since algebraic, geometric, and language abilities had stood out as dominant types in that composite known as mathematical ability, Dr. Rogers selected six from the original seventeen tests to constitute the sextet, and which seemed to afford the most dependable quick measures of these different abilities. The six tests originally selected were considerably revised and simplified in 1921, and it is this revised test which is now available in pamphlet form for testing pupils. In the revision the Geometry and Algebraic Computation Tests were materially changed and simplified, and the Interpolation Test somewhat shortened. The method of scoring was also much simplified.[22]

Thurstone Vocational Guidance Test

Another test of mathematical ability is the Thurstone Vocational Guidance Test which was issued in 1922. It was prepared by Dr. L. L. Thurstone, of the Carnegie Institute of Technology. It consists of five separate tests, one in each of the following subjects: arithmetic, algebra, geometry, physics, and technical information. The tests are designed to be given in the senior year of the high school to students who intend to study engineering. The purpose of the test is to enable the prospective engineering student to obtain in advance a rough measure of his own ability to succeed in the engineering courses of the freshman year of the technical school.

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[21] *Teachers College Record*, Vol. XIX, No. 4, September, 1918, p. 416.

[22] The original six tests, together with the original scoring scheme, are given in Dr. Rogers' dissertation, *Experimental Tests of Mathematical Ability and Their Prognostic Value*, pp. 113-118.

The arithmetic test consists of twenty-two short problems. It is not an arithmetic examination in the narrow sense. While its problems call for arithmetic computation, the test calls for more than that. The problems are all selected with the idea that they will appeal to students with engineering interests. The time allowance for the arithmetic test is thirty minutes.

The algebra test consists of twelve short problems and is not markedly different from conventional examinations in this subject. The algebra test has been standardized by giving it to 7039 engineering freshmen at the time of their admission to college, and the norms are therefore especially useful for the student who wishes to compare himself with the standards of ability of freshmen in engineering colleges. If the candidate scores average or better in this test, he can be reasonably sure that he will be able to carry the freshman engineering mathematics satisfactorily. The only merit which this algebra examination has over other examinations in this subject is that it has been standardized on a large group of engineering freshmen at the time of their admission to college and not during the time when they were studying algebra in high school. The fact that the test is objectively graded facilitates the marking of the papers and avoids ambiguities in the scores. The time allowance for the algebra test is thirty minutes.

The geometry test consists of nine geometric constructions which the student is asked to make. It is a test of the applications of geometry rather than of geometric proof. The standards for this geometry test are based on the results of giving the test to 6232 engineering freshmen. The time allowance for this test is thirty minutes.

A test in physics and one in technical information are also provided.

The test, together with the manual of directions, may be obtained from the World Book Company, Yonkers, New York.

Conclusion

"We can scarcely grasp the full significance of the introduction of scientific measurement in education. We can but dimly discern its probable results. When clairvoyance and divination were for the first time in history replaced by science, by the inductive experimental method, how few were conscious of the momentousness of the event that had taken place! How few realized that a turning point in the history of mankind had been reached! In like fashion we apprehend but faintly that a new era has begun in the records of the human race. The birth of physical science was a world-shaping event, inasmuch as it gave control over the material universe. The application of measurement and experimentation to mental products is as remarkable an occurrence, for it literally means control of the human machine. The last factor in efficiency to be controlled, the human factor, will in all likelihood yield the richest results of all. It seems only yesterday that we ignored the fact that a few unnecessary movements in an activity like sewing add much to the bodily energy consumed in a day, so we now are for the most part unaware of the wastefulness of our methods of habit-formation in mathematical work. As soon as effort is devoted to changing this state of affairs, we shall be magnificently rewarded in the saving in human strength that will result. At this time in the world's history, when the minds of men were never more malleable, when far-reaching adjustments in our social and in our educational institutions are in progress, we should be eager to play a part in a revolution still more fundamental, more invasive and pervasive than any other—the downthrow of opinion and the rise of science in the schoolroom. When we recall how long it took before we had precise instruments of measurement in physical science and how much is expended yearly to maintain bureaus of weights and measures, we shall not grudge the cost of obtaining educational measuring rods, although the magnitude of the task confronting us may temporarily give us pause. It is well to remember, however, that in no subject are circumstances more favorable, in none should progress be so rapid as in mathematics. The work already accomplished, although fragmentary, has already proved its value and gives promise of richer results. Undoubtedly the tests at our disposal have much significance for the mathematics teacher. They have begun the detailed analysis of mathematical abilities; they have rendered possible improved methods of classification, and diagnosis and prognosis of individual ability can now be made by their help. Analysis, classification, and diagnosis—these three outstanding ends they serve; and the greatest of these is diagnosis."[23]

PART V. BIBLIOGRAPHY

Arithmetic

CLEVELAND SURVEY TESTS

1. **Counts, G. S.** Arithmetic Tests and Studies in the Psychology of Arithmetic. Supplementary Educational Monographs, Vol. i, No. 4. Chicago: University of Chicago Press (1917).
2. **Heckert, J. W.** "The Cleveland Survey Tests in Arithmetic in the Miami Valley." Elementary School Journal, xviii: 447-457 (February, 1918).
3. **Judd, Charles H.** Measuring the Work of the Public Schools. Cleveland: The Survey Committee of the Cleveland Foundation (1916).
4. **O'Hern, Joseph P.** "Practical Application of Standard Tests in Spelling, Languages, and Arithmetic." Elementary School Journal, xviii: 662-679 (May, 1918).
5. **Peterson, Joseph.** "Methods of Interpreting Results in the Cleveland Arithmetic Tests." Journal of Educational Research, iii: 280-292 (April, 1921).
6. **Smith, J. H.** "Individual Variation in Arithmetic." Elementary School Journal, xvii: 195-200 (November, 1916).
7. **School Surveys:** Cleveland, Ohio; Grand Rapids, Michigan; St. Louis, Missouri.

COURTIS TESTS

1. **Baldwin, B. T.** "Application of the Courtis Tests in Arithmetic to College Students." School and Society, i: 569-576 (April 17, 1915).

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[23] See *The Mathematics Teacher*, Vol. XI, No. 4, June, 1919, 162-163.

2. **Ballou, Frank W.** Provisional Minimum Standards in Addition, Subtraction, Multiplication, and Division for Pupils in Grades 4 to 8. School Document No. 9. Boston Public Schools, Department of Educational Investigation and Measurement (1914).

3. **Ballou, Frank W.** Educational Standards and Educational Measurements with Particular Reference to Standards in the Four Fundamentals of Arithmetic. School Document No. 10. Boston Public Schools, Department of Educational Investigation and Measurement (1914).

4. **Ballou, Frank W.** Arithmetic: The Courtis Standard Tests in Boston, 1912-1915. An Appraisal. School Document No. 15. Boston Public Schools, Department of Educational Investigation and Measurement (1916).

5. **Ballou, Frank W.** Arithmetic: The Value to the Teacher, to the Principal, and to the Superintendent, of Individual and Class Records from Standard Tests. School Document No. 22. Boston Public Schools, Department of Educational Investigation and Measurement (1917).

6. **Ballou, Frank W.** "Reliable Basis for Establishing Standards from the Results of the Courtis Standard Tests." Educational Administration and Supervision, i: 469-472 (September, 1915).

7. **Boyer, Philip A.** "The Courtis Tests in Arithmetic." The Mathematics Teacher, xi: 121-132 (March, 1919).

8. **Buckingham, B. R.** "The Courtis Tests in the Schools of New York City." Journal of Educational Psychology, v: 199-214 (April, 1914).

9. **Courtis, Stuart A.** Teacher's Manual for the Courtis Practice Tests. Yonkers: World Book Company, 1914; Revised Edition, 1920.

10. **Courtis, Stuart A.** Manual of Instructions for Giving and Scoring the Courtis Standard Tests in the Three R's. Revised Edition. Detroit: Department of Coöperative Research (1914).

11. **Courtis, Stuart A.** Courtis Standard Research Tests. First Annual Accounting. Bulletin No. 1, 1911-1912. Detroit: Department of Cooperative Research.

12. **Courtis, Stuart A.** Courtis Standard Research Tests. Second Annual Accounting. Bulletin No. 2, 1912-13. Detroit: Department of Cooperative Research.

13. **Courtis, Stuart A.** Courtis Standard Research Tests. Third, Fourth, and Fifth Annual Accountings—Capacity, Ability, and Performance—Summary of Tabulations. Bulletin No. 4, 1913-16. Detroit: Department of Coöperative Research.

14. **Courtis, Stuart A.** "Measurement of Growth and Efficiency in Arithmetic." Elementary School Teacher, x: 58-74, 177-199 (October and December, 1909); xi: 171-185, 360-370, 528-539 (December, 1910; March, 1911; June, 1911).

15. **Courtis, Stuart A.** "Standard Scores in Arithmetic." Elementary School Teacher, xii: 127-137 (November, 1911).

16. **Courtis, Stuart A.** "The Reliability of Single Measurements with Standard Tests." Elementary School Teacher, xiii: 326-345; 486-504 (March and June, 1913).

17. **Courtis, Stuart A.** "Educational Diagnosis." Educational Administration and Supervision, i: 89-116 (February, 1915).

18. **Courtis, Stuart A.** "Courtis Tests in Arithmetic: Value to Superintendents and Teachers." Chap. viii, 91-106. Fifteenth Yearbook, Part I, of the National Society for the Study of Education. Chicago: University of Chicago Press (1916).

19. **Courtis, Stuart A.** "Measuring the Child's Capacity." Normal Instructor and Primary Plans, xxv: 31, 76 (October, 1916); xxvi: 53-54, 82 (November, 1916); xxvi: 45-46 (December, 1916).

20. **Courtis, Stuart A.** "Research Work in Arithmetic." Educational Administration and Supervision, iii: 61-74 (February, 1917).

21. **Courtis, Stuart A.** The Gary Public Schools. Measurement of Classroom Products, pp. 176-215. New York: General Education Board (1919).

22. **Courtis, Stuart A. and Parker, F. E.** "Value of Measurements." English Journal, viii: 203-217 (April, 1919).

23. **Courtis, Stuart A. and Thorndike, E. L.** "Correction Formulae for Addition Tests." Teachers College Record, xxi: 1-24 (January, 1920).

24. **Haggerty, M. E.** A Coöperative Study in Educational Measurements. Indiana University Bulletin, Vol. xii, Study No. 27 (1915).

25. **Haggerty M. E.** Studies in Arithmetic. Second Report on the Measure of Arithmetic in Indiana City Schools. Indiana University Bulletin, Vol. iii, Study No. 32 (1916).

26. **Hanus, Paul H. and Gaylord, Harry D.** "Courtis Arithmetic Tests Applied to Employees in Business Houses." Educational Administration and Supervision, iii: 505-520 (November, 1917).

27. **Henmon, V. A. C.** "Improvement in School Subjects Throughout the School Year." Journal of Educational Research, i: 81-95 (February, 1920).

28. **Hinkle, E. E.** "Experiments for Efficiency in Arithmetic. The Courtis Tests." Educational Bi-Monthly, viii: 189-201 (February, 1914).

29. **Mead, Cyrus D.** An Experiment in the Fundamentals, Giving the Results of Tests Made in the Cincinnati Schools with Two Kinds of Practice Material. Yonkers: World Book Company (1917).

30. **Mead, Cyrus D. and Johnson, Charles W.** "Testing Practice Material in the Fundamentals." Journal of Educational Psychology, ix; 287-297 (May, 1918).

31. **Monroe, Walter S.** Report of the Use of the Courtis Standard Research Tests in Arithmetic in Twenty-four Cities. Bulletin, Vol. iv, No. 8, of Bureau of Educational Measurements and Standards, Kansas State Normal School, Emporia, Kansas (1915).

32. **Monroe, Walter S.** "Improvement of Instruction through the Use of Educational Tests." Journal of Educational Research, i: 96-102 (February, 1920).

33. **Sutton, C. W.** Arithmetical Abilities of School Children as Shown by the Courtis Tests. Bulletin No. 1. Cleveland: Board of Education, Division of Reference and Research (November, 1917).

34. **Uhl, W. L.** "The Use of Standardized Materials in Arithmetic for Diagnosing Pupils' Methods of Work." Elementary School Journal, xviii: 215-218 (November, 1917).

35. **Zeidler, R.** "Tests of Efficiency in the Rural and Village Schools of Santa Clara County, California. Courtis Standard Research Tests." Elementary School Journal, xvi: 542-555 (June, 1916).

36. **School Surveys:** Butte, Montana; Salt Lake City, Utah; Leavenworth, Kansas; New York City.

MONROE TESTS

1. **Monroe, Walter S.** "The Ability to Place the Decimal Point in Division." Elementary School Journal, xviii: 287-293 (December, 1917).

2. **Monroe, Walter S.** "The Derivation of Reasoning Tests in Arithmetic." School and Society, viii: 295-299, Part I (September 7, 1918); viii: 324-329, Part II (September 14, 1918).

3. **Monroe, Walter S.** "A Series of Diagnostic Tests in Arithmetic." Elementary School Journal, xix: 585-607 (April, 1919).

4. **Monroe, Walter S.** Report of Division of Educational Tests for 1919-1920. University of Illinois Bulletin, Vol. xviii, No. 21 (1921).

STONE REASONING TEST

1. **Stone, C. W.** "Problems in the Scientific Study of the Teaching of Arithmetic." Journal of Educational Psychology, iv: 1-16 (January, 1913).
2. **Stone, C. W.** Arithmetical Abilities and Some Factors Determining Them. New York: Bureau of Publications, Teachers College (1908).
3. **Stone, C. W.** Standardized Reasoning Tests in Arithmetic and How to Utilize Them. New York: Bureau of Publications, Teachers College (1916, 1922).
4. **School Surveys**: Butte, Montana; Salt Lake City, Utah; Nassau County, New York.

WOODY SCALES

1. **Anderson, C. J.** "The Use of the Woody Scales for Diagnostic Purposes." Elementary School Journal, xviii: 770-781 (June, 1918).
2. **Baldwin, B. T. and Others**. Studies in Experimental Education. Johns Hopkins University Studies in Education, No. 3. Baltimore: Johns Hopkins Press (1920).
3. **Monroe, Walter S.** "An Experimental and Analytical Study of Woody's Arithmetic Scales, Series B." School and Society, vi: 412-420 (October 6, 1917).
4. **Theisen, W. W. and Fleming, C. W.** "Scientific Management in Education." Chap. IX of The State and Public Schools. Two Years' Progress in Education in Wisconsin. Wisconsin: State Department of Public Instruction (1919).
5. **Woody, Clifford**. Measurements of Some Achievements in Arithmetic. New York: Bureau of Publications, Teachers College (1916, 1920).
6. **Woody, Clifford.** The Woody Arithmetic Scales. How to Use Them. New York: Bureau of Publications, Teachers College (1920). Reprinted from 1920 edition of Measurements of Some Achievements in Arithmetic.
7. **Woody, Clifford**. "The Teaching of Educational Measurements." Educational Administration and Supervision, v: 7-14 (January, 1919).
8. **Woody, Clifford.** "Tests and Measures in the Schoolroom and Their Value to the Teacher." School and Society, vi: 61-66 (July 21, 1917).

MISCELLANEOUS REFERENCES

1. **Ballou, Frank W.** "The Function of a Department of Educational Investigation and Measurement in a City School System." School and Society, i: 181-190 (February 6, 1915).

2. **Ballou, Frank W.** "Improving Instruction through Educational Measurement." Educational Administration and Supervision, ii: 354-36 (May, 1916).
3. **Ballou, Frank W.** Arithmetic: Research Tests in Addition of Fractions: Determining the Achievement of Pupils in the Addition o Fractions. School Document No. 3. Boston Public Schools, Department of Educational Investigation and Measurement (1916).
4. **Ballou, Frank W.** Arithmetic: Determining the Achievement of Pupil in Common Fractions. School Document No. 5. Boston Publi Schools, Department of Educational Investigation and Measuremen (1918).
5. **Beeson, Marvin F.** Educational Tests and Measurements. Colorad State Teachers College Bulletin, Series 20, No. 3 (June, 1920).
6. **Brown, J. C.** "An Investigation on the Value of Drill Work in the Fundamental Operations of Arithmetic." Journal of Educational Psychology, ii: 81-88 (February, 1911); iii: 485-492, 561-570 (November and December, 1912).
7. **Finley, George W.** A Comparative Study of Three Diagnostic Arithmetic Tests. Colorado State Teachers College Bulletin, Series 20, No. 4 (July, 1920).
8. **Gray, P. L.** "Norms of Performance in the Fundamental Processes of Arithmetic." Journal of Experimental Pedagogy, iii: 310-318 (June, 1916).
9. **Jessup, W. A. and Coffman, L. D.** Supervision of Arithmetic. New York: Macmillan (1916).
10. **Kallom, Arthur W.** "Analysis of and Testing in Common Fractions." Journal of Educational Research, i: 177-192 (March, 1920).
11. **Klapper, Paul.** Teaching of Arithmetic. New York: D. Appleton and Co. (1916).
12. **Rice, J. M.** "Educational Research: Arithmetic." The Forum, xxxiv: 281-287 (October, 1902); xxxiv: 437-452 (January, 1903); xxxiv: 588-607 (June, 1903).
13. **Starch, Daniel.** "A Scale for Measuring Ability in Arithmetic." Journal of Educational Psychology, vii: 213-222 (April, 1916).
14. **Stone, John C.** Teaching of Arithmetic. Chicago: Benjamin H. Sanborn and Co. (1918).
15. **Studebaker, John Ward.** Teachers' Manual for Studebaker Economy Practice Exercises in Arithmetic. Chicago: Scott, Foresman and Co. (1916).

16. **Thorndike, E. L.** "Practice in the Case of Addition." American Journal of Psychology, xxi: 483-486 (July, 1910).

17. **Thorndike, E. L.** "The Relation between Speed and Accuracy in Addition." Journal of Educational Psychology, v: 537 (November, 1914).

18. **Thorndike, E L.** "Measurements of Ability to Solve Arithmetical Problems." Pedagogical Seminary, xxi: 495-503 (December, 1914).

19. **Thorndike, E. L.** Means of Increasing Efficiency in Arithmetic. Indiana University Bulletin, Vol. xii, No. 4 (1914).

20. **Thorndike, E. L.** The New Methods in Arithmetic. New York: Rand McNally and Co. (1921).

21. **Thorndike, E. L.** Psychology of Arithmetic. New York: Macmillan (1921).

22. **Thorndike, E. L. and Hahn, H. H.** "Some Results of Practice in Addition under School Conditions." Journal of Educational Psychology, v: 65-84 (February, 1914).

23. **Wilson, G. M.** A Survey of the Social and Business Usage of Arithmetic. New York: Bureau of Publications, Teachers College (1919).

24. **Wilson, G. M.** "The Proper Content of a Standard Test." Elementary School Journal, xix: 375-381 (January, 1919).

Algebra

HOTZ SCALES

1. **Cawl, Franklin R.** "Practical Uses of an Algebra Standard Scale." School and Society, x: 88-90 (July, 1919).

2. **Harris, Eleanora.** A Study of the Hotz First-Year Algebra Scales and the Rugg-Clark Standard Algebra Tests. Chicago: University of Chicago Press (1919).

3. **Hobbs, James B.** "Results from Giving the Hotz First-Year Algebra Scale Tests to a Six-Eight Month Group." School and Society, xii: 353-354 (October, 1920).

4. **Hotz, Henry G.** First-Year Algebra Scales. New York: Bureau of Publications, Teachers College (1918).

5. **Hotz, Henry G.** Teacher's Manual for First-Year Algebra Scales. New York: Bureau of Publications, Teachers College (1922).

6. **Survey:** Virginia Public Schools, Part II.

7. **Survey:** Reading, Pa., High School.

8. **Survey:** School System of Mount Holly, N. J.

9. **Biennial Report, 1916-1918,** State Department of Public Instruction, Madison, Wisconsin.

RUGG-CLARK TESTS

1. **Rugg, H. O.** "Experimental Determination of Standards in First-Year Algebra." The School Review, xxiv: 37-66 (January, 1916).
2. **Rugg, H. O. and Clark, J. R.** "Standardized Tests and the Improvement of Teaching in First-Year Algebra." The School Review, xxv: 113-132, Part I (February, 1917); xxv: 196-213, Part II (March, 1917).
3. **Rugg, H. O. and Clark J. R.** "A Coöperative Investigation in the Testing and Experimental Teaching of First-Year Algebra." The School Review, xxv: 346-349 (May, 1917).
4. **Rugg, H. O. and Clark, J. R.** "The Improvement of Ability in the Use of the Formal Operations of Algebra by Means of Formal Practice Exercises." The School Review, xxv: 546-554 (October, 1917).
5. **Rugg, H. O. and Clark, J. R.** Scientific Method in the Reconstruction of Ninth-Grade Mathematics. Supplementary Educational Monographs, Vol. ii, No. 1. Chicago: University of Chicago Press (1918).

MISCELLANEOUS REFERENCES

1. **Childs, Hubert G.** The Measurement of Achievement in Algebra. Bulletin of the Extension Division, Indiana University, Vol. ii, No. 6 (1917).
2. **Dalman, Murray A.** "Hurdles. A Series of Calibrated Objective Tests in First-Year Algebra." Journal of Educational Research, i: 47-62 (January, 1920).
3. **Douglass, Harl Roy.** The Derivation and Standardization of a Series of Diagnostic Tests for the Fundamentals of First-Year Algebra. University of Oregon Publication, Vol. i, No. 8 (April, 1921).
4. **Kelley, Truman L.** "Values in High-School Algebra and Their Measurement." Teachers College Record, xxi: 246-290 (May, 1920).
5. **Monroe, Walter S.** "A Test of the Attainment of First-Year High-School Students in Algebra." The School Review, xxiii: 159-171 (March, 1915).
6. **Monroe, Walter S.** "Measurement of Certain Algebraic Abilities." School and Society, i; 393-395 (March 13, 1915).
7. **Monroe, Walter S.** Report of the Use of the Monroe Research Tests in Algebra in Twenty-four Cities. Bulletin, Vol. vi, No. 7, Bureau of Educational Measurements and Standards, Kansas State Normal School, Emporia, Kansas (1917).
8. **Symonds, Percival M.** "The Psychology of Errors in Algebra." The Mathematics Teacher, xv: 93-104 (February, 1922).

9. **Thorndike, E. L.** "An Experiment in Grading Problems in Algebra." The Mathematics Teacher, vi: 123-134 (March, 1914).
10. **Thorndike, E. L.** "The Nature of Algebraic Abilities." The Mathematics Teacher, xv: 6-15, Part I (January, 1922); xv: 79-92, Part II (February, 1922).
11. **Thorndike, E. L.** "The Psychology of Problem Solving." The Mathematics Teacher, xv: 212-227, Part I (April, 1922); xv: 253-264, Part II (May, 1922).
12. **Thorndike, E. L.** "The Abilities Involved in Algebraic Computation and in Problem Solving." School and Society, xv: 191-193 (February 18, 1922).
13. **Thorndike, E. L.** 'The Permanence of School Learning." School and Society, xv: 625-627 (June 10, 1922).
14. **Thorndike, E. L.** "The Effect of Changed Data upon Reasoning." Journal of Experimental Psychology, v: 33-38 (February, 1922).
15. **Thorndike, E. L.** "The Psychology of the Equation." The Mathematics Teacher, xv: 127-136 (March, 1922).
16. **Thorndike, E. L. and Woodyard, Ella.** "The Uses of Algebra in Study and Reading." School Science and Mathematics, xxii: 405-415, Part I (May, 1922); xxii: 514-522, Part II (June, 1922).
17. **Wood, O. A.** "A Failure Class in Algebra." The School Review, xxviii: 41-49 (January, 1920).

Geometry

MINNICK TESTS

1. **Minnick, John H.** "A Comparative Study of the Mathematical Abilities of Boys and Girls." The School Review, xxiii: 73-84 (February, 1915).
2. **Minnick, John H.** An Investigation of Certain Abilities Fundamental to the Study of Geometry. Philadelphia: University of Pennsylvania (1918).
3. **Minnick, John H.** "Certain Abilities Fundamental to the Study of Geometry." An Abstract. Journal of Educational Psychology, ix: 83-90 (February, 1918).
4. **Minnick, John H.** "A Scale for Measuring Pupils' Ability to Demonstrate Geometrical Theorems." The School Review, xxvii: 101-109 (February, 1919).
5. **Minnick, John H.** "Mathematical Tests: Their Relation to the Mathematics Teacher." The Mathematics Teacher, xi: 199-205 (June, 1919).
6. **Minnick, John H.** "The Scoring of Geometry Test W." Educational Administration and Supervision, vi: 509-511 (December, 1920).

MISCELLANEOUS REFERENCES

1. **Irwin, H. N.** "A Preliminary Attempt to Devise a Test of the Ability of High-School Pupils in the Mental Manipulation of Space Relations." The School Review, xxvi: 600-605 (October, 1918); xxvi: 654-670 (November, 1918); xxvi: 759-772 (December, 1918).
2. **Metzler, W. H.** "The Problems in the Experimental Pedagogy of Geometry." Journal of Educational Psychology, iii: 545-560 (December, 1912).
3. **Stockard, L. V. and J. Carleton Bell.** "A Preliminary Study of the Measurement of Abilities in Geometry." Journal of Educational Psychology, vii: 567-580 (December, 1916).

Mathematical Ability

1. **Mensenkamp, L. E.** "Tests of Mathematical Ability and their Prognostic Values: A Discussion of the Rogers Test." School Science and Mathematics, xxi: 150-162 (February, 1921).
2. **Rogers, Agnes L.** Tests of Mathematical Ability and their Prognostic Value. New York: Bureau of Publications, Teachers College (1918).
3. **Review.** "A New Means of Testing Mathematical Ability of High-School Pupils." Teachers College Record, xix: 415-418 (September, 1918).
4. **Rogers, Agnes L.** Directions for Using the Rogers Test of Mathematical Ability. New York: Bureau of Publications, Teachers College (1921).
5. **Rogers, Agnes L.** "Tests of Mathematical Ability—Their Scope and Significance." The Mathematics Teacher, xi: 145-163 (June, 1919).
6. **Thurstone, L. L.** Thurstone Vocational Guidance Tests. Manual of Directions. Yonkers: World Book Company (1922).

Miscellaneous References

1. **Breslich, E. R.** "Testing as a Means of Improving the Teaching of High-School Mathematics." The Mathematics Teacher, xiv: 276-291 (May, 1921).
2. **Breed, F. S. and Breslich, E. R.** "Intelligence Tests and the Classification of Pupils." Part II. The School Review, xxx: 210-226 (March, 1922).
3. **Buckingham, B. R.** "Suggestions for Procedure Following a Testing Program. Part I—Reclassification." Journal of Educational Research, ii: 787-801 (December, 1920).

4. **Buckingham, B. R. and Monroe, Walter S.** "A Testing Program for Elementary Schools." Journal of Educational Research, ii: 521-532 (September, 1920).
5. **Cody, Sherwin.** Commercial Tests and How to Use Them. Yonkers: World Book Company (1919).
6. **Courtis, Stuart A.** "The Measurement of High-School Mathematics." School Science and Mathematics, xviii: 507-526 (June, 1918).
7. **Dearborn, W. F.** "The Misuse of Standard Tests in Education." School and Society, iii: 493-495 (April 1, 1916).
8. **Douglass, H. R.** "Some Uses and Limitations of the Standard Educational Test." Educational Administration and Supervision, v: 475-490 (December, 1919).
9. **Gates, Arthur I.** "The True-False Test as a Measure of Achievement in College Courses." Journal of Educational Psychology, xii: 276-287 (May, 1921).
10. **Holmes, H. W. and Others.** A Descriptive Bibliography of Measurements in Elementary Subjects. Harvard Bulletins in Education, Vol. v (June, 1917).
11. **First Annual Report of Bureau of Educational Research.** University of Illinois Bulletin, Vol. xvii, No. 9 (October 27, 1919).
12. **Indiana University Conference on Educational Measurements, April, 1919.** Bulletin of the Extension Division, Indiana University, Vol. v, No. 1 (September, 1919).
13. **Indiana University Conference on Educational Measurements, April, 1920.** Bulletin of the Extension Division, Indiana University, Vol. vi, No. 1 (September, 1920).
14. **Kelley, Truman L.** "The Reliability of Test Scores." Journal of Educational Research, iii: 370-379 (May, 1921).
15. **Keyser, Cassius J.** "The Human Worth of Rigorous Thinking." The Mathematics Teacher, xv: 1-5 (January, 1922).
16. **Lincoln, Edward A.** "The Effects of Native Intelligence upon Scores in Standard Tests." School and Society, xii: 441-444 (November 6, 1920).
17. **McCall, William A.** "Criteria for Determining Which Educational Test to Employ." School and Society, vii: 207-209 (February 16, 1918.)
18. **McCall, William A.** "How to Compute the Median." Teachers College Record, xxi: 124-138 (March, 1920).
19. **McCall, William A.** How to Measure in Education. New York: Macmillan (1922).
20. **Monroe, W. S., DeVoss, J. C., and Kelly, F. J.** Educational Tests and Measurements. New York: Houghton Mifflin Company (1917).

21. **Monroe, Walter S.** Measuring the Results of Teaching. New York: Houghton Mifflin Company (1918).

22. **Myers, G. W.** "A Plan for Testing Methods of Secondary Mathematics." The School Review, xxii: 91-97 (February, 1914).

23. **Pintner, Rudolph and Marshall, Helen.** "A Combined Mental-Educational Survey." Journal of Educational Psychology, xii: 32-43 (January, 1921).

24. **Pintner, Rudolph and Marshall, Helen.** "Results of the Combined Mental-Educational Survey Tests." Journal of Educational Psychology, xii: 82-91 (February, 1921).

25. **Rugg, H. O.** Statistical Methods Applied in Education. New York: Houghton Mifflin Company (1917).

26. **Schorling, Raleigh and Clark, John R.** "A Program of Investigation and Coöperative Experimentation in the Mathematics of the Seventh, Eighth, and Ninth School Years." The Mathematics Teacher, xiv: 264-275 (May, 1921).

27. **Starch, Daniel.** Educational Measurements. New York: Macmilla (1916).

28. **Symonds, Percival M.** "Mathematics as Found in Society: With Curriculum Proposals." The Mathematics Teacher, xiv: 444-450 (December, 1921).

29. **Thorndike, E. L.** An Introduction to the Theory of Mental and Social Measurements. New York: Bureau of Publications, Teachers College (1919).

30. **Thorndike, E. L.** "Measurement in Education." Teachers College Record, xxii: 371-379 (November, 1921).

31. **Thorndike, E. L.** "Changes in the Quality of the Pupils Entering High School." The School Review, xxx: 355-359 (May, 1922).

32. **Thorndike, E. L., and Symonds, P. M.** "The Occupations of High-School Graduates and Non-Graduates." The School Review, xxx: 443-451 (June, 1922).

33. **Thorndike, E. L.** "Instruments for Measuring the Disciplinary Values of Studies." Journal of Educational Research, v: 269-279 (April, 1922).

34. **Touton, Frank C.** "The Frequency of Certain Problem Solving in the High-School Curriculum and a Suggested General Method of Solution." School Science and Mathematics, xxii: 330-343 (April, 1922).

35. **Wilson, G. and Hoke, K. J.** How To Measure. New York: Macmillan (1920).

36. **The Seventeenth Yearbook** of the National Society for the Study of Education, Part II. Bloomington, Illinois: Public School Publishing Company (1918).

CHAPTER XIV

The Training of Teachers of Mathematics

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Foreword

The general plan of this selection of material from a very much longer incomplete report appears on inspection of the table of contents; the method of selection is indicated in the introductory paragraph of Part II, page 447. In the longer report is a detailed statement of: (a) The actual standards for certification of teachers of mathematics in secondary schools for every state in the Union, and for each of its cities of not less than one hundred thousand inhabitants; (b) the courses offered in every college in the country for giving the prospective teacher of mathematics in the secondary school his professional and scientific training. Only a mere fraction of this material is here selected.

In order adequately to picture the quality of recently appointed teachers in the secondary schools of a state, it would be necessary not only to indicate the various methods by which such teachers may have been appointed, but also the percentage of teachers added by each of these methods. Too often do we find reports referring to the manner of appointment involving the highest standards only—as if such were characteristic of the state, whereas not five percent of the teachers may have been so appointed. These percentages are naturally very difficult to obtain, and since they could not be procured for all states to which we refer, they are not included in this report. A complete record of the methods of certification in the ground covered has, however, been attempted.

Discussion of the very extensive literature of the subject is entirely omitted; the original section on Accrediting Agencies (Part III, page 502) is greatly abbreviated; and argument and discussion on the Summaries and on the tentative standard formulated at the end are deleted.

The population figures are those for 1920 and are taken from advance sheets supplied by the Bureau of Census.

I. The Training of Teachers of Mathematics for Secondary Schools in Foreign Countries

In 1912 about 150 reports[1] regarding the teaching of mathematics in seventeen foreign countries were presented to the International

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[1] The International Commission on the Teaching of Mathematics stated in July, 1921 (Enseignement Mathematique), that it had published 187 volumes or fascicules which it listed, containing 310 reports of 13565 pages.

Congress of Mathematicians at Cambridge, England. With these as a basis the writer prepared an extensive report, published by the Bureau of Education, Washington, D. C., in 1918, on *The Training of Teachers of Mathematics for the Secondary Schools of the Countries represented in the International Commission on the Teaching of Mathematics.* For each country considered, rather detailed information and full bibliographies were given with reference to the training of teachers to serve in better secondary schools for boys.

A summary of the material thus set forth is here adapted for an introductory chapter which may be suggestive to those who thoughtfully consider in what way desirable reforms in connection with our secondary schools may be brought about. That the contrast in standards and ideals may be more marked, this summary is placed in immediate contiguity to some indications of corresponding conditions in the United States.

In instituting such a comparison it should be borne in mind that there is no concensus of opinion throughout the world as to the periods to be devoted to primary, or to secondary, education. Most foreign countries agree that twelve or thirteen years normally lead to a university. But while the United States and Australia hold that only four years[2] should be devoted to secondary education, seven countries (Austria, Finland, Hungary, Italy, Japan, Roumania, and Russia) set aside eight years, and three countries (France, with her two extra years for the mathematical or military or naval or engineering specialist, Germany, and Sweden) nine years.

In contrast to eight years of elementary education[2] in the United States, France and Germany consider that the best results are obtained when even the three or four years allotted to primary instruction are given in connection with secondary schools. It is then not surprising to find that there is great difference between the scholastic equipment of students coming from these two general types of school. The graduates of the classe de mathématiques spéciales in France, or of the German Gymnasium are about on a par with the youth who has finished his junior year in one of the better American colleges. And in other countries also, like Denmark, Japan, and Sweden, the graduate of a higher secondary school has covered considerable of the work of which the equivalent is offered in colleges in the United States. It is only in the light of such considerations that the full force of, say, Sweden's requirement of ten or eleven years of preparation before a graduate of a Gymnasium may return as a regularly appointed professor can be adequately appreciated.

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[2] The recent movement towards the establishment of junior high schools tends to reduce the elementary period to six years and to extend the secondary period to six years.

But we are anticipating; let us consider our inquiry in connection with each country in turn.

Australia. (Primary education, about 7 years; secondary education, 4 years.)—The standards for the training of secondary school teachers in Victoria and New South Wales are those to which all states of the Commonwealth are approaching. In the Teachers' College and University at Melbourne, Victoria, the usual course of training lasts four years—three years to obtain the B. A. or B. Sc. degree and one year for special professional training. The professional training includes: (*a*) Attendance at lectures in the university on the theory of education with special reference to the method of teaching the various subjects; and (*b*) 120 hours of teaching under supervision in primary and secondary schools. In New South Wales similar requirements of at least four years of training after leaving the high school are maintained. Professional work is taken up after graduation from the university. This consists in part: (*a*) Of study in philosophy, education, principles of teaching, and school hygiene; (*b*) of continuous practice teaching (8-10 hours a week). In neither Victoria nor New South Wales is it necessary that the teacher shall have had special courses in mathematics in the university. On the other hand the better schools prefer graduates with "honors" in mathematics.

Austria. (Primary education, 4 years; secondary education, 8 years.)—The training required for candidates as teachers in the leading Austrian secondary schools, namely the Gymnasium and the Realschule, is practically the same. There must be not less than three and one-half years of scientific and other preparation (not always good) at a university. After five semesters the preliminary examination in philosophy and pedagogy may be taken. When this is passed, application may be made, not earlier than the end of the seventh semester, for the Lehramtsprüfung (teachers' examination), with its Hausarbeiten (theses) and oral examinations. After this, in theory only one year of practical training at a Mittelschulseminar (secondary school seminary) is required, while a second year is permitted. But in practice very few candidates have had more than a minor part of the training of even a single Probejahr (trial year) before taking up actual teaching.

The written and oral examinations for the Lehramtsprüfung may include mathematics as: (*a*) a major, or (*b*) a minor. (*a*)—For a major, the candidate must be familiar with "general arithmetic," the foundations of higher algebra and theory of numbers and their significance for elementary mathematics, elementary geometry, synthetic and analytic geometry, differential and integral calculus and its applications to geometry, the elements of the calculus of variations and the foundations of the modern theory of functions, and the principal results of investigations concerning the foundations of mathematics. (*b*)—When mathematics is a minor, it is demanded that the candidate shall have: knowledge of elementary arithmetic; insight into the structure of the field of real numbers and into operations with them; knowledge of elementary geometry to the extent of what is taught in secondary schools, and exercises in space perception; accuracy and speed in the solution of simple examples applying the idea of a function to functions coming up in secondary school work; and the elements of differential and integral calculus.

The candidate is given three months to prepare the Hausarbeit on some subject not discussed in the lectures at the university. With mathematics as major recent Hausarbeiten have dealt with such topics as: "The theory of Fourier series"; "Theta functions and their applications in the theory of surfaces of the fourth order"; "Method of derivation of large prime numbers"; and "Algebraic treatment of the twenty-seven lines on a cubic surface." With mathematics as a minor: "Weierstrass's theory of irrational numbers," "Properties of the nine-point circle," and "The theorems of Fermat and Wilson."

The Probejahr includes observation of teaching methods, practice teaching under direction, the preparation of reports and weekly conferences with the staff of the seminar on such matters as teaching, school discipline, pedagogy, school hygiene, and new publications of interest to teachers.

It is deplored that teachers of mathematics in the lower sections of the secondary schools may have had only the scientific training required when mathematics has been taken as a minor.

Belgium. (Primary education, 6 years; secondary education, 7 years.)—The graduating diploma from an athénée royal (the leading government secondary school) or from certain of the collèges entitles a student to enter one of the universities at which all teachers of the athénées are prepared. The inspectors of studies and professors in an athénée and the rector (or head) of a lower secondary school must have secured the doctor's degree from a university and the masters (surveillants) in an athénée must have the university diploma of "candidate." (The four universities include two state establishments, the free university at Brussels and the Roman Catholic University at Louvain.)

During the first two years the student prepares for examinations leading to the certificate as "Candidate in physical sciences and mathematics."[3] The examinations of the first year are on: Analytic geometry (plane and solid), descriptive geometry, higher algebra and the elements of the theory of determinants, differential and integral calculus (first part), analytic statics, experimental and laboratory physics. For the second year: Logic and psychology, moral philosophy, projective geometry, integral calculus (second part), elements of the calculus of variations and calculus of differences, pure kinematics, crystallography, and laboratory exercises.

The first examinations for the doctorate occur at the end of the third year. They are on: Higher analysis, dynamics, general mathematical physics, spherical astronomy and elements of mathematical astronomy, elements of the calculus of probabilities with the theory of least squares.

The subjects of the second set of examinations for the doctorate (in the fourth year) include: Mathematical methodology, elements of the history of the physical sciences and of mathematics, and one of the five following groups chosen at pleasure by the candidate: (*a*) Higher analysis; (*b*) higher geometry; (*c*) analytic mechanics and celestial mechanics; (*d*) mathematical astronomy and geodesy; and (*e*) experimental physics and mathematical physics. The examination in the group chosen is somewhat searching, as the thesis has to do with the development of some part of it. The thesis must be publicly defended. Prospective teachers must also give two public "lessons"

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[3] This certificate very nearly corresponds to the bachelor's degree in this country.

(such as would be given to pupils in an athénée), the subjects for which are designated in advance by the jury, and chosen from the program of the athénées.

The young doctor is qualified to teach in an athénée at once without any professional training. In general, however, it is only after several years in some such minor position as surveillant that he may reach the status of a professor in an athénée. Connected with such a chair in the large cities, the salaries, corresponding pensions, and social position are attractive.

Denmark. (Primary education, 5 years; secondary education, 7 years.)—The university course for the scientific training of a teacher for a secondary school usually ends in about six years, with what is known as the skoleembedsexamen. This examination consists of two parts, covering the major and two minors; if the major is mathematics, the minors are astronomy with applied mathematics, and chemistry with physics. But in addition to these at the end of the first year the candidate must pass the "filosofikum," an examination in logic, psychology, and the elements of the history of philosophy. This entitles him to the degree "candidatus philosophiæ." Having passed the skoleembedsexamen he is "candidatus magisterii." After two years of professional training (a year in the state Pedagogic Seminary and a year as assistant, or regular teacher, in a state or private secondary school), or its equivalent, the candidate is finally examined as to eligibility to teach.

In the first part of the skoleembedsexamen the mathematical subjects include: Plane and solid analytic geometry, theory of infinite series, differential equations with a single independent variable, total and partial linear differential equations in three variables, differential geometry, statics, kinematics, hydrostatics, and advanced portions of gymnasial mathematics from a higher point of view. The second part of the examination with mathematics as a major includes questions on such subjects as theory of functions, elementary theory of numbers, methods of descriptive geometry, synthetic and analytic projective geometry, and history of mathematics.

England. (Primary education, about 6 years; secondary education, about 7 years.)—The amount of mathematics taught in English secondary schools varies from such a program as the elements of arithmetic, algebra, and geometry required for a London University matriculation examination, to programs which include trigonometry, synthetic and analytic geometry, theory of equations, differential and integral calculus, statics and hydrostatics—indeed all subjects necessary to prepare for an entrance scholarship examination in mathematics at Cambridge University, for example. In the mathematical training of the teacher there is corresponding variation. In practically all cases it is necessary for a candidate for a position to be a college graduate. But on the one hand he may have received a "pass" degree, in three years, without any college work in mathematics; and on the other hand he may have spent four years in obtaining a degree with first-class honors in mathematics. As to the extent of the professional training, this does not exceed a year in the training college of what we call the graduate school of a university, but even this is not yet always insisted upon. Athletic abilities are a valuable asset for a man seeking a position.

Finland. (Primary education, about 3 years; secondary education, 8 years.)—The steps required in the preparation of the secondary school teacher of mathematics after leaving the lyceum, the leading secondary school of the country, are: (*a*) To pass the examinations for a "candidate of philosophy," or for the "certificate of aptitude in teaching" after four or five years of preparation in the university; then (*b*) to spend a year of professional training in one of the normal seminaries; and (*c*) to pass an examination in theoretical and practical pedagogy. In each of the examinations, grades "approbatur," "approbatur cum laude," and "laudatur," are assigned.

A professor in a secondary school must have the degree of candidate of philosophy with highest grade in the major subjects required by the post. For the assistant professor, the same degree, or certificate of aptitude, with at least the second highest grade in the major subjects required by the position is necessary.

The candidate of philosophy has had courses in spherical trigonometry, analytic geometry, differential and integral calculus, differential equations, theory of numbers, theory of functions of a complex variable, and occasional courses in such subjects as minimal surfaces, theory of groups, and elliptic functions.

France. (Primary education, 4 years; secondary education, 9 years.)—The highest type of secondary school in France is the lycée. The following remarks are with reference to preparation of teachers of this institution. The tendency of recent legislation is to demand that all teachers shall have passed the examination of the agrégation. The requirements of this examination in respect to knowledge of mathematics and dexterous handling of materials are unparalleled in any other country. Professional training, in comparison with the amount required by Denmark or Prussia, is comparatively neglected, and this has been judged by some as a weakness in the French preparation. It should be borne in mind, however, that it is difficult for an intelligent Frenchman (and the agrégés are the élite of the country's mathematical intellectuals) to be otherwise than interesting, effective, and elegant in his exposition of any subject.

The "bachelier" (as the graduate from a lycée is called) who wishes adequately to prepare himself as a mathematical teacher in a lycée, usually undergoes the strenuous training of the "classe de mathématiques spéciales" in mathematics, physics, chemistry, descriptive geometry, drawing, German, and French. This preparation, covering at least two years, equips him for the university, where the three-year course continues to be of a highly specialized character. The best students pass the examination for the "licence" at the end of the first year, other tests at the end of the second, and present themselves for the terrific "concours" of the agrégation at the end of the third year. A concours is a competitive examination for a number of places fixed in advance. In the case of the agrégation for teachers in the secondary schools the annual number of vacancies to be filled between 1885-1909 never exceeded 14. For these places there were usually about 80 competitors, many of them the survivors of earlier competitive examinations. For example, there is a concours on leaving the classe de mathématiques spéciales for entry into the Ecole Normale Supérieure, the greatest training school in the country for the concours of the agrégation; only about eight percent of the annual number of competitors in science can be admitted to the school. A certain amount of attention is here paid to pedagogy, and to practice teaching, but in general such subjects constitute a meager part of the training of the teacher.

As to the mathematical attainments of the agrégé (one who successfully passes the agrégation), they are equaled by those of only a very small percentage of professors of mathematics in American colleges.

Germany. (Primary education, 3 years; secondary education, 9 years.)—Among the nations of the world, Denmark, France, Sweden, and Germany certainly maintain the highest standards in the selection of teachers for their better secondary schools. And no small part of the remarkable training which these teachers get is from master minds and master teachers in the schools themselves. Germany insists much less than France on brilliancy and breadth of mathematical attainment, but, like Denmark and Sweden, decidedly more on theoretical and practical professional training.

The candidate for teaching in a secondary school in Prussia (which includes about two-thirds of the population of Germany) is examined by a commission composed partly of professors and partly of state officials. Before the commission he must place: (*a*) his Maturitätszeugnis (graduating certificate) from a Gymnasium, Realschule, or Oberrealschule (leading types of secondary schools); (*b*) documents to show that he has spent at least three years in study at one or more German universities (in point of fact candidates usually spend four or five years; for three of the six semesters required the mathematician may have attended a technical school of university grade); (*c*) a statement of the subjects (for example, pure mathematics and physics and applied mathematics) and classes (Sexta to Untersekunda, inclusive, or Obersekunda to Oberprima[4], inclusive) for which the candidate hopes to prove his qualifications; and (*d*) a statement of the field in which he wishes to receive subjects for required essays. Not more than three months, ordinarily six weeks, are allowed to prepare the essays. For one of them some publication of note, such as the candidate's doctor's dissertation, may sometimes be substituted.

The written and oral examinations are in philosophy, pedagogy, religion, German literature, and the candidate's special subjects. For first-grade rank in pure mathematics the candidate must show that he is thoroughly acquainted with the principles of higher geometry, arithmetic, algebra, higher analysis, and analytic mechanics and can solve fairly difficult problems without assistance.

Having passed these examinations "pro facultate docendi", the Schulamtskandidat (prospective teacher) must still undergo two years of severe professional training: (*a*) The Seminarjahr, and (*b*) the Probejahr.

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[4] The years in a German Gymnasium, from the lowest to the highest are, in order, called: Sexta, Quirta, Quarta, Untertertia, Obertertia, Untersekunda, Obersekunda, Unterprima, and Oberprima.

When favorable report has been made on the work of the Probejahr the candidate's name is put on a list of teachers eligible to appointment in the higher schools of the province in which the examination was held.

Several German states follow Prussia's method of training teachers for secondary schools. Others differ in various ways: Bavaria, Saxony, Würtemberg, Baden, and Hesse each require four years at a university; in Würtemberg the Seminarjahr is unnecessary; and Bavaria and Baden demand no Probejahr. There are also differences in details, even when general requirements are the same. But throughout the nation the attractions of the teacher's position are sufficiently strong to draw some of the best mathematical talent of the counry.

Hungary. (Primary education, 4 years; secondary education, 8 years.)—The method of training secondary school teachers in Hungary is more adequate than that in Austria. Recruits for such training are drawn from among the graduates of the Gymnasia and Realschulen. The elaborate and searching training is carried on during five years at a university, or in the Polytechnikum, and at a training college. In starting out, the candidate must elect two subjects which he wishes to teach; the selection of a third is recommended. Mathematics, physics, and descriptive geometry are not uncommonly chosen together.

The examinations in the course are threefold: (*a*) The general examination, at the end of the fourth semester; (*b*) the examination on special subjects, at the end of the eighth semester; and (*c*) the pedagogic examination at the end of the next year, which the candidate usually employs in professional practice.

In the general examination the oral part contains questions in plane and spherical trigonometry, analytic geometry, analysis, descriptive geometry, physics, the Hungarian language, and one modern language (English, French, German, or Italian) with its literature.

Five months before the examination in special subjects the candidate is given, in each of his subjects, a topic upon which he has to write a dissertation. These dissertations must indicate originality of treatment, familiarity with the literature, and broad knowledge of the topics. In the oral examinations on special subjects the candidate must show decided proficiency. The oral examination in mathematics is on: (*a*) The mathematics of secondary schools; (*b*) certain parts of geometry, algebra, and analysis common for all candidates; and (*c*) the following five subjects, one of which must be known thoroughly, the others, in a general way: (i) Modern geometry and the theory of algebraic forms; (ii) theory of numbers and higher algebra; (iii) general theory of skew curves and surfaces; (iv) general theory of analytic and elliptic functions; and (v) advanced portions of integral calculus.

The examination in pedagogy consists of a dissertation, and of an oral examination in history of philosophy, logic, psychology, pedagogy, history of pedagogy, and special methods for teaching mathematics. While this examination occurs at the end of the fifth year, most of the courses of study have been taken during the first four years at the university. The fifth year is largely devoted to training and practice teaching in a seminary, such as Kármán's, or the state training college. Teachers do not attain to full service until after three years of probationary activity.

Italy. (Primary education, 4 years; secondary education, 8 years.) —A candidate for a position as teacher of mathematics in a secondary school must possess the degree of "Dottore in matematiche pure" from one of the universities. After the candidate has graduated from a liceo or an istituto (two of the leading types of higher Italian secondary schools) four, sometimes five, years are devoted to securing this degree. With an exception to be referred to presently, the courses of study are wholly in pure and applied mathematics and allied fields. An idea of the breadth of some of these courses may be obtained by recalling that the following works consist mainly of university lectures: F. Severi's *Lezione di geometria algebraica,* 1908; E. Ciani's *Lezione di geometria projettiva ed analitica,* 1912; and G. Castelnuovo's *Lezione di geometria analitica,* third edition, 1913.

The first two years of study lead to the certificate "licenze dell' universitá." During the last two, or three, years the student (*a*) carries five advanced courses (theoretical geodesy being put on a plane with higher analysis); (*b*) writes a dissertation; and, if he wishes to occupy an advantageous position with respect to appointments, (*c*) attends lectures on methods of teaching leading to the "diploma di magistero" (these lectures are given in the scuola di magistero of the university). The final examinations of the university lead to the degree of laureate, of which there are various grades. The dissertations present great variation in value and originality. The candidate is required to publish a full resumé of his results, but not the thesis itself. The better ones find ready acceptance by the mathematical journals or transactions of learned societies.

The first appointment of a teacher in a secondary school is for three years, after which the appointment may be made permanent if the inspector's report is favorable; if not, the appointment is extended for another year, when the teacher is definitely appointed or dropped.

In the preparation of the teacher the lack of professional training is regarded by many as serious; his scholastic equipment in connection with his special subject is probably equal to that of the German teacher in secondary schools.

Japan. (Primary education, 6 years; secondary education, 8 years.) —The usual method of preparation as a teacher in a middle (lower secondary) school is: (*a*) To take the course of four years in one of the higher normal schools whose graduates are granted teachers' licenses; or (*b*) to attend some similar institution and pass the four grades of examinations set by the department of education for the licence. A "gakushi" (graduate) of the faculties of science in the universities of Tokyo and Kyoto, who has no professional training, is entitled to receive a licence without examination. Similar privileges are accorded to graduates of foreign universities.

The courses in the higher normal school which prepare teachers ot mathematics and physics include professional training, teachers' courses in algebra and geometry, higher algebra, plane and solid analytic geometry, and differential and integral calculus. The four grades of examination in mathematics set by the department of education are in: (*a*) Arithmetic, algebra, and geometry; (*b*) trigonometry; (*c*) analytic geometry; and (d) differential and integral calculus.

Teachers in the eight higher middle schools, which are closely allied to the universities in their organization, are nearly all gakushi in mathematics. At the university the gakushi have passed examinations in calculus and differential equations, solid analytic geometry, projective geometry, astronomy and least squares, general physics, general dynamics, and theory of numbers.

Netherlands. (Primary education, 6 years; secondary education, 6 years.)—Gymnasia (six-year) prepare students to enter directly the various departments of the universities, but graduates of higher burgher schools (five-year) must pass university examinations in

Latin and Greek. The first examination after entering the university is the "examination for candidature," at the end of about three years; the second examination, that for the doctorate, occurs at the end of about six years (often more) of study. If, in addition to passing the latter examination, the candidate prepares for the faculty an appropriate thesis, he is entitled to the degree of doctor. It is from among those who have passed the examination for the doctorate that teachers for gymnasia and higher burgher schools are almost wholly recruited. They have had no professional training.

A candidate for appointment as teacher in a three-year or five-year higher burgher school may also prepare himself for the position by passing a series of exceedingly elaborate examinations, the preparation for which is made entirely outside of the universities. For the five-year school, examinations on the following subjects, among others, must be passed: Plane and spherical trigonometry, descriptive geometry, analytic geometry, calculus, theoretical and applied mechanics, physics, chemistry, cosmography, geology, and mineralogy. During the past fifty years exceedingly few have qualified themselves as teachers in this way.

Roumania. (Primary education, 4 years; secondary education, 8 years.)—Graduating from the lycée when about 19 years of age, the student prepares himself to be a secondary school teacher by study for: (*a*) The diploma, "licence ès sciences mathématiques" of a university; and for (*b*) the certificate of a pedagogic seminary. The examinations for the licence are in higher algebra, analytic geometry, descriptive geometry, differential and integral calculus, theory of functions, mechanics, and astronomy. It takes at least three years to prepare for these examinations. Professional training is obtained at a pedagogic seminary.

A candidate for the position of teacher in a secondary school must pass an "examination of capacity," held once in three years. To be admitted to this examination the student must possess the diploma and certificate mentioned above. The examination consists of: (*a*) Three written examinations on the elementary material of secondary education and on the more advanced material of the licence; (*b*) two oral examinations, one on mathematical questions proposed by a jury, the other on pedagogy; and (*c*) two practical examinations—that is, two lessons such as the candidate might deliver to pupils in a lycée.

Russia. (Primary education, 3 years; secondary education, 8 years.)—Graduates of the Gymnasium and of the seven-year "real school" are entitled to register in faculties of the universities. The course for teachers in the Gymnasia, as well as for most teachers in the real schools, occupies four years and leads to a diploma of the first or second grade.[5] The courses for the mathematics-analytic mechanics group at the University of Petrograd include spherical trigonometry, analytic geometry, calculus, partial differential equations, theory of numbers, theoretical mechanics, hydrostatics and hydrodynamics, and astronomy and celestial mechanics.

It was not till recently that any provision was made to give professional training to candidates for teaching positions in state secondary schools. This beginning was made in 1909, when a year course including lectures in logic, psychology, pedagogy and its history, as well as practice teaching, was organized for the district of Petrograd.

Spain. (Primary education, 4 years; secondary education, 6 years.)—The "bachiller en artes" (graduate) of an instituto (the chief Spanish secondary school) may prepare himself to teach mathematics in an instituto at any one of the three universities; Barcelona, Madrid, or Zaragoza. The courses in the faculties of science lead to the "licenciatura" certificate in about four years; the apt scientific investigator may procure the doctorate in another year. Every candidate for a professorship in the institute must have a licenciatura certificate; the best positions are obtained by the doctors in mathematical sciences.

The mathematical courses leading to examinations for the licenciatura are as follows: Mathematical analysis, metric geometry, analytic geometry, infinitesimal calculus, cosmography and physical geography, projective geometry, descriptive geometry, rational mechanics, and spherical astronomy and geodesy. (This list of subjects strongly reminds one of those required for the licenza in Italy.) The examinations take on different forms and imply a certain amount of professional training.

To prepare for the doctorate the student must (a) attend courses in higher analysis, higher geometry, astronomy of the planetary system, and mathematical physics; (b) present a memoir on a subject selected by the candidate; and (*c*) successfully defend the memoir before a tribunal.

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[5] This is not a degree; the degrees of magister and doctor require longer preparation.

Sweden. (Primary education, 3 years; secondary education, 9 years.)—The very thorough course of the Gymnasium concludes with the Studentsexamen (graduating examinations), the sole means by which a student may enter one of the universities. Mathematics is there taught in the faculty of philosophy. The courses for secondary-school teachers are very elaborate and lead to: (*a*) The examination called "Filosofisk ämbetsexamen," at the end of about four years of study; and (*b*) the examination "Filosofie Licentiatexamen," after about eight years of preparation. Candidates for positions as teachers in the Realskolor (secondary school, first five years) are required to pass the first of these examinations; those seeking higher posts in the Gymnasia (secondary school, last four years), the second; but in this latter case the candidate must also defend a thesis for the degree of doctor.

In each case the candidate has to spend a probationary year in connection with some one of five special schools which organize excellent professional training. Moreover, before regular appointment as professor or assistant professor in the Government secondary schools, the candidate must have spent at least two years in successfully performing the duties attached to the position. It is probably no exaggeration to state that Sweden leads the world in the extent of scientific preparation required for her professors in the Gymnasia.

The subjects of the Ambetsexamen include: Projective geometry, theory of equations, calculus, analytic geometry, theory of numbers, theory of probabilities, ordinary differential equations with constant coefficients; and, generally, for the highest predicate, elements of the theory of differential equations, analytic functions, and differential geometry. In the preparation for the Licentiatexamen the candidate is introduced into various important fields of modern mathematics, including ordinary and partial differential equations of mathematical physics.

Switzerland. (Primary education, 6-4 years; secondary education, 7-9 years.)—Considering the country as a whole, there are no fixed examinations for secondary-school teachers; nor are there any colleges for training them, although there is a department of pedagogy in the University of Geneva. For most of the teaching positions in the secondary schools the majority of the cantons demand that the candidate shall have completed a university course, or its equivalent, of four or five years. Typical of such cantons are Fribourg and Geneva. Candidates with two years' training in a university, or its equivalent, can usually only be appointed as teachers in the lower classes of the better schools; an exception requiring the qualifying word is Canton Vaud. There is a general tendency to favor candidates who have had technical training such as a student derives from an assistantship in the Federal Polytechnic School.

On the whole, high scientific standards of training are demanded on the part of secondary teachers, but their professional training is almost entirely neglected.

Subjects included in examination requirements for secondary school teachers in courses of the Polytechnic School and the University of Bern are of the following types: Descriptive geometry, calculus, analytic geometry of three dimensions, projective geometry, theory of functions, Fourier's series and integrals, differential geometry, and theory of transformation groups.

SUMMARY

These very brief sketches of the conditions in foreign countries can suggest only in imperfect outline some outstanding features as to the manner in which prospective teachers of mathematics in the secondary schools are trained. After leaving the secondary schools the training is derived, broadly speaking, from two sources: I. Courses in a university or a similar higher institution; and II. Professional training.

I. All the countries require some university training on the part of candidates for appointment as secondary school teachers.[6] The maximum requirements are in Denmark and Netherlands, each 6 years, and in Sweden, about 8 years. On the other hand, for minor positions in the athénées of Belgium and regular positions in Canton Vaud, Switzerland (where most Cantons require 4 or 5 years), only 2 or 2½ years of attendance at a university are compulsory. The complete record is as follows: Australia (Victoria and New South Wales), 3 years; Austria, 3½ to 4; Belgium, 4 to 5, 2 for minor positions in athénées; Denmark, 6; England, 3 to 4; Finland, 4 to 5; France, not less than 3, in addition to 2 years in classes de mathématiques spéciales; Germany, 3 to 4, but rarely 3 and often 5 are taken; Hungary, 4; Italy, 4 to 5; Spain, 5, for lower positions, 4; Sweden, usually 8; Switzerland, 4 to 5 years for the most part, in one or two cases 2 to 3.

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[6] So far as this statement concerns Japan, reference is made to the higher middle schools.

Let us consider an example to bring out more clearly the implications of these statements. Since the future mathematical teacher entering a German university is about on a scholastic par with the student who has finished the junior year at a college in the United States, we may state, roughly, that the German teacher has generally had at least three years more of scientific training than the prospective teacher going out as a college graduate in the United States. Moreover, in contrast, his training has, as a rule, been based upon a secondary education relentlessly thorough, and conducted by those who are masters of their profession; in the university his line of work has been very specialized. Such specialization is characteristic in the case of most of the foreign countries considered above in connection with the preparation of their teachers for secondary schools.

The lists of courses taken by such students indicate how the majority of the universities emphasize instruction in such subjects as plane and solid analytic geometry, calculus, functions of a real variable, differential equations, differential geometry, descriptive geometry, mechanics, and physics. Projective geometry is prominent in such countries as Belgium, Denmark, France, Italy and Spain; various topics of higher algebra in Hungary, Italy, the Netherlands, Roumania and Russia; foundations of mathematics in Austria, Belgium, Japan, and Spain; and history of mathematics in Belgium and Denmark.

It is to be borne in mind that mere names of courses mentioned in connection with different countries do not, without much elaboration, convey a very definite idea. Calculus as a requirement for the prospective teacher in Japan is vastly different from that in Belgium where the equivalent of de la Vallée Poussin's treatise on analysis may be the basis for part of the first two years of work in a university. What is given in our first and second courses of calculus in American colleges is covered in special secondary courses of France, so that the student at a university starts in with courses in analysis and differential geometry such as are outlined in the first volume and part of the second of Goursat's treatise on analysis.

II. In addition to attendance at universities, some countries require professional training. Australia (Victoria and New South

Wales), England (generally), Finland, Roumania, and some states of Germany each require one year (it is only in theory that Austria requires a year); other states of Germany and Denmark require two years each; in addition to a year in a seminary, Sweden requires two years of probation as teacher before regular appointment and in a similar way Hungary requires three; and in Italy after four years of trial a teacher may be dropped. In France the professional training may possibly be estimated at half a year. In six countries no professional training is made compulsory. These countries are: Belgium, Japan (in higher middle schools), the Netherlands, Russia, Spain, and Switzerland. Details regarding the nature of professional training in different countries, and as to its comparative efficiency, may be found in the report referred to in the first paragraph of this section, page 431.

II. The Training of Teachers of Mathematics for Secondary Schools in the United States

Having set forth the general standards for the appointment of teachers of mathematics in the better secondary schools of seventeen foreign countries, the higher standards in various sections of the United States will now be considered. The states of the Union with the lowest standards (which are indeed deplorable) will not here be divulged. The selection of states and cities for consideration has been made from each of the educational "Divisions" employed by the Bureau of Education at Washington, D. C. These Divisions, and the states which they embrace, are as follows:

A. North Atlantic Division: Connecticut, Maine, Massachusetts, New Hampshire, New Jersey, New York, Pennsylvania, Rhode Island, Vermont;

B. North Central Division: Illinois, Indiana, Iowa, Kansas, Michigan, Minnesota, Missouri, Nebraska, North Dakota, Ohio, South Dakota, Wisconsin;

C. South Atlantic Division: Delaware, District of Columbia, Florida, Georgia, Maryland, North Carolina, South Carolina, Virginia, West Virginia;

D. South Central Division: Alabama, Arkansas, Kentucky, Louisana, Mississippi, Oklahoma, Tennessee, Texas;

E. Western Division: Arizona, California, Colorado, Idaho, Montana, Nevada, New Mexico, Oregon, Utah, Washington, Wyoming.

A. North Atlantic Division

(Connecticut, Maine, Massachusetts, New Hampshire, New Jersey, New York, Pennsylvania, Rhode Island, Vermont)

In this Division we select for consideration the state of Pennsylvania, the city of Cambridge, Mass., and facilities at Harvard and Columbia Universities for training prospective teachers of mathematics for secondary schools. We shall see that these universities offer also courses valuable for secondary school teachers who have already taught.

The section concludes with a note on practice teaching at Brown University.

PENNSYLVANIA

(Population, 8,720,159; area 45,126 square miles.)

State Requirements. The prospective teacher of mathematics in a high school of this state must obtain either a "college provisional certificate" or a "college permanent certificate." The first of these certificates entitles the holder to teach for three years the subjects prescribed for a public high school of the third class[7] or to teach in any public high school of the commonwealth the subjects indicated on its face. The applicant for this certificate must be a graduate of an approved college or university and must have successfully completed at least 200 hours of work of college grade in education. Each subject mentioned on the face of the certificate is one in which the candidate has had in college at least twelve semester hours including professional training in the teaching of the subject. An applicant for a "permanent college certificate" must not only have met the same scholastic requirements as above, but also must have taught successfully for three years in the public schools of Pennsylvania. Evidence of successful experience includes reports on: (a) Personality (appearance, voice, power, character, coöperation); (b) preparation

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[7] There are four classes of high schools in Pennsylvania. A high school of the first class maintains four or more year courses of study, of not less than nine months in each year, and must employ at least three properly qualified teachers during the entire term. A high school of the second class maintains a three year course of study, of not less than eight months in each year, and must employ at least two properly qualified teachers during the entire term. A high school of the third class maintains only two years of study, of the fourth class one year, for graduation.

(command of English, specific knowledge, general scholarship, professional equipment); (c) technique (room conditions, selection and organization of subject matter, resourcefulness, motivation, conclusiveness); and (d) pupil reaction (efficient functioning of habits and skills, command of subject matter, thinking ability, expression, tastes, appreciations and ideals).

There are thirteen state normal schools with high school graduation for matriculation and regular courses of two years. Of such schools graduates, who have taken also six semester hours of college work, may teach in ninth grades.

In the early part of 1921 the General Assembly of Pennsylvania enacted certain notable educational measures including some which are of special interest for our inquiry. A brief summary of these is as follows:

(a) There is established a State Council of Education to consist of nine business and professional men and women, the Superintendent of Public Instruction being the chief executive officer and the president of the State Council of Education.

(b) After September 1, 1927, no person may be employed in any school in the state who does not show graduation from an approved college or university, or an approved state normal school, or who does not furnish evidence of equivalent education. (Provision was made, in 1921, for summer courses of nine weeks each in the thirteen state normal schools and in twelve of the colleges and universities of the state to enable those now in the service to obtain such further education and training as may be required to meet the standards set for 1927.)

(c) The certification of teachers is transferred from county and city superintendents to the State Department of Public Instruction.

(d) There is established a state wide salary schedule with increments for teachers, but making salaries and increments contingent upon qualifications and efficient service. (In high schools of the first class the minimum salary for a high school teacher is $1800 and there are 8 annual increments of $175. It is explicitly stated, however, that "high school teachers of exceptional achievement in scholarship and in teaching experience, to be determined by regulations prescribed by the State Board of Education and such additional regulations as the local board of education may prescribe, shall be advanced from $3200 to at least $3600.")

(e) The length of the minimum school term in the state is increased one month.

(f) Sufficient funds are provided for the proper organization of the State Department of Public Instruction. (This organization includes the appointment of directors of special subjects. For example, Mr. J. A. Foberg, the director of mathematics, has charge of all the mathematics taught in the public schools of the state.)

Philadelphia. In the largest city, *Philadelphia* (population, 1920, 1,823,158), a candidate as teacher of mathematics in a high school must pass examinations as follows: (*a*) Oral; (*b*) on methods of teaching and management; (*c*) on algebra, plane and solid geometry, plane and spherical trigonometry, analytic geometry, and calculus. The names of those who receive an average of 70% or over are placed on the eligible list. To have been admitted to the examinations, the candidate must have presented proof of graduation from an approved college, or of equivalent education.[8]

Colleges and Universities. Since the prospective teacher in the high school must be a graduate of a college or university, it is of interest to note than in Pennsylvania there are 43 institutions of this kind—more than in any other state of the Union. Of these, the following offer no course in the theory of education and no course in mathematics, primarily for prospective teachers: Carnegie Institute of Technology, Drexel Institute, Dropsie College, Haverford College, Pennsylvania Military College, St. Joseph's College, and St. Vincent College and Ecclesiastical Seminary. The remaining 36 colleges and universities—10 are for men only (one colored), 7 for women only and 19 are co-educational—offer courses in the theory of education and in mathematics through the calculus. Twelve provided in recent years special courses, in mathematics and the teaching of mathematics, for prospective teachers; 11 gave courses in descriptive geometry, 8 in projective geometry, and 9 in the history of mathematics. Six gave in summer schools regular courses in mathematics for secondary schools, and 2 offered special courses in mathematics for teachers in such schools.

The three institutions, at which are to be found the most elaborate facilities in the state for the training of prospective teachers, are the University of Pennsylvania, Pennsylvania State College, and the University of Pittsburgh. A detailed statement is given concerning the first of these.

UNIVERSITY OF PENNSYLVANIA[9]

The School of Education was established in 1914 and the requirements of admission to the School are the same as those for the College.

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[8] This, and more, is of course set forth in the *state* requirements for its high school teachers.

[9] Authorities: Bulletin: (a) Catalogue 1920-1921, February 15, 1920; (b) The School of Education, Announcement for 1920-1921, November 13, 1920: (c) Graduate School, 1920-21. September 15, 1920; (d) The College, College courses for Teachers, 1920-1921, May 15, 1920; (e) The College, Summer School Courses, July 5-August 13, 1921, March 17, 1921.

In the four year course, leading to the degree of Bachelor of Science in Education, the following subjects are required of all students:

English, Psychology, and a *Foreign Language* (each 12 semester hours); *Introduction to education* (2 semester hours, first year.—To make students realize the importance of their professional studies, to familiarize them with educational terms and to guide them in the proper selection of their college courses); *Introduction to mathematical statistics* (3 semester hours.—Curve plotting, frequency curves, averages, form of the distribution, normal probability curve, theory of correlation); *Zoology* or *botany,* and *History* (each 6 semester hours); *Social institutions and organizations, History of education, General educational principles,* and *School administration and secondary education* (each, 4 semester hours); *Physical education and hygiene* and *Observation and supervised teaching* (each, 8 semester hours.—The observation is carried on through the coöperation of local schools and is supervised by experts. Students are required to spend five school periods a week, for one term in the public schools).

These required courses occupy 42½ year hours or units. Of the remaining 23½ required for the degree, 13½ are free electives and 10 are to be devoted to specialization in one or two subjects. When a student specializes in mathematics only, the number of units which may be required by the department of mathematics is 12, this amount including the regular required course in mathematical statistics. In addition to these 12 units, the student takes 2 units in *Methods and practice in teaching mathematics,* given by the department of education. This course considers (*a*) the specific educational values of mathematics, (*b*) the selection and organization of subject matter, and (*c*) the methods of presentation which will best realize these values.

Six additional units are required either in mathematics or in a related subject. Twenty units are, therefore, required for each student who specializes in mathematics only. When a student specializes in two subjects, one of which is mathematics, he is required to take 9 units in mathematics and 2 units in the course in Methods and practice in teaching mathematics referred to above.

Eighteen courses are given by the department of mathematics in the School of Education. Students majoring in mathematics are advised by the department to arrange their work in the first two years so that they may begin the study of the calculus not later than the beginning of the junior year. *College algebra* (1½ units), *analytic geometry*

(1½ units) and *calculus* (3 units) are always included among courses taken by those who specialize in mathematics. Other courses are:

(a) *Synthetic geometry* (1½ units—Continuation of elementary geometry both plane and solid; study of methods of proof; discussion of originals; modern developments of geometry of the circle and triangle).
(b) *Foundations of geometry* (1½ units—A considerable degree of mathematical maturity is required for admission to this course).
(c) *A review of elementary mathematics* (given in 1921 for the first time, and to be taken only as a free elective.) The course was introduced not only to meet the needs of students who are called upon to pass competitive or classifying examinations before beginning their teaching, but also for the specific purpose of giving a comprehensive view of the important principles of elementary mathematics up to and including subjects which are usually treated in a first course in the calculus.
(d) *History of mathematics* (1½ units—Course in calculus prerequisite).
(e) *The teaching of mathematics* (1 unit—In exceptional cases a student may take this course at the same time as the calculus).
(f) *Mathematical theory of errors* (1½ units)—Discussion of the theory of probability, the Gaussian and other laws of error, adjustment of observations and derivation of empirical formulas. Introduction to mathematical statistics, a prerequisite.

The courses in mathematics offered at the University of Pennsylvania and preparing teachers for secondary schools are given in the following named schools or departments: A. The School of Education (open to men and women, work in which has been described above); B. The College, College Courses for Teachers (open to men and women); C. The College (open to men only); D. The Graduate School (open to men and women); E. The Towne Scientific School (open to men only).

B. The College Courses for Teachers are intended primarily for those engaged in teaching. The classes meet in the afternoons and evenings and on Saturdays. The instruction corresponds in every respect to that given in the regular college course. In no year are more than five courses in mathematics offered; for example: solid geometry (1½ units), plane trigonometry (1½ units), advanced algebra (1½ units), analytic geometry (1½ units), and calculus (3 units). Any student in the College Courses for Teachers who has completed a first course in calculus may be admitted to certain courses in the Graduate School. For several years the College has been giving regular courses at a number of local centers during the academic year. Courses in college algebra and analytic geometry have been given regularly at the West Chester Normal School. These courses have been taken not only by those who intended later to pursue regular college work, but also by those who were to begin their actual teaching upon graduating from the normal school.

C. All students entering the College are required to proceed to their degrees under the provisions of a curriculum which includes six units in any one, or three units in each of any two, of the three subjects mathematics, physics, and chemistry. A course in plane trigonometry is a prerequisite for the work in physics. Before the end of the second year in College each student must select a major subject in which he is required to complete nine units of work before graduation. Mathematics is the major subject for all students preparing to teach that subject. No course in *methods of teaching* is given in the College, where twenty courses in mathematics are regularly taught. They include a course in *Projective geometry*, and courses in *History of mathematics* and *Foundations of geometry* as listed above in connection with the School of Education. Five courses in education are also offered in the College, namely: *The beginnings of educational thinking, General educational principles, History of educational practices, Elements of school administration and management* and *The secondary school.*

D. The students who pursue work in mathematics in the Graduate School are, as a rule, either teachers of mathematics in colleges or high schools, or those who are preparing to teach mathematics in such schools. Courses in *History of mathematics* and *Synthetic projective geometry* (second course) are the only ones which need be named.

E. Graduates of the Towne Scientific School, requirements for admission to which are the same as those to the College, are sometimes preferred as teachers of mathematics in high schools where emphasis is placed more on applied than on pure mathematics. Thirteen courses are given by the department of mathematics in this School. There are also courses in descriptive geometry.

In the Summer School, are offered many courses in education and in mathematics suitable for the prospective teacher in the high school. In education such courses are: *Educational measurements, Observation* (in the Demonstration High School), and *The teaching of mathematics in secondary schools;* in mathematics: *Solid geometry, plane trigonometry, college algebra, analytic geometry, differential calculus, and integral calculus.*

CAMBRIDGE, MASS.

(Population, 109,694).

To be eligible for examination and to be rated in experience, an applicant for a high school certificate must have: (a) A degree from an approved college or university, with evidence of professional training, or conclusive evidence of special fitness; and (b) three years of successful experience, for non-residents, in teaching high school studies; two years, for residents.

The Board of Examiners holds examinations annually. Applicants for teachers' certificates are rated by the Board under three heads: scholarship, experience, and personal fitness. In order to secure a certificate an applicant must be rated 75% or higher in each of these.

A rating in scholarship is obtained in one of the following ways: (a) By averaging the marks if the applicant is required to take a written examination; (b) in case the applicant is not required to take a written examination, by averaging the marks received at college, university, certified technical high school, state normal school or Boston normal school, or the marks given in a teacher's certificate granted by the Board of Examiners of any city or any state in the United States and which may be accepted by the Board of Examiners of Cambridge. Rating in personal fitness is determined after an oral examination of the applicant and the consideration of the evidence relative to his character, previous teaching experience, and the medical examiner's report relative to health.

Candidates as teachers of mathematics are required not only to have specialized in this subject in college but also to have had experience in teaching it.

HARVARD UNIVERSITY[10]

The following semester courses are offered by the faculty of arts and sciences to both undergraduates and graduates: *Principles and problems of modern education*—contemporary educational conditions (introductory course), *Public school administration in state and city*—contemporary tendencies and problems, *History of education*, and *Educational psychology and mental hygiene.* Primarily for graduates are the semester courses: *Principles of secondary education* (lectures, prescribed reading, reports, discussions, and a thesis; school visits and observation) and *Teaching* for which the previous course is prerequisite (teaching under supervision in the schools of Newton, Brookline, Arlington, Medford, Somerville, Belmont, Watertown, and Cambridge; lectures, exercises and discussions on problems of class-room practice).

In the Graduate School of Education, among the various curricula offered, are: Curricula for teachers in secondary schools including junior high schools, and curricula for principals of secondary schools.

Prospective secondary-school teachers, as yet without experience in teaching are expected to take the following semester courses:

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[10] Authorities: Official Register of Harvard University (a) Graduate School of Education, 1920-21, May 17, 1920; (b) Division of Mathematics, 1920-21, June 7, 1920; (c) Announcement of Courses of Instruction offered by the Faculty of Arts and Sciences, 1920-21, September 20, 1920; (d) Announcement of the Summer School of Arts and Sciences and of Education July 5-August 13, 1921, March 10, 1921.

(a) *Principles of secondary education.*

(b) *Principles of teaching in the secondary school,* designed to provide a systematic training in the principles of teaching in junior and senior high schools.

(c) *Apprentice teaching in the secondary school.* Each student is assigned to a post in a chosen school where he first observes and gradually takes over responsibility for various phases of the class work as his experience and development justifies. Thus the student gains practical experience, under guidance, in the preparation of materials of instruction, in the management of a class, in the conduct of study, in the control of recitations, and in other phases of class teaching and management. That experience he interprets in conference with the instructors and teachers and in class meetings of course (b).

(d) Such other courses either in the Graduate School of Education or in the Graduate School of Arts and Sciences as may be approved. No student will receive approval as qualified for secondary school teaching whose program of professional preparation does not conform to these requirements.

Teachers already in service or having had experience in secondary-school teaching will be expected to take course (a) above, before entering any advanced course in the field of secondary education. Other courses open to such students are:

(b) As above.

(e) *Secondary school administration.*

(f) *The junior high school and plans for the reorganization of secondary education.*

(g) *Problems of secondary education.* This year course is intended for advanced students and is primarily a course in investigation. It is limited to fifteen graduate students who have pursued other courses in secondary education and have had experience in teaching or administration in the secondary school.

(h) *Tests and scales for the measurement of secondary-school instruction.* A year course designed as a laboratory course for the development of standard tests and scales for the measurement of instruction in the junior and senior high schools.

There are also: A year course in *Individual research in secondary education* (offering to advanced students an opportunity to pursue independent investigations under the guidance of the instructor), and various courses in educational statistics and educational measurements.

The regular courses include: *solid geometry, logarithms, plane and spherical trigonometry, higher algebra, calculus, introduction to modern geometry, descriptive geometry,* and *the subject matter of elementary mathematics.* The object of the last named course is to make a careful study, from the higher mathematical point of view, of certain fundamental topics in elementary mathematics. Special stress is laid upon the study of various sorts of numbers, integers, fractions, quadratic surds, general irrationals, and complex numbers, and of their different applications to algebra and geometry. The course is especially suitable for students who expect to teach elementary mathematics in school or college, but is open only to those who have passed the regular year course in calculus and analytic geometry.

Among the courses offered in the summer schools are the following: *General principles of education, The principles of educational psychology and mental hygiene, The measurement of intelligence, The history of education, Principles of school administration, Principles of secondary education, Secondary school administration, The junior high school or intermediate school* (for teachers in junior high schools, for those intending to teach in such schools and for principals and superintendents in charge of their administration), *The teaching of mathematics* (lectures, readings, discussions, and reports), *Mathematics in the junior high school, with demonstrations of work with junior high school pupils* (lectures, readings, discussions, and reports). There are also courses in *trigonometry, analytic geometry* and *calculus*.

COLUMBIA UNIVERSITY[11].

In Teachers College, the following semester courses are offered:

(a) *The teaching of algebra in secondary schools* (three hours). It is the purpose of this course to present the best modern practice in America and abroad in the teaching of algebra. The course includes a discussion of the present problems relating to the position of algebra in the curriculum, the reorganization of its subject-matter, the sequence of topics, the aims in teaching, the reasonable chances of improving the curriculum in the immediate future, a consideration of the investigations that have been conducted on the teaching of algebra, and the modern tests for measuring progress and attainments in this subject. Attention is given to the presentation of typical methods of recitation and to the applications of algebra in various lines.

(b) *The teaching of geometry in secondary schools* (three hours). This course aims to present the best modern practice in America and abroad in the teaching of geometry. It includes such topics as the introduction to geometry, geometric drawing, applications of geometry in life about us, the foundations of geometry, the nature and types of geometric reasoning, and its relation to reasoning in other fields, systematic methods of attacking exercises, typical methods of presenting geometry, the current tendencies in subject-matter, the fusion of algebra and geometry, recent experiments to improve the teaching of geometry, modern tests for measuring the results of geometric study, the use of models and measuring instruments, the teaching of trigonometry, the high school mathematics library, and the mathematics club. The adaptation of geometric subject-matter to the needs of the junior high school is also discussed. In connection with the work in methods there is opportunity for observation of actual teaching in high school geometry, especially in classes where the newer methods are being tried.

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[11] Authorities: Columbia University, Bulletin of Information: (a) Barnard College Announcement, 1920-1921, February 28, 1920; (b) Columbia College Announcement, 1921-1922, March 12, 1921; (c) Division of Mathematical and Physical Sciences. . . Announcement 1921-1922, April 2, 1921; (d) Teachers College Bulletin, School of Education Announcement, 1921-1922, March 12, 1921; (e) Summer Session Announcement. . . , 1921, February 5, 1921.

(c) *Advanced course in the teaching and supervision of mathematics* (four hours). Discussion of special problems of mathematical education, such as the theories of teaching the science, the development and present status of the methods of attacking problems considered in the light of modern mathematics, the various methods of treating certain topics in the secondary school and in the college, and the work of the International Commission on the Teaching of Mathematics and of the National Committee on Mathematical Requirements. The work is largely individual, after the manner of a seminar. Special attention will be given to a study of the junior high school, including the question of required and elective mathematics.

(d) *Advanced course. Experimentation in the teaching of mathematics in secondary schools* (four hours). The movements to reorganize the teaching of mathematics have resulted in a certain amount of experimentation to improve curricula and methods of teaching in both the junior and senior high schools, the results of which will be studied and discussed in this course. In addition, there will be an opportunity to observe significant experiments in progress in schools of New York City and vicinity, including the Horace Mann and Lincoln Schools. New lines of experimentation will be considered, and the students will be expected either to conduct experiments themselves in such classes as may be available, or to present detailed outlines of a new experiment. The modern testing movement and its bearing upon experimentation will be included.

(e) *History of mathematics* (two semesters, three hours each). This course is designed to give a general view of the historical development of the elementary branches of mathematics—arithmetic, algebra, synthetic and analytic geometry, trigonometry, and the differential and integral calculus—from the earliest times to the present. The rise and growth of the higher mathematics, chiefly in the nineteenth century, are also considered briefly. The course is recommended especially to those who expect to teach mathematics, showing, as it does, the science in evolution and tracing the causes that have led to its development or stagnation in various epochs. This course is so arranged that either half may be taken. Students wishing to do more intensive work in the history of mathematics should register for (f) instead of (e).

(f) *Practicum in the history of mathematics* (two semesters, four hours each). This is course (e), together with two extra hours per week. It is a research course open only to those who desire to study exhaustively the history of one or more special topics. The work will consist chiefly of the critical study of early treatises which are not accessible in English, and therefore should not be undertaken without a fair knowledge of two foreign languages. This course is so arranged that either half may be taken.

(g) *Seminar in the history of mathematics* (two semesters, four hours). Research and discussion. Open to those who have credit for (f).

In Columbia College are given courses through the *calculus* and in *projective geometry. Descriptive geometry* is taught in the School of Engineering.

In the Faculty of Pure Science there is a semester course, *Introduction to mathematical philosophy,* and a year course entitled *Philosophy of mathematics.* In the first of these, the aim is to consider questions respecting the logical bases, philosophical bearings, and psychological aspects of mathematics; to explain, with a minimum use of technical symbols, the chief concepts among those which characterize mathematics; to indicate their significance as prototypes for thought in outlying fields; to signalize their function in the science and art of criticism; to study their efficacy in the education of human beings as such; in summary, to view mathematics in perspective relation with the other cardinal enterprises of the human spirit.

The Barnard College curriculum includes courses through the *calculus,* semester courses in *projective geometry* and *graphic methods in mathematics,* and a three hour year course in *The fundamental concepts of modern mathematics.* The last named course deals with some of the results of recent investigation, especially in their bearing on elementary mathematics. Among the topics treated are the following: Euclidean and higher constructions; dimensionality and coördinates; the geometry of motion, transformations, groups, and invariants; the concept of number and function; infinity and the theory of assemblages; the theory of relativity.

In the Summer School, apart from courses in *logarithms and trigonometry, solid geometry, college algebra, analytic geometry, differential calculus,* and *integral calculus,* the following five courses, among others, were given in Teachers College:

(h) *Theory and practice of teaching algebra in secondary schools,* much as course (a) above.

(i) *Theory and practice of teaching geometry in secondary schools,* an abridgment of course (b) mentioned above.

(j) *Demonstration class in plane geometry,* a class in beginning demonstrative geometry taught in the tenth year of the Demonstration School, illustrating the methods of teaching outlined in course (i).

(k) *Practicum in the teaching of mathematics.*

(1) *Commercial mathematics for teachers in high schools and business colleges.*

Course (k) is devoted to the problems of the reorganization of instruction in mathematics in secondary schools, making a special study of the recent reports of the National Committee on Mathematical Requirements. The topics include those affecting reorganization of materials such as the mathematics of the junior and senior high schools; curricula for special types of schools including commercial, manual training, and girls' high schools; prescribed and elective courses. Questions relating to method and content of courses are also discussed such as intuitive and demonstrative geometry; formal discipline; the function concept in high school mathematics; the results of recent experiments to improve mathematics teaching; standardized tests in high-school mathematics; laboratory methods; the use of surveying instruments; and the vitalization of problem material.

Course (1) deals with the subject-matter and teaching of commercial mathematics as taught in these schools. The work will include routine calculations of business, including short methods, calculations involved in accounting and business statistics, graphs, sinking funds, depreciation, obsolescence and replacement, calculating machines, annuities, bond valuation, and the problems of finance and investment. Much of the subject-matter considered yields valuable illustrative material for teachers of algebra. A consideration of business forecasts and reports, such as Babson's, Dun's, Bradstreet's, and the *Annalist* will receive attention. The course will also include a study of the curriculum in commercial schools and of the methods of presenting the work.

Note on Practice Teaching at Brown University

The outstanding feature of the training given prospective teachers at Brown University is the practice teaching, open only to graduate students, conducted under actual conditions in the Providence schools, where continuous instruction is given by each student for 5 to 15 periods a week throughout the year. This practice teaching is under continuous direct or indirect supervision by an experienced teacher who knows the school, and it is closely correlated with university work. Further details in this regard may be found in *School and Society,* April 8, 1916, and in the writer's *Bulletin* (*l. c.,* p. 204f.).

This scheme has won high praise from such prominent authorities as J. F. Brown (*The Training of teachers for secondary schools in*

Germany and the United States, New York, 1911, p. 242f.), and W. S. Learned (*The Oberlehrer, A study of the social and professional evolution of the German schoolmaster,* [Harvard studies in Education, vol. 1], Cambridge, Mass., 1914, p. 131f.).

B. North Central Division

(Illinois, Indiana, Iowa, Kansas, Michigan, Minnesota, Missouri, Nebraska, North Dakota, Ohio, South Dakota, Wisconsin).

The states of this Division are all embraced in the territory covered by the North Central Association of Colleges and Secondary Schools which is the most active organization of its kind in the country (see the section on "Accrediting Agencies" in this report p. 502). Since the teachers of mathematics, appointed in recent years by all high schools accredited by this Association, must have a training equivalent to graduation from a standard four-year college, which is more than is absolutely required by any state law of the Division, the general standard for appointment is not here the lowest among the Divisions. For special treatment we have chosen the state of Illinois which includes Chicago, the largest city, and the University of Chicago —perhaps the most notable center for the training of teachers of mathematics for secondary schools, in the Division. The standards in eight cities (one from each of eight states), of the 19 cities of the Division of over 100,000 inhabitants each, are indicated below. Contributions made towards the training of mathematics teachers by the Universities of Michigan, Minnesota, and Wisconsin, as well as by the notable normal schools of Michigan, Missouri, and Ohio, are also set forth. The literature dealing with educational problems of the Division is extensive; only a very few titles are cited in the pages which follow.

ILLINOIS

(Population, 6,485,098; area, 56,665 square miles).

State requirements. Official documents state that every teacher (including the principal) in a recognized high school ought to have an educational preparation equivalent to: (a) Completion of a four-year high school course; and (b) completion of at least two years beyond a high school course in a higher institution of learning (normal school, or college, or university). The principal of any recognized high school must hold either a *high school certificate,* or a *supervisory certificate,* or a *first grade certificate* endorsed for high school work by the county superintendent. *Second grade elementary certificates* when endorsed by the county superintendent are made valid by the law for teaching in the ninth and tenth grades, but such a certificate is not deemed suitable for a teacher in a recognized high school.

A *county* supervisory certificate, valid for three years for teaching and supervising, cannot be granted without an examination in English, educational psychology, history of education, and school administration. Two classes of applicants may be admitted to this examination: (a) Those who have completed the two-year course of a recognized normal school and have taught or supervised successfully for two years; (b) those who have completed two years of work in a recognized higher institution, one of which is in a normal school or an equivalent preparation, and who have taught or supervised successfully for two years.

A *state* four year supervisory certificate is valid for teaching and supervising in all grades of the state. The requirements of this certificate are: (a) Graduation from a recognized high school and from a recognized normal school, or an equivalent preparation; (b) three years of successful supervision, two of which shall have been in the state of Illinois on a county supervisory certificate; (c) a successful examination in English, educational psychology, sociology, history of education, and school management, administration and supervision; (d) the preparation of a thesis on one or more problems of school administration, the subject or subjects of which shall be selected from a list prescribed by the Superintendent of Public Instruction.

To be admitted to the examination for a *county* high school certificate, valid for three years, an applicant must present credentials showing the completion of two years of successful work in a recognized higher institution with the following credits: English (6 semester hours), educational psychology (3 hours), principles and methods of teaching—pedagogy (3 hours), and electives (48 hours). The applicant must pass an examination in English and pedagogy, and in six high school subjects, three majors and three minors chosen as follows—one subject from each of the groups: (i) mathematics, (ii) history; (iii) science; (iv) language or literature; and two other subjects chosen from the list of high school subjects. This county certificate is granted without examination to those who have presented the following certificates: (a) Graduation from a recognized college; (b) faculty recommendation of ability to teach in the high school; (c) credits for English (8 semester hours), Education (6 hours), three majors of not less than 16 semester hours each in three high school subjects or group of related subjects; (d) Electives sufficient to make up the remaining number of 120 semester hours required for graduation.

The requirements for the *state* four year high school certificate, valid for teaching and supervising in any high school, and in the seventh and eighth grades of any district in the state, are: (a) Graduation from a recognized college or university, or the completion of an equivalent preparation; (b) three years of successful teaching, two of which shall have been in the state on a first grade, a high school or a supervisory county certificate; (c) a successful examination in English, educational psychology, and the principles and methods of teaching; and (d) the preparation of a thesis on one or more secondary school problems, the subject or subjects of which shall be selected from a list prescribed by the Superintendent of Public Instruction.

To be admitted to the examination for a first grade county certificate the applicant must present evidence of graduation from a recognized four year high school, or equivalent preparation. and six months of successful teaching. The examination is in: Orthography (including spelling), civics, Illinois history, physiology, arithmetic, pedagogy, English, algebra, general history, and any three of the following natural sciences: botany, zoölogy, physics, chemistry, and physiography. A general average of 80%, with a minimum of 60%, is required in the examinations. The certificate is also issued to a graduate of a recognized normal school, or of an institution offering an equivalent preparation, provided the applicant has had one year of successful practice teaching and applies for the certificate within three years after graduation. The equivalent of graduation from a recognized normal school is attendance of two years in a recognized higher institution in which the following credits have been earned: English (6 semester hours), mathematics or natural science (6 hours), history or social science (6 hours), education (6 hours—educational psychology, history of education, school administration or principles and methods of teaching), practice teaching (6 hours), and electives (30 hours). The practice teaching must be in a training school under competent and close supervision.

A four year state elementary school certificate is valid for teaching and supervising in the elementary schools and in the first and second years of the high school of any district in the state. The requirements for the certificate are: (a) Graduation from a recognized high school and from a recognized normal school, or an equivalent preparation, (b) three years of successful teaching, two of which shall have been in the state on a first grade county certificate; (c) a successful examination in English, educational psychology, and the principles and methods of teaching; (d) the preparation of a thesis on one or more elementary school problems, the subject or subjects of which shall be selected from a prescribed list.

For the second grade elementary certificate (county), the applicant must present evidence of graduation from a recognized two-year high school or an equivalent preparation. The applicant for this certificate is examined in: Orthography, civics, Illinois history, physiology, penmanship, reading, grammar, geography, United States history, arithmetic, general science, pedagogy, and the principles and methods of the state course of study. A general average of 75%, with a minimum of 60%, is required. The second grade certificate may be granted without examination to those who have completed the equivalent of the first (junior) year's work in a recognized normal school. The equivalent of this work is attendance in a recognized higher institution in which the following credits have been earned: English (6 semester hours), mathematics or natural science (6 hours), history or social science (6 hours), education (6 hours—educational psychology, history of education, school administration, or principles and methods of teaching), and electives (6 hours).

Chicago Requirements. There is only one city of over 100,000 inhabitants in the state, namely, Chicago (population 2,701,705). No certificate to teach an academic subject in a high school is issued to a candidate who has not passed the examinations of the Board of Education, conducted with special reference to the position sought. The examination in any subject may include questions in methods of teaching and may be, in part, oral; if so divided the oral examination does not exceed more than 20% of the total points required for passing the written examination. To be admitted to the examination the candidate must present credentials showing: 1 (a) Graduation from an accredited college (having a four-year course beyond the regular four-year high school course) and (b) two years of successful experience in graded schools; or 2 (a) Graduation from an accredited college, and (b) completion of the required work in The Chicago Normal College,[12] and (c) one year of successful experience in graded schools. The examination is in one major and three minors. When the major is mathematics the minors are English, a professional study, and any language or science minor. When physics or chemistry is a major, mathematics or a science may be a minor. A general average of 80% must be obtained and no subject below 50%; the major subject must count as two-fifths in the examination. A general knowledge of the subjects such as shall indicate sufficient familiarity to enable the candidate to teach them is expected in all minor subjects. In major subjects it is assumed that the candidate has made a specialty of the one selected and a more special and exhaustive test is made. The usual college courses, with their expository lectures, added to what may be presumed from the candidate's years of experience as an instructor, indicate the ground that may be covered in an examination.

Universities and Colleges. There are in Illinois 30 institutions of this type. Of these the Bradley Polytechnic Institute and St. Viator College offer no mathematics or education courses primarily for the prospective teacher of mathematics in secondary schools. McKendree College has ignored repeated requests for copies of its catalogue. Of the remaining 27 institutions, all offer courses through the calculus, and all offer various courses in education. Projective or modern geometry is taught at 13, descriptive geometry at 9, history of mathematics at 6, and special mathematics courses for teachers at 12 institutions. Ten of these 12 institutions (the University of Chicago and the University of Illinois will be considered separately) offer courses as follows:

EUREKA COLLEGE, *Teaching of mathematics in the high school* (2 semester hours) for seniors who expect to teach mathematics and who have had 16 semester hours of mathematics.—GREENVILLE COLLEGE, *The teaching of mathematics* (2 hours) with courses through differential calculus as prerequisites. The topics are: fundamental concepts of algebra and geometry that present special difficulties in teaching; educational value of high school mathematics, and literature of elementary mathematics.—ILLINOIS COLLEGE, *Teaching of mathematics* (2 hours) with the course in analytic geometry prerequisite, is required of all prospective teachers seeking the recommendation of the department for teaching. It is stated that the course deals with the educational thought of the day relating to the teaching of high school mathematics and requires recitations, reports on reference reading, and informal discussion.—HEDDING COLLEGE, *Methods of teaching mathematics* (3 hours) with the course in analytic geometry as prerequisite.—KNOX COLLEGE, *Teachers' course* (2 hours) with the course in integral calculus prerequisite. The topics of the course are: Value of study of mathematics; mathematical methods of demonstration; chief methods of teaching; modern movements in the teaching of mathematics (correlation, etc.); miscellaneous points in class-room work;, preparation of the teacher; qualification of the teacher; the curriculum; the foundations of mathematics; the teaching of algebra and the examination of texts; brief comparison of American and foreign methods and systems. At the end of the course each student presents a thesis on some problem of American secondary education.—Loyola University, *Methods of teaching mathematics* (2 hours) with courses in Solid analytic geometry and Differential calculus as prerequisites.—Monmouth College, *The teaching of mathematics* (2 hours) with courses in college algebra, and plane and spherical trigonometry, prerequisite.—Northwestern College, *The teaching of mathematics* (2 hours) with four semester courses in college mathematics, and principles of teaching in the department of education, as prerequisites. The text-book used is that of D. E. Smith or J. W. A. Young. This course is required of those who expect the recommendation of the department as teachers of secondary mathematics.—Rockford College, *Fundamental concepts in mathematics with applications to the teaching of algebra and geometry* (2 hours).—Shurtleff College, *Teachers' course*, with a course in trigonometry and analytic geometry as prerequisites.

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[12] Nothing but elementary school mathematics is taught in this "College" (Announcement, Chicago Normal College, 1918-1919).

University of Chicago.[13] Courses in the history and the teaching of elementary mathematics are offered by the department of mathematics and the school of education. "These courses embody the conviction that elementary students need to have their mathematics made not easier but more perfectly intelligible and attractive."

Those who expect to teach mathematics as a major subject in secondary schools are advised to complete at least the following courses in their undergraduate career; (a) Trigonometry, college algebra, plane analytic geometry, differential and integral calculus and its applications, theory of equations, and the synoptic course in advanced mathematics (a general historical and logical survey of a considerable portion of pure and applied mathematics); (b) descriptive astronomy, mechanics, and general physics; (c) principles of education and methods of education; (d) practice teaching of mathematics in the university high school, for which the foregoing courses in education are prerequisite; (e) courses in the School of Education on the teaching of secondary mathematics, and on the history of secondary mathematics.

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[13] Authorities: University of Chicago, Circular of the Departments of Mathematics, Astronomy and Astrophysics, Physics, Chemistry, 1921-22; letters from members of the University of Chicago.

The following four term courses are offered primarily for prospective high school teachers in the department of mathematics:

(a) *Solution of numerical equations*: Isolation of the real roots by graphic methods and by use of Sturm's functions, Newton's and Horner's methods of solution, algebraic solution of cubic and quartic equations—Dickson's text with calculus as a prerequisite;

(b) *Selected topics in mathematics.* Based on certain parts of *Monographs on Topics of Modern Mathematics Relevant to the Elementary Field,* edited by J. W. A. Young;

(c) *Critical review of secondary mathematics.* A brief survey of the subject-matter of secondary mathematics from the modern point of view, aiming both to organize the theory of the whole scientifically and to gather the products of this work for use in teaching;

(d) *Topics of geometry.* The topics are selected from the following: Foundations of geometry, nature and value of geometric reasoning, methods of solution of geometric problems, limits in geometry, anharmonic ratio, complete quadrilateral, polars, duality, modern geometry of the triangle, geometric conics, and others,—with much work in actual solution of problems. The course is very definitely intended to give prospective teachers a collegiate extension of secondary school geometry analogous to college algebra as an extension of secondary school algebra.

In the course on *History of mathematics,* the teaching phases of the subject are emphasized, and the period to Euler and Lagrange is considered. The text used is either the larger work of Ball or Cajori, but collateral reading is assigned in a number of other standard works. All students are regularly required (a) to make reports on assigned readings and topics, and (b) to prepare a chart showing graphically the significance, the placing in time, and the contributions of historic schools and chief personnel.

Some of the topics in the course on *The teaching of secondary mathematics* are as follows: Aims and purposes of secondary mathematics, methods of teaching and gauging progress, need for reorganization of high school mathematical courses, suggested ways of improving existing courses, educational needs that must be met, treatment of definitions and early theorems and principles, the routine and the rational in mathematics, ways of attacking problems and theorems, teaching mathematical applications, directed observation in high school classes for four weeks. Specific problems and questions brought out by the directed observation take up the rest of the time of the course.

Directed observation is started by giving students a three page blank guide with spaces for filling in answers to questions.[14] Two sorts of classes are observed, namely demonstration exercises and regular class exercises. Three or four demonstration lessons are arranged to exhibit some special feature of teaching practice, and the entire college class with the instructor attends. The next day this observation is discussed critically by the teachers in-training, to get started in "how to see". For the rest of the four weeks the college class is distributed among the high school classes, not more than three or four to one class. These collegiate groups become an organic part of the high school class for the time being, the high school teacher having charge of them and supervising their work even though the college instructor is present as a visitor. Once a week the college class meets as a class under the regular college teacher and reports critically on what has been observed. Redistributions and detailed instructions for the succeeding week are here given in the light of developments to date. The high school teachers are invited and welcome at these meetings and discussions. When present, they always participate. In addition to these weekly reports a general report of all the observation is required at the close of the observing period of four weeks.

University of Illinois.[15] The basis for the award of a state high school certificate has been set forth above; in the state university special provision is, therefore, made for meeting such requirements. It offers two three-hour year courses, *Technic of teaching* and *Educational psychology*, embodying the materials usually covered in this connection by the state examinations, and these courses must be taken by students desiring the official recommendation of the appointments committee for teaching positions in secondary schools. The first of these courses treats of types of class-room exercises and preparation of teaching plans, the hygiene of instruction, class room management, and professional ethics; it offers observation of teaching in neighboring schools. In the second course through lectures, readings, and demonstrations the following topics are discussed: Instinct; habit and acquisition of skill; perception and memory, conception, judgment, and reasoning; applications to school subjects. A four-hour course, *Introduction to psychology*, is prerequisite for this course.

The university makes special note of the fact that the above mentioned state requirement for theoretical professional training does not meet the requirement of the North Central Association of Colleges and Secondary Schools in connection with its accredited high schools. Newly appointed teachers here must have completed courses in education aggregating at least eleven semester hours. Another three-hour year course, on *Principles of secondary education,* requires knowledge of the above mentioned courses as a prerequisite. It deals with: Evolution of high schools and of the fundamental concepts of secondary education; proposed reorganization; relation of high schools to the state systems; legal status; articulation with the elementary school, the college, technical school, the community, and the home; the teaching staff; reconstruction of curricula, "controls" of instruction; direction of "student activities".

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[14] Three of the questions, one from each page, are as follows: III. Was the lesson based on a topic or a definite problem? If on a topic what was it? If on a problem, what was it? In what specific situation in the lives of the pupils will the subject-matter be needed? Did the pupils recognize this use?—VII. Did the teacher constantly seek to place responsibility for inference with the pupils or did he require or permit them to accept conclusions merely because they were to be found in the book? Give an example.—XII. Did you get any expressions of pupils to show what you regard as a sincere expression of their attitude toward the subject, or the teacher? If so, give samples.

[15] Authority: University of Illinois Bulletin, Annual Register 1920-1921, February 28, 1921.

Although the School of Education of the university controls a high school in Urbana[16] the only practical experience in teaching classes is offered in connection with agriculture, home economics, and physical education.

In the department of mathematics a *Teachers' Course* (2 semester hours) is offered for those who have 18 hours of credit in mathematics. The topics are: Secondary algebra and geometry, educational value, position in course, methods of teaching, correlation, American and foreign methods, order and importance of topics, text-books, and literature. The conduct of the course is by lectures, discussions and reports.

Other two-hour semester courses with the same prerequisite are: *History of mathematics* and *Fundamental concepts of mathematics.* In the former course by lectures and reports on assigned reading are taken up: The elementary subjects; rise and growth of higher mathematics, chiefly in the nineteenth century; and biography. In the latter the topics are: The number concept; unity, aggregate, order, and correspondence; irrationals and limits, transcendence of e and π; parallel axiom and non-euclidean geometries; constructions with ruler and compasses; function; logic of mathematics.

A three-hour semester course in *Constructive and projective geometry* deals with: Methods of geometric constructions and constructive solution of problems; approximate constructions; discussion of possibilities of construction by ruler and compasses, by double-ruler, and instruments of higher order; elements of inversion geometry in plane and space; stereographic projection, isometric projection, orthographic projection, perspective, geometric principles involved in projective processes, introduction to projective geometry; analytic formulation of geometric methods.

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[16] Cf. University of Illinois School of Education, Bulletin No. 7: Practice Teaching in the School of Education, . . . 1893-1911, by F. Morehouse, 1912.

Many other courses in mathematics are offered. Indeed the University of Illinois is probably one of the three universities in the country where most ample opportunities are offered for advanced mathematical study.

CITIES

The standards for the certification of teachers of mathematics in Chicago have been indicated above. Similar indications in connection with cities in seven other states are given below.

Cleveland, O. (population, 796,836).—A candidate as teacher of mathematics in a senior high school must, in general, have had "four years of approved education beyond a four year high school course, or the equivalent of a full standard college course.[17]" In making a selection among several possible candidates, consideration of comparative fitness to render the best service shall determine each case of first appointment. The first three years of service are strictly probationary. Reappointment is based not only on success already achieved, but also on evidence of growth warranting the expectation of service commensurate with any automatic increases in salary provided by the schedule.

An elaborate education survey of the city was published in 15 volumes: *The Cleveland School Survey* [of 1915], 1917-1920.

The Cleveland School of Education, in cooperation with Western Reserve University, gives regular courses, also extension, and summer session courses, which are primarily for the development of teachers of mathematics in secondary schools. Apart from various courses in education and through the calculus, at Western Reserve, a year course on *Foundations of geometry and algebra* is offered at the College for Women. There is also a course in *projective geometry*. In the summer school of the Cleveland School of Education the following courses were offered in 1921: *Mathematics of investment*, *Advanced algebra and analytic geometry* and *The elements of calculus*.

Detroit, Mich. (population, 993,739).—Every candidate as teacher of an academic subject in a high school must have had: (a) The equivalent of a bachelor's course, with a good record, in a four year college or university; (b) 11 semester hours in education; (c) one year of successful experience in teaching. There are no special requirements for a teacher of mathematics as distinguished from a teacher of another academic subject; as in Grand Rapids, however, it is supposed that the candidate has had "special preparation in mathematics". No attempt has been made yet to designate the distribution of the eleven hours of education. Some members of the Board of Education believe that these hours should be spread over: Educational psychology, tests and measurements, philosophy, history and administration of education.

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[17] The state professional requirements in Ohio are indicated below in our note on the Kent State Normal College.

At the *University of Detroit* the prospective teacher may take courses in college algebra, plane trigonometry, analytic plane geometry, differential calculus and integral calculus, in addition to courses for theoretical professional training

Indianapolis, Ind. (population, 314,194)—The prospective teacher of mathematics must have an A. B. degree, or its equivalent, from a standard college and in addition a year or two of successful teaching experience. Most of the teachers have their master's degree. In addition they are required to take an examination which "covers the field of secondary and college mathematics" For the *state* of Indiana there are no legal requirements as to qualifications of candidates for teachers of mathematics excepting those pertaining to the teaching certificate. For the ordinary certificate at least two years of college work with freshman mathematics completed and a minimum successful grade on the state examination in English, algebra, geometry, and the science of education, are required.

Kansas City, Mo. (population, 324,410)—According to printed regulations High School Certificates may be granted on the basis of approved credentials or examinations. In the latter case the prospective teacher of mathematics mus pass in the following subjects, and also in one science as a minor: algebra, geometry, trigonometry, and analytics. Approved credentials call for graduation from a college or university approved by the Superintendent as having courses required for graduation equivalent to those in the University of Missouri, and satisfactory evidence of two years of successful teaching experience under approved conditions. In the absence of courses especially fitting the candidate for the work for which he applies, or in case these courses seem inadequate, he must write a thesis upon, or take an examination on, any subject special or general, that may be required by the superintendent of the examining board.

While printed regulations indicate that a newly appointed teacher of mathematics need not be a college graduate, the standards of recent years make this an indispensable condition for consideration.

Milwaukee, Wis. (population 457,147)—Applicants for high school positions must hold a state license,[18] and must be graduates of some accredited college or university. After appointment, and satisfactory service for four years as a regularly appointed teacher, the appointment is made permanent.

Minneapolis, Minn. (population, 380,582)—In order that a candidate be eligible as a senior high school teacher he must be a graduate of a four-year collegiate course of the grade furnished by the University of Minnesota, and have made special and adequate study of the subject or subjects to be taught Further the candidate must have had two years of successful experience in teaching under approved conditions. Such experience must have been gained either in the Minneapolis public schools or in schools of a grade corresponding to that of the Minneapolis high schools. Excess of one type of qualification—whether of education or of experience—beyond the minimum requirement, will not be accepted as a substitute for the lack of the minimum qualification of the other type.

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[18] Details in this regard are given below under the heading: University of Wisconsin.

Omaha, Neb. (population, 191,601)— The prospective teacher of mathematics in a high school must be a college graduate who specialized in mathematics at least to the extent of taking courses through a course in analytic geometry, and preferably through a course in calculus also. The teacher must also have had some training in the pedagogy of mathematics. Omaha does not require state certification of its candidates as teachers.

In Omaha there are two universities: Creighton University and the University of Omaha. At CREIGHTON UNIVERSITY a number of courses are offered in education and in mathematics through differential and integral calculus. There is also a two-hour semester course, *Methods of teaching mathematics* offered in the department of mathematics. The preparation of teachers includes observation of expert teaching in the Creighton University High School, and eaching of thirty lessons in the high school, under the supervision of a critic teacher. Similar opportunities for theoretical and practical professional training, and for scientific training, are also available at the UNIVERSITY OF OMAHA, with the exception that there is no course in mathematics primarily for teachers in high schools.

UNIVERSITIES, COLLEGES, AND NORMAL SCHOOLS

A survey in the Division of leading institutions which contribute o the preparation of teachers of mathematics in secondary schools would be incomplete, and tend to create a wrong impression, were not he work done at a few other institutions set forth.

University of Michigan.[19] The *Teacher's Diploma* is given to a student in connection with his bachelor's degree, provided he has taken eleven semester hours in education including *Introductory course* (2 hours credit—lectures, readings, reports, discussions), and *Observation and special methods in mathematics* (1 hour credit application of the principles of class management to concrete situations; the aims and values of mathematical study; the course in mathematics for the secondary school; classification and critical consideration of special methods and modes; comparison of text-books; 20 hours of high school observation). Among numerous other courses offered are: *History of education* (2 hours), *psychology of education* (2 hours), *secondary education, introductory course* (2 hours). At public schools of Ann Arbor students in education are given opportunity to study methods of eaching the various subjects of the curricula in the high school. Students seeking a Teacher's Diploma are classified into groups and required to observe expert teaching of their major and minor subjects.

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[19] Authorities: University Bulletin—New Series. University of Michigan (a) College of Literature, Science and the Arts, Annual Announcement 1920-1921, August 21, 1920; (b) Summer Session, July 1-August 26, 1921, March 5, 1921.

Observation is accompanied by written reports, conferences, and collateral reading. No opportunity is offered for the student to do actual teaching.

In mathematics, prospective teachers in high schools are advised to take: Plane trigonometry and algebra (4 hours), plane analytic geometry (4 hours), calculus (4 year hours), *Teachers' seminary, algebra* (2 semester hours), *Teachers' seminary, geometry* (2 hours), and five or more additional hours to be selected after conference with the head of the department. Other courses, among many, are: *Projective geometry* (3 hours, synthetic 3 hours; analytic), *descriptive geometry* (3 hours), *mathematical theory of statistics* (2 year-hours), *introduction to the mathematical theory of interest and insurance* (3 hours), and *history of mathematics* (2 year-hours).

In the summer session the several courses include: *Psychology of education* (2 hours), *principles of education* (2 hours), *high school administration* (2 hours), *secondary education—advanced course* (2 hours), various courses through the calculus, and *mathematical theory of statistics* (2 hours).

Under authorization of the state legislature, the Board of Regents of the University of Michigan gives a Teachers' Certificate to any person who takes a bachelor's, master's, or doctor's degree and also receives a Teacher's Diploma. The Certificate serves as a legal certificate of qualification to teach in any school of the state.

Michigan State Normal College,[20] Ypsilanti, Mich. For the four year curriculum leading to a B. A. degree 48 units constitute the course. They must be taken in accordance with the following requirements: (a) 4 units in education, 4 units in teachers' courses, and 2 units in teaching; (b) 2 units in composition and rhetoric, one of which must be taken in the freshman year; and (c) 5 units from each of the following groups—I: Ancient language, modern language, English language and literature; II: Mathematics, astronomy, physics, chemistry, geology, zoölogy, botany, and geography; III: history, political science, philosophy, and education. The remaining 21 units offer opportunity for freedom of election and for specialization in one or more subjects (upon the principle of major and minor subjects).

More than a score of courses are offered in education. The mathematical courses offered include: *Solid geometry; methods in geometry* (a review of plane and solid geometry with special attention to methods and presentation of the subject); *history of mathematics* (designed to show the student how the subjects he is to teach have developed; *differential calculus; integral calculus; theory of equations; solid analytic geometry; differential equations; theoretical*

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[20] Authority: Michigan State Normal College Bulletin. Catalogue Number. 1920-1921, May, 1921.

mechanics; studies in mathematical education; mathematical theory of investment, and *introduction to the theory of statistics.*

University of Minnesota.[21] Every prospective mathematics teacher in the Minnesota public schools must have a certificate; the one of interest to the university student being a *first-grade professional certificate* which qualifies him to teach in any high school of the state. The *University teachers' certificate* is valid for two years as a first grade professional certificate. Holders are eligible for all positions open to holders of the first-grade professional certificates. After two years of successful experience the University teacher's certificate may become a *life certificate* upon endorsement by the State Department of Education, and the president of the university. These two years of experience must be gained within Minnesota.

The following course, leading to the degree of Bachelor of Science (in Education) in the College of Education of the University, is prescribed for the University teacher's certificate: *Brief course in history of education* (5 term credits) or *History of education* (9 term credits), *Educational sociology* (3 credits), *Elementary educational psychology* (3 credits), *Technic of teaching* (3 credits), *Practice teaching* (5 credits; the preceding courses prerequisite), and *Special methods* (at least 3 credits). Prerequisites for this work are two years of college work including a course in *General psychology* and *Introductory laboratory psychology.* Students registering in the College of Education are expected to earn a minimum of 22 credits in education courses. Eleven of these credits should be completed in the junior year and eleven in the senior year.

Opportunity for practice teaching is provided in part by the University High School, and in part by the Minneapolis city schools. Prerequisites are the 14 to 18 credits in education indicated above, and a statement from the department of mathematics recommending the candidate. Such a recommendation is only granted when he has 5 credits each for courses in trigonometry, college algebra, analytic geometry, calculus 1, and calculus 2, as well as for higher algebra and solid geometry, unless they were taken in high school. These are for a minor recommendation for a teacher's certificate; for a major recommendation, 8 credits are required in addition to those just mentioned for a minor. A term course in *History of elementary mathematics* primarily for prospective teachers in high schools, presents a brief course in the history of arithmetic, algebra, and geometry. Courses in *Advanced analytic and synthetic geometry* and *Descriptive geometry* are also given.

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[21] Bulletin of the University of Minnesota: (a) The College of Education. Announcement for the year 1921-1922, September 6, 1921; (b) The College of Science Literature and the Arts Announcement for the year 1921-1922, May 5, 1921; (c) The Summer Session Announcement of Courses 1921, June 20 to July 30, May 21, 1921.

Practice teaching includes directed observation and actual teaching, under supervision. One to two weeks of carefully directed, well-digested observation on the part of the student, is regarded as the best preparation for his teaching. Lesson plans are first made of the work in the recitations observed. The student begins teaching only after the critic is reasonably sure that the student has clearly in mind the object to be attained, and has presented a satisfactory plan of execution. It is supposed that each person, during the semester that he pursues the course in practice teaching, shall conduct at least 36 recitations or perform[22] equivalent work.

In the Summer Session a *Teachers' course in mathematics* was offered to students preparing to become teachers of secondary school mathematics. The course consisted of: Lectures, readings, discussions, methods of presentation, assignments, lesson plans, examinations, plans for beginning courses in algebra and plane geometry, aims and outcomes of education.

Missouri and its State Teachers Colleges. Blanket certificates to teach are issued by the state educational institutions. These certificates make it legal for the holder to teach any subject in the public school curriculum. The only check is a minimum requirement set by the State Department of Education in connection with the approval of secondary schools. The work of a teacher of high school mathematics will not be approved if the teacher has completed less than 7½ semester hours of college work in mathematics. The School of Education of the University of Missouri requires that students selecting mathematics as their subject of specialization for a certificate to teach complete not less than 15 semester hours in this subject.

The state issues a *five-year certificate* to those who have had two years of standard college work, and who have passed successfully examinations in any one subject under each of the groups I, II, III, and IV, and three subjects under group V. I, History and government American history, European history, English history, oriental history economics, sociology, Missouri history and government, political parties in the United States; II, Rhetoric and literature: English composition, exposition and argumentation, American literature, English literature, college rhetoric; III, Mathematics: College algebra, trigonometery, solid geometry, advanced arithmetic, analytic geometry; IV, Sciences: Physics, chemistry, geology, physiography, botany, advanced agriculture; V, Professional Subjects: Educational psychology, history of education, primary teaching and kindergarten, elementary courses of study, Missouri school system, supervision and administration, high school teaching, theory and methods of teaching, current reading circle books, principles of education, and pedagogy.

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[22] Further details may be found in an article, from which quotations ar made above, "Practice teaching at the University of Minnesota," by R. A. Kent School and Society, July 22, 1916, vol. 4, pp. 140-144.

A *life certificate* may be granted to a teacher who, with a five-year certificate as above, has taught successfully for forty months, and passed an examination in five professional subjects which were not used in securing the five-year certificate.

Applicants who hold the A. B. degree or its equivalent from any standard college may be granted a five-year certificate when at least twenty hours of work in educational subjects have been completed. The subjects must include: General elementary psychology, educational psychology, methods or principles of education, practice teaching, history of education, secondary education, and school administration.

A *three-year certificate* will be granted to graduates of recognized senior colleges which have completed the work in education (18 hours minimum) just outlined. After sixteen months of successful teaching in Missouri, this certificate may be converted into a life certificate.

There are other forms of certificates, but enough has been given above roughly to indicate the standards of legal requirement.

The State Teachers Colleges are situated at Cape Girardeau, Kirksville, Marysville, Springfield, and Warrensburg. All of the colleges offer four-year courses leading to the B. Sc. and (except the one at Kirksville) B. A. degrees. All offer courses in mathematics through the calculus and all (except the one at Cape Girardeau) offer special mathematics courses for prospective teachers in secondary schools as well as courses in the history of mathematics. The college at Cape Girardeau offers a course in descriptive geometry. The special courses offered are as follows:

Kirksville—*The teaching of mathematics in high schools* (2½ term hours). "This course shows the value and place of secondary mathematics in the curriculum. It helps to organize the material of algebra and geometry and to correlate these subjects with allied subjects". *History of mathematics* (2½ term hours). "This course gives a historical survey of the science of mathematics. It enables the student to follow the genesis of this science, to grasp the essential facts of mathematics, and to utilize them in the teaching profession."

Marysville—*Teaching of mathematics* (2½ term hours, with trigonometry and college algebra prerequisites). *History of mathematics* (2½ term hours with a course on analytic geometry prerequisite). A study of the development of the topics included in a high school course.

Springfield—*The teaching of high school mathematics* (2½ term hours; course on Principles of teaching prerequisite). "A thorough investigation of the subject-matter and the method of high school algebra, geometry, and trigonometry". *History of mathematics* (2½ hours credit). "A general view of the development of the elementary branches of mathematics".

Warrensburg—*Teachers' mathematics* (2½ term hours). "A discussion of the teaching of mathematics in high schools, including a course of study for junior high schools. The correlation of subjects and practical laboratory methods form an important part of the course". *History of mathematics* (2½ term hours). "The study of the development of the various topics included in the ordinary high school course, together with a study of some of the men who have contributed largely to the development of the science of mathematics".

In concluding this note, attention is directed to an elaborate report, of over 470 pages, published in 1920 by the Carnegie Foundation for the Advancement of Teaching, entitled: *The Professional Preparation of Teachers for American Public Schools. A Study based upon an Examination of Tax-supported Normal Schools in the State of Missouri*, by W. S. Learned, W. C. Bagley, and C. A. McMurry, G. D. Strayer, W. F. Dearborn, I. L. Kandel, H. W. Josselyn.

Kent State Normal College,[23] **Ohio.** This normal college of northeastern Ohio provides programs of study, both for those who have graduated from four-year high schools and from four-year standard colleges. The course for graduates of high schools leads to the degree of B. S. in Education, in four years. Of the 192 term hours in the program, 96 must be distributed in different departments as follows: Education (28 hours—in psychology, principles of education, history of education, school administration, sociology); practice teaching (12 hours—under direction in the college high school); special methods (8 hours); English (12 hours); history (12 hours); science or mathematics (12 hours); manual training, or household science, agriculture (12 hours). Physical training is required twice a week throughout the course. At the end of the first year each student must choose a major subject in which he must complete a minimum of 36 hours. On consultation with the head of the department in which he majors, the student must select at least one minor in which he is to complete 24 term hours.

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[23] Authority: Kent State Normal College Quarterly, Ninth annual catalog 1920-1921, Announcements, 1922.

For a major in mathematics the student is recommended to take in his freshman year: Trigonometry, college algebra, and analytic geometry (12 hours); in his sophomore year: Calculus (12 hours); in his junior year: Junior high school mathematics (4 hours), senior high school mathematics (4 hours—a discussion of the present day tendencies in arranging courses in mathematics for senior high schools), and tests and measurements (4 hours—a discussion of the mathematical theory underlying the marking system together with a discussion of the standard tests for measuring ability and performance in mathematics); and in his senior year: Teaching of algebra (4 hours); teaching of geometry (4 hours); and, as an elective, history of mathematics (4 hours). Four-hour courses in advanced analytic geometry and theory of equations are also offered.

All three state normal colleges of Ohio confer degrees.

The school code of the state of Ohio requires that a candidate for a high school provisional or life certificate shall have had at least 30 semester hours of professional training, the subjects being: General psychology, educational psychology, history of education, principles of education (or science of education), school administration. Methods of teaching (either general or special), and supervised practice teaching. It is not required that methods courses should have been in the majors or minors. The teacher is not required to teach the subject he has made a major or minor. Graduation from a college or university is not, by law, necessary for appointment.

University of Wisconsin.[24] In accordance with the statutes of Wisconsin, *University teachers' certificates* are issued to all graduates, of a regular collegiate course, who complete the course in pedagogical instruction prescribed by the university. This certificate, when presented to the state superintendent, entitles the holder to receive a license to teach in any public school in Wisconsin for one year, renewable for a second year after one year of successful experience. Graduates who present satisfactory evidence of good moral character and of two years of successful teaching in the Wisconsin public schools after graduation are entitled to receive from the state superintendent an unlimited state certificate.

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[24] Authorities: Bulletin of the University of Wisconsin: (a) Announcements, 1919-1920, May, 1920; (b) Summer Session, 1921 (June 27-August 5).

Graduation from the School of Education and the recommendation for the University teachers' certificate is based upon the following conditions and requirements: (a) A total of 124 credits exclusive of physical training and military drill; (b) satisfactory fulfillment of the requirements of preparation for teaching: (i) a major subject[25], or (ii) a major subject[26] and one minor subject, or (iii) a major subject and two minor subjects; (c) the recommendation of the departments of the major and minor subjects, or of the responsible authorities of the special courses as to fitness for teaching; (d) The presentation of a certificate of physical health and fitness from the university medical examiner; (e) the completion of the following professional requirements (15 credits): (i) Psychology (3 credits), educational psychology or mental development (3 credits), principles of education (3 credits), technique of education (2 credits), departmental teachers course (2 credits), and educational practice (2 credits).

The professional requirements for the University teachers' certificate are in advance of the minimum fixed by law for a license to teach. The legal requirement is twelve hours in psychology and education[27]. Attention is called to the fact that the minimum requirement of the North Central Association is eleven hours in education.

For mathematics as a major in the School of Education, a minimum includes a three-hour year course in calculus, a three-hour semester course in *Determinants and analytic geometry of three dimensions,* a semester course in *Differential equations* or *Theoretical mechanics* and the courses that necessarily precede them. *The content of secondary mathematics* (2 semester hours) is open to those who offer mathematics as a major, and is recommended as a desirable course for the junior year. A knowledge of solid geometry, obtained in the secondary school or elsewhere, is also required.

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[25] "To fulfill the requirements of preparation for teaching the selected major a student must present credits in amount and kind as prescribed by the department of such major subjects. This departmental prescription must in all cases include at least twenty credits, and, in addition, credit for the departmental teachers course."

[26] "To fulfill the requirements of preparation for the teaching of the selected minor subject or subjects a student must present credits in amount and kind as prescribed by the department of such minor subject. This departmental prescription must contain ten credits, exclusive of the departmental teacher's course."

[27] Graduation from a standard college or university is not necessary for the certification of a high school teacher in Wisconsin. For example, a student who takes the three-year course for the preparation of high school teachers and principals at the State Normal School of La Crosse, receives (Bulletin of the State Normal School, La Crosse, June, 1920) a special diploma, which after two years of successful teaching in the state, entitles the holder to an unlimited state certificate. For mathematics as a major here the electives in mathematics will be 25 units and in science 10 units. Among the courses in mathematics offered are the following seven, each for five credits: College algebra, trigonometry, analytic geometry, elementary analysis 1, elementary analysis 2, calculus 1, and calculus 2.

For mathematics as a minor: (a) A minimum of twelve credits from various courses through the calculus, determinants and analytic geometry, commercial algebra, mathematics for agricultural students, projective geometry, and descriptive geometry. Six of these credits must be earned at the University of Wisconsin "with a grade of at least fair". (b) *The teaching of mathematics* (2 hours), required of all those who offer mathematics as their principal minor; *The content of secondary mathematics*,[28] mentioned above, is also open to such students. (c) A knowledge of solid geometry obtained in the secondary school or elsewhere.

The practical professional training at the University of Wisconsin has attracted wide attention and has been made the subject of elaborate report.[29]

> The plan involves "directed teaching" in the Wisconsin High School, and "directed and supervised observation" both there and in the Madison High School. The outstanding idea is[30] to "avoid as much as possible any interference with the work of the high school classes upon which the cadet teachers practice, and yet to give the apprentice teacher a chance to show his skill. The method of accomplishing these much-to be desired results may be illustrated as follows: Students in the university who desire to teach mathematics, let us say, become regular members of the high school class in algebra. They sit with the high-school pupils, are held rigidly responsible for the subject matter of the course, are called upon by the class instructor to recite and to participate fully in all the activities of the class. In addition, these students are frequently asked by the instructor to assume the role of teachers—it may be to demonstrate a problem, to correct errors in blackboard work, or to conduct a class discussion. Moreover, these apprentice teachers are required to keep a record of the concrete class-room situations as they arise. Upon these situations as solved or unsolved by the instructor in charge, they make critical comments as to what ought to have been done or ought not to have been done to meet the situations. These records are carefully examined by the principal of the high school and by the instructor in mathematics and are used as the basis of the frequent and regular personal consultations by these officers with the apprentice teachers. The records are filed in the principal's office, where they may be examined by superintendents who come to the university in search of promising teachers."

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[28] These two courses are the "departmental teacher's courses" referred to above in the general syllabus of professional requirements.

[29] H. L. Miller, "The University of Wisconsin plan for the preparation of high-school teachers," pages 7-165 of The Eighteenth Year book of the National Society for the Study of Education. Part I. The Professional Preparation of High School Teachers, Bloomington, Ill., 1919. ("Bibliography on the training of high-school teachers," pp. 355-358.)

[30] Or rather, was, in 1915. Quotation is here made from School Review, vol. 23, (1915), pp. 199-200; see also vol. 24, (1916), pp. 515-520.

"The feature which is especially commendable in this plan is the excellent basis for critical observation. A visit for one day, a visit for a number of days, for that matter, does not give even the trained observer sufficient data upon which to judge the merit of either class or instructor; but several months of daily observation, backed by participation in the activities of the class, plus incidental work as a teacher, give even the most inexperienced college student a real opportunity for securing a practical working knowledge of how to solve teaching problems."

At the Summer Session of the University of Wisconsin were offered several courses in education leading to the University teachers certificate. In mathematics, apart from various courses including those through the calculus, were: *The teaching of high school mathematics, survey of elementary mathematics* (history and development of algebra, geometry, and trigonometry), *descriptive geometry,* and *projective geometry,* synthetic treatment."

C. South Atlantic Division

(Delaware, District of Columbia, Florida, Georgia, Maryland, North Carolina, South Carolina, Virginia, and West Virginia).

Of these states the District of Columbia has been selected as exhibiting the highest standards of the Division in connection with the appointment of its high school teachers of mathematics. To indicate another standard, notes on certification in Virginia are culled from a recent survey report. It was found necessary to neglect all consideration of South Carolina, since repeated requests for information were met by the state superintendent of education with evasions. Of the six cities in the division, with more than 100,000 inhabitants, it seemed desirable to give a brief sketch of the organization in Atlanta, Ga., in addition to that of Washington, D. C. A report on contributions made by The Johns Hopkins University, by the University of North Carolina, and by the University of West Virginia, are added in conclusion.

DISTRICT OF COLUMBIA

(Population, 437,571; area 69.2 square miles).

District Requirements. The city of *Washington* is coterminous with the District of Columbia. According to regulations of 1906, no person without a degree from an accredited college, or a graduation certificate from an accredited normal school, such normal school graduate having had at least five years of experience as a teacher in a high school, is allowed to teach any academic subject (of which mathematics is one) in a high school. No such teacher is appointed until he has passed an examination prescribed by the Board of Examiners. This examination is both written and oral. The written examination for the prospective teacher in mathematics is on mathematics as a major and on two minors. For the major the examination is in the following subjects: Elementary and college algebra, plane and solid geometry, plane trigonometry, elements of conic sections, elements of differential and integral calculus. The prescribed minors are in: (a) English; and (b) psychology and pedagogy (which includes (i) history of education, (ii) principles of education, and (iii) practical methods of application of these principles). The purpose of the English minor is to test the applicant's ability to use the English language, and includes a theme of about three hundred words to be written in good clear idiomatic English on a subject to be designated by the Board of Examiners at the time of the examination.

The oral examination is prescribed to enable the examiners to estimate the applicant's general personal fitness for the position as a teacher, and involves considerations of knowledge, culture, education, experience, resourcefulness, record as a student and teacher, etc.

The entire examination must be taken at one time. Of a total 100 credits, 60 and 40 are respectively assigned to the written and oral examinations. The oral examination of a candidate is not held unless he makes at least 70% in his major and at least 40 out of the 60 credits in his written examination. The examination in its entirety is not considered unless the applicant makes at least 70% in: (a) Written and oral English; (b) record as a teacher; (c) personality. The names of those who attain to a final total of at least 70% are placed on an eligible list, arranged according to rank, from which appointments are made in order of their rank as vacancies occur in the position for which the candidates have respectively qualified. Any name placed upon the eligible list remains on the list without further examination for a period of two years.

Licenses to teach are issued for a period of two years. No one is licensed unless he passes a physical examination conducted by a physician authorized by the health officer of the District of Columbia.

Colleges and Universities. The following institutions in Washington or the District of Columbia offer courses in mathematics through the calculus, and courses in education contributory to the preparation of prospective teachers in high schools: Catholic University of America, George Washington University, Howard University, Trinity College, and Washington Missionary College. Of these, two, only, offer special courses in mathematics for prospective teachers, and the same two offer courses in projective geometry and history of mathematics. Only two institutions offer courses in descriptive geometry, and but three provide opportunities for practice teaching.

CATHOLIC UNIVERSITY OF AMERICA (*Year Book*, 1920-1921, May, 1920)—Among a dozen courses offered in Education and Psychology are the following: *General psychology, experimental educational psychology, philosophy of education, history of education* and *school administration and management.* There is no course dealing specifically with secondary education.

In mathematics a five-hour year course in trigonometry, advanced algebra, and anayltic geometry leads to a four-hour course in calculus for three semetsers. Among several "graduate" courses offered, is one on *higher geometry* (lectures and problems).

GEORGE WASHINGTON UNIVERSITY (*Bulletin, Catalogue Number*, June, 1921)—Among a number of different courses offered in education in the Teachers College are: *Secondary education* (4 semester hours; a study of the administration and pedagogical aspects of the junior and senior high school); *observation* (2 semester hours; observation of experienced teachers, under guidance, followed by discussions), and *practice teaching* (2 semester hours; teaching under supervision and criticism). The observation and teaching are carried on in the public schools of the city.

For the candidate who wishes to procure a "teacher's diploma," Bachelor of Arts (in Education), 20 semester hours in professional subjects are required. Apart from courses mentioned above, the candidate takes: *History of education* (4 hours), *principles of education* (2 hours), *principles of teaching* (2 hours), *school hygiene* (2 hours), and electives in education (4 hours). Further, the student must complete at least 18 semester hours in a major, and at least 10 semester hours in a minor, in addition to the first *required* course, if any, in the same subject.

For all kinds of Bachelor of Arts degrees 6 semester hours of mathematics are required. The prospective teacher majoring in mathematics takes 24 hours in the subject. This might be taken in connection with the following courses, for example: College algebra, solid geometry, plane trigonometry (3 year-hours), trigonometry, analytic geometry (3 year-hours), differential and integral calculus (3 year-hours), *an elementary course on the theory of equations; the number-system of algebra* (3 semester hours), and *mathematical logic* (3 hours—The nature of mathematical reasoning; the fundamental concepts of mathematics; mathematical systems, consistency, independence, and completeness of postulates; postulates for geometries, algebras, and logic; the algebra of logic). Several other courses in mathematics including a course in descriptive geometry are also offered.

In the Summer School were courses in college algebra, plane and spherical trigonometry, solid geometry, and analytic geometry.

Howard University (*Bulletin, Annual Catalogue* 1920-1921, June, 1921)—This institution for colored students offers a considerable variety of courses in education and mathematics. In the department of education are the following courses: *Teaching of applied and practical mathematics* and *teaching of secondary school mathematics.* The first of these is designed to meet the needs of teachers of mathematics in manual training and industrial schools of secondary grade. Some of the topics are: The use of the transit, level, plane table, slide rule, planimeter; the theory of approximation and errors; the construction of genuine applied problems selected according to the needs of students; graphic solutions; the history of the teaching of mathematics in manual training high schools.

The second course is a study of the pedagogical and educational aspects of secondary school algebra, geometry and trigonometry. Some of the topics treated are: The function of mathematics in the secondary school, methods of arranging and developing the subject matter, modes of presentation, current mathematical literature, typical parts of algebra, geometry, and trigonometry.

Through an arrangement with the city schools the department of education provides opportunity for observation and practice teaching under actual school conditions.

Among a number of courses in the department of mathematics, apart from regular courses through the calculus, are the following: *Descriptive geometry, projective geometry* (college algebra and analytic geometry prerequisite), and *the history of mathematics* (6 units of college mathematics prerequisite). In the second of these courses the treatment is chiefly synthetic in character, and the work is based on Dowling's book. The purpose of the last named course is to outline, in a general way, the historical development of the elementary branches of mathematics, algebra, geometry and trigonometry from ancient times to the present.

Trinity College (*Catalogue*, 1920-1921)—This Catholic institution for the higher education of women offers no courses dealing specifically with secondary education. A certificate is given to those who have attained an average of 75% in the six courses offered together with courses in logic, general psychology, and ethics; and who have had at least 20 hours of observation work in the city schools in connection with the courses: *The psychology of education,* and *general and special methods.*

Apart from semester courses offered in solid geometry, trigonometry, advanced algebra, analytic geometry, theory of equations and determinants, and a three hour course in calculus for three semesters, there are courses in *projective geometry* (3 semester hours), *history of mathematics* (1 year hour; two semesters of calculus prerequisite), and *teachers' course* (2 year hours; two semesters of calculus prerequisite; a critical review of algebra and geometry with a view to modern methods of teaching).

Washington Missionary College (*Calendar*, 1921-22)—Courses are offered in algebra (4 semester hours), trigonometry (4 hours), analytic geometry (4 hours), and calculus (8 hours). Among a number of courses offered in education are: *Secondary education* (2 year hours), and *observation and teaching* (4 semesters, 2 hours each). Each normal student is required to spend one period each day during both the junior and senior years in the practice department, in observation or in actual teaching under careful supervision.

CERTIFICATION IN VIRGINIA

(Population, 2,306,361; area, 42,627).

Virginia issued thirty-six separate varieties of certificates to teachers before 1918, when the number was reduced to seven[31]. Three of these have interest for us: (a) *Collegiate Professional Certificate;* (b) *Collegiate Certificate;* and (c) *Normal Professional Certificate.*

The collegiate professional certificate is the highest grade of certificate provided, and is issued to graduates of a standard four-year college, university, or normal school, whose college work has included fifteen per cent, or nine session hours, devoted to purely professional study and practice teaching. It is a "blanket" certificate qualifying the holder to teach any subject or subjects in any grade of any public school in the state. It is valid for ten years and renewable for a similar period.

The requirements for the collegiate certificate are the same as for the collegiate professional certificate except that there is no requirement for professional study and practice. Its holder is entitled to the same privileges, but this certificate is valid for five years instead of ten, and renewable for ten if, in the meantime, the holder has met the requirement for professional study and practice teaching.

The normal professional certificate is issued to graduates of the two-year professional course in a standard normal school based on the completion of a standard four-year high school course. It qualifies the holder to teach any grade or grades in the elementary school, and, if a graduate of the two-year course offered at Virginia state normal schools for the special training of high school teachers, the holder may receive a special certificate permitting him to teach any or all subjects in the first two years of the high school course. The normal professional certificate is valid for ten years and renewable for a similar period from time to time.

A *"local permit"* may also be issued by a division superintendent on his written request to employ a person who does not hold a state teacher's certificate. It may be terminated at the pleasure of the division superintendent, and in no case is it valid for a period lasting longer than the last day of June following its date of issue.

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[31] The information here and in what follows is taken from Virginia Public Schools, A Survey of a Southern State Public School System. By the Virginia Education Commission, and the Virginia Survey Staff. Part one. Yonkers-on-Hudson, World Book Company, 1920.

In 1917-18, 665 schools in Virginia offered some kind of high school work. Of these only 227 (about one-third) were four-year accredited high schools, 184 (about 28 per cent) were three or four year unaccredited schools, and 194 (about three-tenths) were schools offering less than three years of high school work. In 1918-19 there were 147 schools (nearly three-tenths of all) which had a total enrollment each of not more than fifteen pupils.

In 1916-17, more than one-third of the high school teachers in the state had received less than two years of college education. In 1918-19 only a little more than one-quarter of the non-city high school teachers had received four years of college education; and nearly fifteen per cent had received no regular education above the high school.

ATLANTA, GA.

(Population, 200,616).

According to Rules of the Board of Education in 1920, high school teachers are here grouped as: (1) Principal, (2) Assistant principal, (3) Heads of departments, (4) Honor group of associates, (5) Associates. To qualify as an associate a candidate must have: (a) A baccalaureate degree from a standard college; (b) equivalent of 11 semester hours in education; (c) three years of successful experience in junior or senior high schools. Those in the honor group of associates must have: (a) A superior degree from a standard university; or (b) six summers of certified work in at least two subjects per summer in a standard school; or (c) unusual skill, activity, and power in teaching as shown by marked and notable results. This type of teacher without the qualification of (a) or (b) would not be considered for the honor group until the end of the third year in the Atlanta system. Excellent teaching must be an accompaniment of (a) and (b). Heads of departments are recommended by the principal and chosen if possible from the honor group of associates. Principals must have: (a) A Master's degree from a standard college; or (b) long service proving special aptitude for the work of principal; (c) special training in supervision and administration.

There are no special legal requirements for a teacher of mathematics as distinguished from a teacher of some other academic subject.

THE JOHNS HOPKINS UNIVERSITY [32]

In this university a number of courses in education are given in both the graduate and the undergraduate departments. The degree of Bachelor of Science, established in 1915, is offered to those completing the required program of studies in the College Courses for Teachers. The program requires 120 points with a minimum of 24 points in the college courses for teachers. All students are required to take: (a) Two courses in English, three courses in at least two foreign languages, one course in history, and one course in science; (b) a major consisting of three courses in one subject, and of two courses in one or two cognate subjects. The required courses may not be counted as part of the work in the major.

The State Board of Education has provided that the minimum amount of college work in education which will meet the requirements of the state law shall include: History of education, educational psychology or principles of education, principles of teaching and special methods in secondary school subjects, and secondary school organization and class-room management. Courses in all of these subjects are given in the undergraduate department of Johns Hopkins. One graduate course in secondary education is also offered.

Graduates may qualify for the *Maryland State High School Teacher's Certificate* in such an academic subject as mathematics, provided "that during their college course at least two high school branches were pursued for two years", and "provided that they had not less than two hundred recitation hours of instruction in education, including the aims of secondary education, and in the methods, observation and practice teaching of high school studies." Opportunity for such preparation is provided in connection with university work.

No courses in mathematics are offered primarily for teachers in secondary schools, but a large number of other courses, including those in projective geometry and descriptive geometry, provide opportunity for advanced work.

In the summer session, no elementary course is offered by the department of mathematics, but more than a dozen courses are offered in secondary education. These include: (a) *The teaching of algebra and geometry in secondary schools;* (b) *the teaching of mathematics in the junior high school.* In the first of these courses the following topics are considered: Educational values and place in the curriculum, courses adapted to senior high schools, discussion of the merits of recent and of well-established texts, class-room methods and practice; important reports and references to journals, diagnostic tests and their use. Each student is requested to be provided with some high school algebra and geometry and with one of the following reference books: D. E. Smith, *The Teaching of Elementary Mathematics;* J. W. A. Young, *The Teaching of Mathematics;* A. Schultze, *The Teaching of Mathematics in the Secondary School.*

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[32] The Johns Hopkins University Circular; (a) University Register, 1920-1921; (b) Summer Courses July 5-August 12, 1921.

UNIVERSITY OF NORTH CAROLINA[33]

In North Carolina, the requirements for a high-school teacher's certificate in mathematics without examination call for: (a) Sixty semester hours of standard college work above high-school graduation, and three hours of general professional work; (b) at least twelve semester hours of college work in mathematics and English.

The course for Bachelor of Arts in Education at the State University requires 27 courses in the college of liberal arts and 9 courses in the school of education. Of the latter not less than two courses must be selected from the group in principles of education; not less than two from the group in methods; and at least one from the group in history of education. All students must take the courses: *Principles of secondary education* (5 term hours), and *General methods in secondary education* (5 term hours).

In the department of mathematics apart from regulation courses in college algebra, trigonometry, analytic geometry, calculus, descriptive geometry, and projective geometry, there are not only other courses in pure and applied mathematics, but also courses for prospective teachers in public schools. The course on *Principles, history, and methods of teaching elementary mathematics* (3 hours for two terms—courses in college algebra, plane and spherical trigonometry prerequisite), reviews high school and college mathematics with special emphasis on the essentials; the inherent relationship between arithmetic, algebra, and geometry is fully exhibited; and investigation and study is made of the methods of instruction in the best modern high schools, and of conditions as they exist in North Carolina.

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[33] The University of North Carolina Record: (a) The Catalogue, 1920-1921, Announcements, 1921-1922, March, 1921; (b) The Summer School, June 21-August 4, 1921, April, 1921.

The course *Introduction to modern higher algebra* (5 term hours) is designed for those who desire further to pursue the study of algebraic methods; and in particular for those who purpose to become teachers. Selected topics of higher algebra are studied, including determinants, polynomials, matrices, invariants, covariants and quadratic forms.

There is also a course on *The foundations of geometry* (5 hours, for two terms—courses in analytic geometry, elementary calculus, and projective geometry prerequisite) which is designed to meet the need of students experienced in mathematical thought-processes, who wish to lay broad and solid the basis of geometrical knowledge. It includes a survey of the contributions of Pasch, Peano, Hilbert, and the modern school to our knowledge of the foundations of geometry, study of the axioms, formation of systems of geometry, etc.

In the summer session, various courses are given in education, and in mathematics through the calculus. A course *The teaching of mathematics in the high school* is a study of the content and arrangement of the material of high school mathematics in accordance with the recommendations of the National Committee on Mathematical Requirements.

The course in *analytic geometry,* offered primarily for teachers of high school mathematics, aims to coördinate algebra and geometry—since intelligent teaching of graphic algebra requires a knowledge of coördinates, equations of loci, the straight line, the conic sections and other curves.

Another course on *advanced plane geometry* is intended to give teachers an opportunity to gain a wider knowledge of the field of Euclidean geometry than is possible in the high school classes. It consists of a brief review of the methods of elementary geometry followed by an introduction to projective geometry. No preparation was presupposed beyond the usual high school courses in plane and solid geometry.

UNIVERSITY OF WEST VIRGINIA[34]

Two thirds of the teachers employed in high schools of the first class in West Virginia must be graduates of standard colleges or universities; the other third must have completed two full years of standard college work. It is expected that all high school teachers shall be especially prepared in subjects they teach, but there is no legal requirement setting forth the amount of preparation teachers of mathematics must have in order to be licensed to teach in the high schools of the state.

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[34] West Virginia University Bulletin: (a) Catalogue and Announcements 1919-1920, July, 1920; (b) Summer School, June 13-July 23 and July 23-August 27, 1921, April, 1921.

The numerous courses in education at the University are open only to juniors and seniors except *Principles of study* (3 semester hours; primarily for freshmen), and *psychology of learning and teaching* (3 hours; open to sophomores). Other courses are the following: (a) *Philosophy of education* (3 hours); (b) *educational psychology* (3 hours) which are prerequisite to (c) *the principles of teaching* (3 hours—required of all students who are candidates for recommendation for teachers' certificates), (d) *directed teaching* (2 hours required for recommendation for teachers' certificates), and (e) *problems of secondary education* (2 hours). Graduates from the University, with not less than 20 semester hours of approved courses in education, and who have demonstrated their fitness by successful teaching in courses (c) and (d), are recommended for high school certificates.

The department of education advises those preparing to teach academic subjects to fit themselves to give instruction in as many as three subjects. Courses are offered in: Solid geometry (3 hours), advanced algebra and trigonometry (5 hours), trigonometry and analytic geometry (5 hours), and calculus (6 hours). For those who major in mathematics the following courses are also required: *Differential equations* (3 hours), *theory of determinants and analytic geometry of space* (2 hours), *advanced topics in calculus* (2 hours) and *projective geometry* (3 hours). Numerous other courses in advanced mathematical work are offered. *Descriptive geometry* (3 hours) is taught in the college of engineering.

In the summer school there are no courses in education dealing primarily with matters relating to senior high schools. In mathematics various courses through the calculus are offered as well as a course in the *mathematical theory of investment* and several advanced courses.

D. SOUTH CENTRAL DIVISION

(Alabama, Arkansas, Kentucky, Louisiana, Mississippi, Oklahoma, Tennessee, Texas)

In this Division we have selected the state of Kentucky for special consideration. While here as in other states of the Division, the state law requires of its high school teachers only two years of college work after graduation from a four-year high school, the facilities provided for amplifying such training, in the case of the prospective teacher of mathematics, seem more highly organized. Outside of this state the cities of Birmingham, Ala., and Dallas, Texas, appear to have standards as high as those of any of the seven other cities in the Division with not less than 100,000 inhabitants each.

KENTUCKY

(Population, 2,416,013; area, 40,598 square miles).

State Requirements. The candidate as teacher in a high school must give satisfactory evidence of scholarship equivalent to graduation from a four-year standard high school and, in addition thereto, the equivalent of two years' work in college or normal school, a part of which should be in professional or pedagogic subjects. Exceptions to this requirement may be made with those who teach and supervise in second and third class high schools [35] when one year in college or normal school in addition to high school graduation may be accepted as satisfactory, temporarily.

A candidate may also become a teacher in a high school on the basis of a special high school certificate obtained by examination in English, mathematics (arithmetic, algebra, and geometry), U. S. history, civil government, and pedagogy.

Many school boards (including those of nearly all the classified cities and towns of the state) require that the candidate as teacher in a high school shall have graduated from a college or university requiring a four-year high school course for entrance, and offering a four-year course for graduation, and a certain amount of theoretical professional training. Such a candidate must hold one of the following four certificates entitling him, without examination, to teach in any public school of the state: (i) *Elementary Teacher's Certificate* (good for two years); (ii) *Intermediate Certificate* (good for four years); (iii) *Advanced Certificate* (good for three years); and (iv) *Life Certificate*. It is stated that the holder of each of these certificates must have been a graduate of a college or university, as indicated above.[36] The first of these certificates, which may be granted on the completion of one year of study in the department of education of the state university, for example, calls for courses equivalent to not less than three year-hours in extent and including instruction in history of education, principles of education, principles of pedagogy, and special study of secondary education.—The second certificate, granted after two years of study in a department of education, requires the student to have taken an equivalent of not less than six year-hours in educational subjects including special study of secondary education, educational psychology, history of education, methods in education, and high school administration. One-fourth of this work may be offered in practice teaching under the direction of an experienced supervisor.—The third certificate is granted after three years of work in professional and pedagogic subjects equivalent to twelve year-hours (one-third for principles of education, one-third for history of education, and one-third for secondary education). At the end of three years a Life Certificate may be substituted for an Advanced Certificate, on the presentation of satisfactory evidence of his teaching ability and moral character.

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[35] Second class high schools are those maintaining a three-year course; third-class, those maintaining a two-year course.

A candidate as teacher of mathematics in high schools of LOUISVILLE (the only city of the state of over 100,000 population, 234,891) must have at least an A. B. degree, and one year of successful teaching experience,—not necessarily in mathematics. Further, the candidate must have had mathematics as either a major or a minor subject while taking the A. B. course.

Colleges, Universities, Normal Schools. Of the ten colleges or universities of the state seven have departments of education for preparing students to obtain such certificates as are described above. These are: University of Kentucky, Georgetown College, University of Louisville, Berea College, Centre College, Transylvania College, and Kentucky Wesleyan College. Of these the first three named offer special facilities for the preparation of the prospective teacher of mathematics in secondary schools.

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[36] This statement is on the authority of Kentucky Department of Education, Rules and Regulations, June, 1916, page 1. It seems to be contradicted by state normal school catalogues (see later).

UNIVERSITY OF KENTUCKY[37]—Prospective teachers are advised to consider the following seven semester courses in mathematics as a minimum preparation for their work: Solid geometry (3 semester hours), plane trigonometry (5 hours), college algebra (5 hours), analytic geometry (5 hours), differential calculus (5 hours), integral calculus (3 hours), spherical trigonometry and the method of least squares (2 hours), Teacher's course (3 hours; history and pedagogy of mathematics, based principally on J. W. A. Young's *Teaching of Mathematics* with a number of other books for reference and report work. There is also a certain amount of practice and observation work in the model high school at which there are about 120 pupils). Numerous other mathematical courses, including those in *projective geometry*, and *descriptive geometry*, are also offered. The department of education is a training school for teachers and among many courses offered are those in the philosophy and history of education, educational psychology, administration and method. A practice high school maintained in connection with the university affords practice teaching for advanced students (juniors and seniors) under the direction of trained teachers. Among courses offered in the summer session were the following: history of education, educational psychology, solid geometry, trigonometry, college algebra, analytic geometry, differential calculus, and integral calculus.

GEORGETOWN COLLEGE[38]—Prospective teachers of mathematics are informed that they should take the following courses as a minimum preparation: solid geometry (3 semester hours), college algebra (3 hours), plane trigonometry (3 hours), plane and solid analytic geometry (3 year hours), mathematics of investment (3 year hours), differential calculus (3 semester hours), integral calculus (3 hours), and teacher's course (3 hours; this course deals with some of the pedagogical problems encountered in secondary algebra and geometry such as their educational value, position in the curriculum, methods of teaching, comparison of American methods with those of foreign countries, order and importance of topics, text-books, literature, etc.).

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[37] Authority: Bulletin of the University of Kentucky: (a) Catalog, 1919-1920; (b) Summer Session, 1921, June 20-July 30.

[38] Authority: Georgetown College Bulletin, Catalogue Number 1920-1921.

Among several other mathematical courses offered is a semester course in projective geometry. Theoretical, but no practical, professional training is offered in the department of education.

UNIVERSITY OF LOUISVILLE[illegible].—Apart from regular courses through the calculus two half-year-courses are offered in the history of mathematics and mathematical pedagogy. The first of these courses consists of a series of lectures intended to be useful not only to the student and teacher of mathematics but also to the general public. It touches upon the development of mathematics from the time of the Greeks down to the present, considerable attention being paid to the relation of mathematics to philosophy, particularly to the great problem of [illegible]". Theoretical and practical professional training are carried on in connection with the Louisville Normal School.

THE EASTERN KENTUCKY STATE NORMAL SCHOOL (*Announcements of Courses*, 1920-1921, October, 1920, page 13) and the WESTERN KENTUCKY STATE NORMAL SCHOOL (*Catalog*, 1920-1921, page 37) each provide courses of two years, for pupils who have graduated from a four-year high school, leading to the Advanced Certificate (see Kentucky above). In the former the regular mathematical courses are (pages 77-80): Arithmetic, algebra, geometry and trigonometry; college algebra, analytic geometry, spherical trigonometry, and a geometry course intended for prospective teachers are also offered. This last named course emphasizes method, giving special attention to original problems and theorems. It includes also the more difficult portions of the model demonstrations of the text especially those dealing with indirect proof and methods of limits. In the Western Kentucky State Normal School calculus is also taught in the third and fourth terms of the second year.

BIRMINGHAM, ALA.

(Population, 178,270).

An applicant for a position as head of a department of mathematics in a high school must be qualified as follows: (a) He must be a graduate of a standard college or university, "which requires for admission the equivalent of a good high school course". (b) He must

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[illegible] Authority: University of Louisville Bulletin, Catalogue Number, Announcements 1920-1921.

present evidence that he has had two years of graduate work in a university, with mathematics as a major topic. In lieu of graduate work, experience of not less than three years in teaching the subject, together with summer work at a university, may be accepted.

To qualify for the position of Instructor in a mathematics department the applicant must be a graduate of a standard college or university with two years of teaching experience subsequent to graduation.

DALLAS, TEXAS

(Population, 158,976).

A candidate as teacher of mathematics in a high school is required to be a "graduate of a standard college, and to have had two years of successful experience as a teacher." There is no fixed requirement as to the amount or kind of previous mathematical study or professional training.

E. Western Division

(Arizona, California, Colorado, Idaho, Montana, Nevada, New Mexico, Oregon, Utah, Washington, Wyoming.)

In this Division we select for consideration the state of California, which is, on the whole, notably higher in its standards for the preparation of teachers for high schools than any other state of the Division. Outside of this state the city of Portland, Oregon, seems to have standards as high as those of any of the four cities of not less than 100,000 inhabitants.

CALIFORNIA

(Population, 3,426,536; area, 158,297).

State Requirements. A candidate for a certificate to teach in a high school (a) must have received [40] a bachelor's degree from a standard college requiring not less than eight years of high school and college training; (b) must have submitted evidence that in addition to the academic and professional courses required for the bachelor's degree, he has completed at least one year of graduate study, 26 units (semester hours), doing full regular work, though not necessarily a candidate for a degree, in an approved teacher training institution.[41]

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[40] Here, and later, the regulations of October, 1920, are quoted.

[41] In defining "approved teacher training institutions" the regulations contain the following statement: "The California State Board of Education has accredited for the purpose of granting recommendations for the high school credential

Such graduate study shall include at least one full year course of advanced or graduate work in at least one of the subjects taught in the high school. (c) He must also submit evidence that he has completed in undergraduate or graduate standing, or the two combined, not less than fifteen units of work, in courses listed in the department of education in the institution in which the graduate work is completed, or courses in other departments of that or other institutions accepted as preparation for teaching by the department of education.[42]

The required fifteen units of work in the department of education must include the following courses: (i) School and classroom management, or equivalent work (1 unit, minimum); (ii) work in actual practice teaching, with conferences[43] (4 units, minimum); (iii) a teacher's course[44] in at least one subject in which the candidate expects to be

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the following schools of education of California universities and colleges; and schools or colleges of education, or (for institutions in which neither such exists) departments of education, of other universities or colleges which belong to the Association of American Universities or the Association of State Universities, and which have been approved as offering graduate academic and professional preparation equivalent in quality to that offered by the school of education of the University of California." (Here follows a list of 5 California, and 25 other, institutions.)

"Recommendations for the California secondary credential may be issued by teacher training institutions, recently accredited for the purpose by the California State Board of Education, only to persons who were graduated within eight years of the date of issuance of the recommendation.

"Persons who were graduated from these accredited institutions more than eight years prior to such date may apply directly to the State Board of Education for a California secondary credential, provided that they have had successful teaching experience of at least seventeen months."

[42] Candidates who have taught for not less than seventeen months in California, or any other state, and who present credentials showing that they have done so with reasonable success, may be excused from one-half of the year of graduate work laid down in (b), and all of the work in (c).

Similar exemption applies to graduates of California state normal schools accepted by the State Board of Education as of equal rank, except as to the course in education, and the teacher's course or courses in the special subject or subjects offered by the candidate for certification, if such be offered in the institution.

Again a candidate may complete one-half of the year of required postgraduate work in an approved teacher training institution, and the other half-year in any California state normal school which offers a special half-year (two terms) of full-time instruction, especially selected for college graduates preparing for high school certification in California, and approved by the State Board of Education, and be entitled to the same exemption as the normal school graduate referred to in the last paragraph.

Finally, the holder of a bachelor's degree granted by the University of California, Leland Stanford Junior University, the University of Southern California, Pomona College, or Mills College, who has completed all the required work in education except practice teaching and classroom management and who has also completed one major and one minor in subjects taught in the high schools, may, with the approval of the head of the department of education of the accredited institution granting the recommendation, elect to spend at least one year in a California state normal school devoting full time to special preparation for teaching a subject in which such a normal school is authorized to grant a recommendation for the special certificate of secondary grade.

[43] Candidates who have taught for not less than one year (eight months) in any elementary or secondary school, and who present credentials showing that they have done so with reasonable success, may be excused from 4 of the 15 units of education represented by practice teaching, and may also be allowed to substitute other work in educaton for the course in school and classroom management.

[44] The regulations note that each "teacher's course" shall be a bona fide teacher's course and shall be made as concrete and practicable as possible, and shall have for its purpose the preparation of teachers to give intelligent instruction in the subject in the high schools of the state.

recommended for certification, if such course be given in the institution and be accepted as or listed under, the work in education (3 units, maximum, for all such courses); (iv) A course in secondary education, presenting particularly the purpose and attainable goals of high school work (2 units, minimum); (v) such other courses relating to the theory, function and administration of education, as are necessary to complete the required 15 units.

It is required that the practice teaching shall be done under the general supervision of the department of education of the college or university in which the year of graduate work is taken, and may be done in schools of elementary or secondary grade, although preferably in secondary school work of the kind the candidate is preparing to teach, and under the direction of competent instructors in such work. The work in practice teaching may also be done in connection with the training school of any California state normal school, of which there are six (at Chico, Fresno, San Diego, San Francisco, San Jose, and Santa Barbara). There is nothing of special interest for prospective teachers of mathematics offered at these normal schools.[45]

When the graduate work required for the recommendation for the high school certificate is completed to the satisfaction of the department of education of the institution in which such work is done, the accredited school or college or department of education prepares a formal recommendation to county, or city and county, boards of education, stating that the candidate has satisfactorily completed all the requirements established by the State Board of Education for recommendations for the high school certificate in California, and that, in the judgment of the school or college or department of education issuing the recommendation, the candidate is worthy to receive a high school certificate in any county, or city and county, of the state.

Each recommendation states on its face the major and minor subjects[46] which the holder thereof has taken in the institution or the kind of work the candidate is qualified to do.

In connection with the above statement of what at first may seem to be very high standards, it should be borne in mind that the law does *not* require that the teacher shall teach either his major or his minor subjects. This is entirely a matter for local authorities to determine. The teacher of mathematics in a California high school may, therefore, so far as the law is concerned, have had no direct preparation whatever for teaching his subject. At present there is in the state an insufficient supply of well-trained teachers.

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[45] Except, possibly, in the case of the San Francisco normal school (the catalogue of which has not been seen).

[46] A major is defined as not less than 24 units of work in a subject, of which 12 units must be in advanced courses (that is, courses taken in the junior, senior, and graduate years). A minor is defined as not less than 12 units of work in a subject, of which 6 units must be in advanced courses.

Colleges, Universities, Institutes. There are 14 institutions of higher learning in California: 9, coeducational; 4, for men only; and 1 for women only. Two, at least,[47] of those for men, California Institute of Technology and St. Mary's College, offer no course in the theory of education, and no course in mathematics primarily for the prospective teacher in the high school. In six other institutions at least a few courses in the theory and history of education, and in mathematics through calculus, are offered; in four of these there are courses in descriptive geometry; in two of them, courses in projective geometry; and in one, the College of the Pacific, two two-unit courses: History of mathematics and teachers' course. The first of these, open to students who have taken work in mathematics for at least two years, is "a general culture course in the development and progress of the more recent mathematical theories." The second course, for which the first is a prerequisite, gives practice in teaching both algebra and geometry, provides a rapid review of algebra, includes study of Euclid, of methods of teaching geometry, and treats of leading text books on plane geometry.

The schools of education of the remaining five institutions have been already listed (University of California, University of Southern California, Pomona College, Mills College, and Leland Stanford Junior University) as those accredited by the California State Board of Education for the purpose of granting recommendations for high school certificates.

UNIVERSITY OF CALIFORNIA[48]—The School of Education publishes the following statement: "The preparation of teachers of mathematics in secondary schools should include substantial courses in analytic geometry, differential and integral calculus, projective geometry, theory of equations, the history and teaching of mathematics, elementary physics, and surveying. The teacher should be familiar with modern elementary researches, as well as the fundamental concepts of algebra and geometry with such modern applications of elementary mathematics to sciences and arts as will be of interest and value to high school pupils. The teacher should possess the ability to perform rapidly and accurately elementary computations and transformations and to express himself clearly in oral exposition. The teacher should have formed the habit of reading the current numbers of standard journals on the teaching of elementary mathematics."

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[47] The catalogue of St. Ignatius University, also for men, was unprocurable.

[48] Authorities: University of California Bulletin: (a) The School of Education Announcement for 1920-21, September, 1920; (b) Announcement of Courses, 1920-21, July, 1920; (c) Intersession, May 9-June 18, and Summer Session June 20-July 30, 1921, April, 1921; (d) Summer Session Los Angeles, June 20-July 30, 1921, March, 1921.

The following courses are planned especially for candidates for the recommendation for the high school certificate:

(a) *Elementary geometry for advanced students* (3 semester hours), for which a five-hour course in analytic geometry and a three-hour course in projective geometry are prerequisite.
(b) *Elementary algebra for advanced students* (3 semester hours), for which the course in analytic geometry and a five-hour course in calculus are prerequisite.
(c) *History of mathematics* (2 year hours), a non-technical course, open to students who have some knowledge of the fundamental ideas of analytic geometry and calculus.
(d) *Logic of mathematics* (2 year hours), analysis of the foundation principles of geometry and algebra.
(e) *Teaching of mathematics in secondary schools* (3 semester hours), a course for seniors and graduate students on the history of methods of teaching mathematics and a critical inquiry into present day tendencies.

In addition to these the candidate must take one semester course, such as Advanced calculus, or Theory of algebraic equations. A large number of other mathematical courses are offered.

At the University of California the School of Education makes both general and specific requirements in connection with those seeking recommendations for high school certificates.

The general requirements are: (a) That the candidate has a command of spoken English adequate to purposes of instruction; (b) that the University physician presents a satisfactory certificate of the candidate's health; (c) that the candidate take the oath of allegiance before a notary public; (d) that the minimum amount of work for two semesters is 18 units, and for one semester 12 units; (e) that all schedules are passed upon by the director of the school of education before application is made for the high school certificate.

Apart from requirements of the State Board there are certain specific academic requirements of the school of education: (a) A graduate course, other than a teacher's course, of not less than 2 units, in a department in which the candidate has completed not less than 12 units of major courses, or an equivalent. (b) A major and three minors[49], or two majors may be accepted. In every case, however, two of the following divisions of university studies must be represented: (1) English; (2) ancient foreign language; (3) modern foreign languages; (4) philosophy, education; (5) history, political science, anthropology; (6) jurisprudence; (7) economics; (8) mathematics, astronomy, physics, chemistry; (9) geography, geology, mineralogy, palaeontology; (10) botany, zoölogy, anatomy, physiology; (11) pathology, bacteriology, hygiene, physical education; (12) architecture, drawing, art; (13) music; (14) agriculture; (15) engineering; (16) house economics (household science and household art); (17) commerce.

Among various 2-unit intersession courses offered are: The principles of secondary education; integral calculus; plane analytic geometry; elementary algebra for advanced students; higher geometry. Among 2-unit courses in the summer session there are: The history of education; the history of secondary education; graphic algebra; mathematical theory of investment; introduction to calculus; elementary geometry for advanced students; history of mathematical science; theory of algebraic equations, and mathematics of economics. In the summer session of the Southern Branch of the University of California such courses as the following are offered: Problems of secondary education, elementary astronomy, plane analytic geometry, and elementary algebra for advanced students.

Less elaboration of opportunity is in evidence at the other institutions of the group we are considering. At the *University of Southern California*[50] the department of mathematics requires for its approval of a candidate's "recommendation", that he successfully complete 10 units (including differential and integral calculus) for a minor, and 30 units for a major. Among the units required for a major are six three-hour courses (theory of algebra, solid and spherical geometry, trigonometry, plane analytic geometry, differential calculus, and integral calculus), a two-hour course in *advanced trigonometry* and a two-hour *Teachers' Course* (lectures on teaching mathematics and a review of elementary mathematics from the standpoint of preparing teachers for secondary schools; the history of mathematics is also studied). Six more units may be selected from the following unit courses: *Descriptive astronomy, theory of numbers, mathe-*

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[49] A major comprises not less than a total of 24 units and not more than a total of 30 units of which at least 12 consists of major or graduate courses in one department of 12 units of correlated major or graduate courses in more than one department. A minor comprises not less than 6 units of major courses in one department, which ordinarily implies not less than 12 units including prerequisites.

[50] Authorities: Bulletin: (a) College of Liberal Arts Year Book for 1920-1921, March, 1921; (b) Summer Session, June 20-July 29, 1921, April, 1921.

matical theory of investment, advanced analythic geometry and differential equations, and *selected topics in mathematics.* The last named course includes the study of certain phases of advanced mathematics which have points of contact with the elementary field and the treatment of topics in the foundations of geometry, modern pure geometry, non-Euclidean geometry, the fundamental propositions of algebra, the algebraic equation, the function concept and the fundamental notions of the calculus, the theory of numbers, constructions with ruler and compasses, and the history and transcendence of π.

The department of mathematics does not offer any course in projective geometry.

In the summer school the seven two-unit mathematical courses offered included *differential calculus, selected topics in mathematics* and *teachers' course.*

At POMONA COLLEGE apart from regular courses through the calculus there are a year course in *projective geometry,* and semester courses in *theory of equations* and *projective geometry.* For high school certification the department of mathematics requires 26 units in mathematics (beyond the more elementary courses of solid geometry, trigonometry, and college algebra): 12 in analytic geometry and calculus; 6 hours in theory of equations, and differential equations or higher analysis; 6 hours in projective geometry or other modern geometry; and 2 hours in the teaching of mathematics. In addition a working knowledge of either French or German is required, also 6 units of college work in physical science.

At MILLS COLLEGE for women a semester of *synthetic projective geometry* is placed in the lower division of courses; differential calculus, integral calculus and theory of equations are among the seven courses in the upper division; *the teaching of mathematics in secondary schools* and *lectures on the history of mathematics,* are the two graduate courses in the division.

At STANFORD UNIVERSITY, the mathematical requirements for a high school teacher's recommendation are much lower than at the University of California. For a major the requirements are[51]: Trigonometry (3 1/3 units), solid geometry (2⅔ units), algebra (3 1/3 units), coördinate geometry (6⅔ units), differential and integral calculus (3 1/3 units), plane geometry (3 1-3 units); for a minor the same courses except calculus are required. Courses in *projective geometry* (3⅓ units), and *Non-Euclidean geometry* (2⅔ units) are also offered.

The course in plane geometry is intended to extend the geometrical horizon both of mathematical students and prospective teachers. In connection with the course a 13 page pamphlet syllabus of plane and solid geometry is used. Some of the headings are: General properties of space, manifoldness, examples of conditions determining figures, axioms, postulates, classification of theorems, theory of parallels, projection, centroid, loci, application of fundamental laws of algebra, symmetry, principle of duality, tangents by methods of limits, orthogonal circles, nine-point circle, regular inscribed polygons, radical axis and radical center, proportion, similar figures. In a similar way the synopsis for the course in solid geometry is from a higher stand-point.

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[51] Stanford University Bulletin: (a) Register 1920-21, August, 1921; (b) Summer Quarter, 1921, March 15, 1921.

In the summer quarter apart from an advanced course in mathematics, the following were given (each for 2⅔ units credit): Algebra, coördinate geometry, trigonometry, and calculus.

Cities. Of the three cities in California with more than 100,000 inhabitants, the largest is *Los Angeles* (population, 576,678). The system of selecting teachers there is declared to be based solely upon merit, which is determined by written and oral examination of applicants who hold the state high school certificate. In this way an eligible list of prospective teachers is prepared after special consideration of their "education, experience, personality and efficiency, and comprehension and attitude regarding educational problems." The eligible list is revised and reconstructed after each examination and candidates' names are ranked in the order of the excellence of their grades. All appointments of candidates are made in the order in which their names appear upon the eligible list.

PORTLAND, OREGON

(Population, 258,288).

In Oregon, one-year state certificates may be issued upon application, to *any* graduate of a standard college or university (that is, such as is standardized by the Bureau of Education, Washington, D. C.). The holder of a one-year state certificate may, after six months of successful teaching experience in the state, and upon recommendation of the county superintendent of the county in which the applicant last taught, receive a five-year state certificate. The holder of a five year state certificate may, after thirty months of successful teaching experience in this state and upon the recommendation of the county superintendent of the county in which the applicant last taught, receive a state life certificate. Each of these certificates entitles the holder to teach in any high school of the state. Into the question of others who may also teach in high schools, we shall not enter.

In Portland all applicants for positions in the high schools are required to have at least two years of teaching experience. The teachers are not appointed on the competitive examination plan employed by many cities; they are elected by the Board of Education upon the recommendation of the Board of Superintendents. Prospective teachers of mathematics are not required to present credentials differing from those of prospective teachers in other subjects. The following "good rule of action" recommended by Professor Cubberley[52] is thus disregarded: candidates as teachers in city high schools should be graduates from a college or university of standing, and furnish evidence of special preparation for the particular work done.

III. Accrediting Agencies

A comprehensive survey of agencies in the United States which tend to improve the standards for the training of teachers in secondary schools should not omit reference to the following organizations:

The Association of Colleges and Preparatory Schools of the Middle States and Maryland[53] (organized in 1888 and embracing the six states: Delaware, District of Columbia, Maryland, New Jersey, New York, and Pennsylvania);

The Association of Colleges and Secondary Schools of the Southern States[54] (organized in 1895, and embracing the thirteen states: Alabama, Arkansas, Florida, Georgia, Kentucky, Louisiana, Mississippi, North Carolina, South Carolina, Tennessee, Texas, Virginia, West Virginia);

The New England Association of Colleges and Secondary Schools (organized in 1884 and embracing the six states; Connecticut, Maine, Massachusetts, New Hampshire, Rhode Island, Vermont);

The New England College Entrance Certificate Board[55] organized in 1902);

The North Central Association of Colleges and Secondary Schools[56] (organized in 1892 and now embracing the eighteen states: Arizona, Colorado, Illinois, Indiana, Iowa, Kansas, Michigan, Minnesota, Missouri, Montana, Nebraska, New Mexico, North Dakota, Ohio, Oklahoma, South Dakota, Wisconsin and Wyoming);

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[52] The Portland Survey. A textbook on city school administration based on a concrete study. Yonkers-on-Hudson, 1915. A Report of Supplementary Survey of Portland Public Schools, by P. W. Horn, was published in April, 1917.

[53] A volume of the Proceedings of the thirty-fourth annual convention, held in November, 1920, was published in 1921.

[54] A volume of the Proceedings of the twenty-fifth annual meeting held in December, 1920, was published in 1921.

[55] The seventeenth and eighteenth Annual Reports were published together in 1920.

[56] Two Bulletins, published by the Bureau of Education, Washington, deal with the field of activity of this Association. These are: (a) A Study of the Colleges and High Schools in the North Central Association, Bulletin, 1915, No. 6; and (b) The Accredited Secondary Schools of the North Central Association, Bulletin, 1919, No. 45.

In Accredited Secondary Schools in the United States (Bulletin 1916, No. 20) was given a complete list of such schools in 1916.

The Northwest Association of Secondary and Higher Schools (embracing the states: Idaho, Oregon, Washington).

These associations include all of the states except California, Nevada, and Utah. In California exceptional standards in connection with its high schools are maintained by the State Board of Education. In Nevada the accrediting agency is the State University. In Utah the standardizing agency is its Department of Public Instruction.

The New England Association of Colleges and Secondary Schools is the original organization of its kind. In connection with secondary schools its purpose is "to establish high standards of administration and instruction in conformity with accepted practice among the best secondary schools of New England, and thus to coöperate in the development of a sound public policy in education." The purpose of the New England College Entrance Certificate Board is to receive, to examine, and to act "upon all applications of schools in New England that ask for the privilege of certification to the colleges belonging to the Board." The other "Associations" prepare lists of high schools which satisfy the standards which they have formulated and it seems probable that the New England Association will soon adopt this policy.

Although the Associations have no legal status, the power which they wield may be very great. This is notably true of the North Central Association. In formulating their standards for high schools the Associations take account of such things as: Purpose of the school, instruction, program of studies, qualifications of teachers, organization and administration, equipment, health, and buildings and grounds. What is implied in this connection may, perhaps, best be illustrated by quotation from "Standards" adopted by: (I) the North Central Association for 1921-1922, and by (II) the New England Association of Colleges and Secondary Schools in December, 1921.

I. North Central Association

A. *Buildings.* 1. The location and construction of the buildings, the lighting, heating, and ventilation of the rooms, the nature of the lavatories, corridors, closets, water supply, school furniture, apparatus, and methods of cleaning shall be such as to insure hygienic conditions for both pupils and teachers.

All schools whose buildings are inexcusably inadequate and lacking in modern equipment may expect to have North Central Association accrediting privileges withheld from them.

2. The laboratory and library facilities shall be adequate to the needs of instruction in the subjects taught. The Association recommends the appointment of a trained librarian for each high school employing ten or more teachers.

B. *Organization and Administration.* 3. No school shall be accredited which does not require fifteen units for graduation. The school year shall consist of a minimum of thirty-six weeks. [A unit course of study in a secondary school is defined as a course covering an academic year that shall include in the aggregate not less than the equivalent of 120 sixty-minute hours of class room work, two hours of shop or laboratory work being equivalent to one hour of prepared class room work. More than twenty periods per week of academic subjects, or twenty-five periods including vocational subjects, exclusive of choral music and physical training, should be discouraged except in the cases of pupils having more than average ability.]

4. The efficiency of instruction, the acquired habits of thought and study, the general intellectual and moral tone of a school are paramount factors, and therefore only schools which rank well in these particulars, as evidenced by rigid, thorough-going, sympathetic inspection, shall be considered eligible for the list.

5. No school shall hereafter be accredited whose salary schedule is not sufficient to command and retain teachers whose qualifications are such as required by this Association[57]. The interpretation of this requirement shall be a matter of special responsibility for the State Committee.

C. *Preparation of Teachers.* 6. All schools accredited by this Association shall maintain the following standards respecting teachers:

a. The minimum attainment of a teacher of any academic subject, and likewise of the supervisors of teachers of such subjects, shall be equivalent to graduation from a College belonging to the North Central Association of Colleges and Secondary Schools requiring the completion of a four year course of study or 120 semester hours in advance of a standard four year high school course. Such requirements shall not be construed as retroactive.

b. The minimum professional training of a teacher of any academic subject shall be at least eleven semester hours in education. This should include special study of the subject matter and pedagogy of the subject to be taught. Such requirements shall not be construed as retroactive.

The Association advises that the following types of courses should be offered as meeting the spirit of the standard: Educational psychology, principles of secondary education, theory of teaching, special methods in subjects taught, observation and practice teaching, history of education, and educational sociology.

D. *The Teaching Load.* 7. a—The number of daily periods of class room instruction given by any teacher should not exceed five. The Commission will reject all schools having more than six recitation periods per day for any teacher.

b—The minimum length of a recitation period shall be forty minutes exclusive of all times used in the changing of classes or teachers. . . [For schools having some definite plan of supervised study, not more than five classes per day should be assigned to any teacher. The Association advises that the maximum be four.]

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[57] At its annual meeting in Chicago, March 18, 1921, the North Central Association "Resolved, that,

(a) It is unalterably opposed to the lowering of the standards for teachers in the North Central high schools.

(b) The only way these standards can be maintained is through an adequate increase in salaries.

(c) An increase of at least 100% over the high school salary base in 1914-15 is imperative at this time.

8. No school whose records show an excessive number of pupils per teacher based on average attendance, shall be accredited. The Association recommends twenty-five for a maximum. In general, no teachers of academic subjects should be assigned more than 150 student-hours of class room instruction per day, organized so as not to exceed six classes per day.

E. *Program of Studies.* 9. The Association recommends that every accredited school offer units of work in mathematics, social sciences, languages (including English), natural sciences, the fine arts, and physical training. It further recommends the introduction of vocational subjects such as agriculture, manual training, household economics, and commercial subjects into schools where local conditions render such introduction feasible. The Association will hold that a sufficient number of qualified teachers must be provided to care adequately for all instruction offered. Not less than the equivalent of the full teaching time of three teachers may be given to academic subjects.

II. New England Association

Appended to its statement of standards (adopted at the annual meeting, December 3, 1921), this Association rightly remarks:

"In the application of these standards it should not be forgotten that there are certain intellectual and spiritual forces in education which cannot be measured. A school may be so rich in these elements, because, perhaps, of the presence of the personality of the administration or the teaching staff, that, despite grave defects in equipment and organization, the product of the school is good. Even when this occurs, however, the school would be greatly improved by the correction of its defects. On the other hand it is obviously possible for a school to meet most of the standards herein set forth and yet to fail of real effectiveness. Perfection in mechanism and equipment will not ensure educational excellence.

"Emphasis is therefore placed upon the necessity for a sympathetic and intelligent administration of these standards. Your committee has thought best to recommend as standards only what experience and the best usage have shown to be definitely attainable rather than what might seem ultimately desirable. With wise administration these standards, subject to discussion and amendment as occasion requires, should bring about improvement of the schools and should thereby enable the schools to render more effective service."

The standards of *Qualifications for teachers* are formulated as follows:

(a) Teachers shall give evidence of adequate preparation in subjects to be taught.

(b) Teachers of academic subjects beginning service in New England in September, 1922, or thereafter, shall have had at least four years in study in institutions of collegiate grade, or the equivalent. In the opinion of the Association five years of such study is desirable.

(c) Teachers of academic subjects beginning service in New England in September, 1924, or thereafter, shall have had professional training equivalent to twelve semester hours.

The following types of courses are recommended as meeting the spirit of this requirement: General introduction to education, educational psychology, principles of secondary education, teaching of particular subjects, observation and practice teaching, history of education.

(d) Teachers of special subjects beginning service in New England in September, 1921, or thereafter, shall have had at least two years of study beyond the secondary school.

(e) Satisfactory evidence of successful experience may be accepted in partial fulfillment of requirements (a), (b), (c), and (d).

Note: It is understood by the Association that the foregoing requirements do not apply to teachers in service in New England previous to September, 1922.

Under the heading of *Organization and administration* are the following clauses concerning the teacher's work:

A teacher's schedule should not include more than 625 pupil-periods per week.

The schedule shall provide for each teacher an average of at least one period a day free from class instruction. In the opinion of the Association the schedule should provide for one period a day free from class instruction and study hall supervision.

No teacher shall give class instruction for more than six periods in any one day.

A teacher's schedule shall not include subjects from more than three of the following fields of knowledge: English, mathematics, social studies, science, commercial subjects, practical arts, and foreign language (each foreign language counting as a separate field).

Teachers shall be so selected and assigned that they shall, in general, give instruction in those fields in which their preparation is specific and adequate.

IV. A Tentative Standard for the Training of Teachers of Mathematics, and Courses Primarily Intended for such Teachers

We have noted the very wide divergence in the United States of standards regarding the certification of teachers of mathematics in senior high schools. Is it possible at the present time to set up *any* standard that would be generally acceptable? The answer must be unqualifiedly in the negative. Indeed, in states where the standards are legally very high, it has recently been found extremely difficult to secure a sufficient number of teachers with the desired qualifications. (see the remarks of National Committee, pp. 15-18).

There is, however, a general belief that, with public realization of the fundamental importance of secondary education, the economic condition of the secondary school teacher will materially improve and the attractions of his work essentially increase. Looking forward then to a happier period, it seems decidedly advisable to formulate a standard for certification which would be generally desirable if it could be put into effect—a formulation suggestive to colleges and universities throughout the land in raising standards to a reasonable status, even though this be far below that of several European countries. It will be observed that the standard here suggested is practically equivalent to that demanded by one state of the Union acting in coöperation with its state university. It is hoped, therefore, that other states will be the more ready to consider the adoption of a policy approximating the following tentative ideal:

To receive *permanent* appointment as a teacher of mathematics in a senior high school a candidate should satisfy the following requirements, or their equivalent:

1. Graduation from a standard four year college, or university, or from an institution offering courses of at least equal difficulty and educational value.
2. Credit for at least the following mathematical courses (given by teachers of mathematics in colleges or universities):
   (a) Plane and spherical trigonometry;
   (b) Plane analytic geometry and the elements of analytic geometry of three dimensions;
   (c) College algebra (1 semester[58]);
   (d) Differential and integral calculus, with applications to geometry and mechanics (3 semesters);
   (e) Synthetic projective geometry (1 semester);
   (f) Scientific training in geometry (2 semesters—(i) *first semester*: Text, J. Petersen's *Methods and Theories for the Solution of Problems of Geometrical Constructions*[59] and accompanying lectures to present the history of the famous problems, and the history of elementary geometry; (ii) *second semester*: Texts, F. Klein's *Famous Problems of Elementary Geometry* (except chapters on transcendence of e and π), J. W. Young's *Lectures on Fundamental Concepts of Algebra and Geometry* (selected chapters), and, for half the semester, J. Hadamard's *Leçons de Géométrie Elémentaire*, vol. 1, *Géométrie plane*, vol. 2, *Géométrie dans l'espace*, especially the chapters on proportional lines, areas, regular polygons, dihedral and polyhedral angles, polyhedra, cylinders, cones, and spheres, and the notes on Euclid's postulate, notions of area, definitions of volumes, regular polyhedra, and groups of rotations.

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[58] Here, and in what follows, it shall be supposed that the term "semester" implies about 45 lectures or recitations—3 hours a week for 15 weeks.

[59] Specific books are mentioned, here and later, in order to give definiteness to the suggestions. It is to be understood, however, that other texts are not necessarily regarded as less desirable.

(g) Scientific training in algebra (2 semesters)—Lectures on the history of elementary algebra and topics from the following texts: J. W. Young's *Lectures on Fundamental Concepts of Algebra and Geometry* (selected chapters), Fine's *College Algebra* (especially the parts on numbers, fundamental theorem of algebra, h. c. f. and l. c. m., symmetric functions, convergence and divergence of series), F. Klein's *Famous Problems of Elementary Geometry* (chapters on transcendence of e and π).

3. Credit for at least the following scientific courses: Theoretical and practical physics (3 semesters), chemistry (2 semesters).
4. Credit for at least the following theoretical professional courses (4 semesters; given by teachers of education): History of education, Principles of education, methods of teaching (including the teaching of elementary algebra and geometry), educational psychology, organization and function of secondary education.
5. Satisfactory performance of the duties of a teacher of mathematics in a secondary school for a period of not less than 10 year, or 20 semester, hours. [It is considered by many competent authorities that the most satisfactory conditions for this practical professional training are in connection with a year of postgraduate work in a school of education organized so that continuous directed teaching of classes in public schools throughout the year is available to students.]

It is believed that college semester-courses in rational solid geometry, descriptive geometry, analytic projective geometry, theory of statistics, mathematics of investment, surveying, practical and descriptive astronomy, and in as many other mathematical topics as possible, are also desirable.

It is generally conceded as further desirable that the prospective teacher should have studied during the college course the following subjects: History, economics, sociology, political science, general psychology, philosophy, and ethics.

Finally, it is suggested to state departments of education that the best interests of its teachers, and their charges, would be conserved, if during the first twenty years of service, an appreciable portion of periodic salary increases of the teacher were made dependent upon continued scientific development (for example, by means of summer-school courses), and upon active coöperation with colleagues in promoting the interests of mathematics and the ideals and purposes of mathematical organizations.

CHAPTER XV.

Certain Questionnaire Investigations

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1. Change of Mind between High School and College as to Life Work

BY

Arthur R. Crathorne

University of Illinois

If every high school pupil knew at the beginning of the freshman year what his or her life work was to be, the planning out of a course of study would be a much more definite problem than it is in reality. Included in this question of one's probable life work is the question of whether or not one can go to college. All college instructors know men and women who are rearranging their course of study on account of a change of mind as to future life work. Many men and women find themselves in college after spending four years in a high school which they attended with no intention at that time of proceeding further with their studies.

The object of this investigation is to measure this change of mind as to life occupation and to find out from actual college students something of their foreknowledge of college attendance. The following questionnaire, made out after discussion with Professor E. L. Thorndike of Columbia University, was answered by freshman students in eleven colleges and universities in eleven widely scattered states:

> 1. If on the day you entered high school you had been asked the question, "What do you expect to do as your life work", what would have been your answer? ....
> ....
> ....
>
> 2. What do you now expect to do as your like work? ....
> ....
>
> 3. Did you know that you were going to college when you entered high school?
>
> In answer to this question check off one of the following answers:
> (a) I was certain of going to college.
> (b) I was fairly sure of going.
> (c) I thought it probable.
> (d) I thought it possible.
> (e) I had no expectation of going.
> (f) I was sure I could not go.
>
> Signature ....

After throwing out those answers which were due to war conditions and a few showing merely an attempt at humor over 2000 answers were available for the study. This group is a good random sample of American college freshman. The questionnaire was printed on cards and in most cases was signed by the student. The institutions represented were:

Beloit College in Wisconsin
Colorado College in Colorado
Dartmouth College in New Hampshire
University of Illinois in Illinois
University of Kansas in Kansas
University of Maine in Maine
University of Nevada in Nevada
Oberlin College in Ohio
University of Oregon in Oregon
Stanford University in California
Washington University in Missouri

From some of these institutions answers were received from practically the whole freshman class. From others answers were received from the classes of one or two instructors. The students attending the summer session of the University of Illinois, being of a somewhat different type from the regular students, were classified separately.

The answers to Questions 1 and 2 were arranged in six groups as follows:

Group 1. The answers to both questions were "undecided" or an equivalent answer was given.

Group 2. The answer to question 1 was "undecided" while some definite occupation was given as the answer to question 2. This was the largest group.

Group 3. The answers to both questions were definite occupations the same for both questions.

Group 4. The answers to both questions were definite, but there were distinct differences between the two answers. For example: "Architect," "Chemist"; "Lawyer," "Engineer"; "Medicine," "Business"; are examples of pairs of answers in this group.

Group 5. The answers to both questions were definite and distinct, but the differences were not so marked as in Group 4. The change from one occupation to the other would not be difficult. The high school course suitable as preparatory to one would be suitable for the other. "Civil Engineer," "Electrical engineer"; "Botanist," "Forestry"; "Clergyman," "Social settlement work," are examples of answers.

Group 6. A decided answer was given to question 1, while the answer to question 2 was "undecided."

Of the 2083 pairs of answers 1576 were by men and 507 women.

Table I gives the above classification into six groups for each institution and for the total number of students, as well as a classification of the answers to question 3, for which there were 2069 answers. This table gives the actual numbers of students for the various colleges and groups. Table II covers the data in Table I with the actual numbers changed to percentages.

From these tables it should be noted that for every institution the numbers in groups 1, 5, and 6 are less than the corresponding numbers in groups 2, 3, 4. Of the 2083 students, 43.1 per cent had made decision as to life work upon entrance to high school while 56.9 per cent thought they had leanings toward some definite occupation. For the freshman year in college these figures are 15.1 per

TABLE I

Answers to Questionnaire by Men and Women

INSTITUTION	No. and Ans.	Questions 1 and 2						Question 3					
		Group 1 1. Undecided 2. Undecided	Group 2 1. Undecided 2. Decided	Group 3 1. Decided 2. Same Decision	Group 4 1. Decided 2. Different Decision	Group 5 1. Decided 2. Slightly Different Decision	Group 6 1. Decided 2. Undecided	(a)	(b)	(c) (2069	(d) Ans	(e) wers)	(f)
Beloit	53	3	16	15	10	3	6	16	15	10	1	6	2
Colorado	41	3	8	9	18	3	0	19	14	4	1	2	1
Dartmouth	405	64	125	91	78	17	30	175	132	44	33	15	6
Illinois	587	43	246	129	103	49	17	137	180	91	71	91	12
Ill. Summer	120	7	37	40	24	6	6	37	32	20	18	12	1
Kansas	135	10	57	19	31	9	9	19	51	22	17	15	7
Maine	147	9	56	19	41	19	3	20	38	26	33	28	2
Nevada	103	3	33	29	25	9	4	25	29	17	9	18	5
Oberlin	277	36	57	56	86	19	23	124	73	32	29	11	8
Oregon	93	8	29	15	30	3	8	27	22	12	12	15	4
Stanford	58	5	19	11	16	3	4	19	18	12	7	2	0
Washington	64	7	17	16	12	4	8	17	23	5	9	6	3
Totals	2083	198	700	449	474	144	118	635	627	295	240	221	51

cent and 84.9 per cent respectively. The total number of students who thought they had made a definite decision as to life work upon entering the high school is 1185, i. e., the sum of the totals for groups 3, 4, 5, 6. Of these 449 were of the same mind and 144 had made a slight change by the time of the freshman year in college. This leaves 592 (sum of the totals of groups 4 and 6) out of the total of 1185 who had made a radical change.

That is practically 50 per cent of the students changed from one definite choice of occupation to another between entrance to high school and entrance to college.

Table III gives the percentage of these students who had changed their minds for each institution and for the men and women of all the institutions. The percentages vary from 40 in the case of Illinois to 68 in the case of Oregon.

TABLE II

Answers to Questionnaire by Men and Women

(Numbers given are percentages)

INSTITUTION	No. of Ans	QUESTIONS 1 AND 2						QUESTION 3					
		Group 1	Group 2	Group 3	Group 4	Group 5	Group 6	(a)	(b)	(c)	(d)	(e)	(f)
		1. Undecided 2. Undecided	1. Undecided 2. Decided	1. Decided 2. Same Decision	1. Decided 2. Different Decision	1. Decided 2. Slightly Different Decision	1. Decided 2. Undecided			(2069	Answers)		
Beloit	53	5.6	30.2	28.3	18.9	5.6	11.3	32.0	30.0	20.0	2.0	12.0	4.0
Colorado	41	7.3	19.5	21.9	43.9	7.3	0.0	46.3	34.2	9.8	2.4	4.9	2.4
Dartmouth	405	15.8	30.9	22.5	19.3	4.2	7.4	43.2	32.6	10.9	8.1	3.7	1.5
Illinois	587	7.3	41.9	22.0	17.5	8.3	2.9	23.6	30.9	15.6	12.2	15.6	2.1
Ill. Summer	120	5.8	30.8	33.3	20.0	5.0	5.0	30.8	26.7	16.7	15.0	10.0	0.9
Kansas	135	7.4	42.2	14.1	23.0	6.7	6.7	14.5	38.9	16.8	13.0	11.4	5.3
Maine	147	6.1	38.1	12.9	27.9	12.9	2.0	13.6	25.8	17.7	22.4	19.0	1.4
Nevada	103	2.9	32.0	28.2	24.3	8.7	3.9	24.3	28.2	16.5	8.7	17.5	4.9
Oberlin	277	13.0	20.6	29.0	31.0	6.9	8.3	44.8	26.4	11.6	10.5	3.9	2.9
Oregon	93	8.6	31.2	16.1	32.2	3.2	8.6	29.3	23.9	13.0	13.0	16.3	4.3
Stanford	58	8.6	32.8	19.0	27.6	5.2	6.9	32.8	31.0	20.7	12.0	3.4	0.0
Washington	64	10.9	26.6	25.0	18.8	6.2	12.5	27.0	36.5	7.9	14.3	9.5	4.8
All Colleges	2083	9.5	33.6	21.6	22.8	6.9	5.7	30.7	30.3	14.3	11.6	10.7	2.5
Highest		Dart.	Kan.	Ill. Sum.	Col.	Me.	Wash.	Dart.	Kan.	Stan.	Me.	Me.	Kan.
Lowest		Nev.	Col.	Me.	Ill.	Ore.	Col.	Me.	Ore.	Wash.	Bel.	Stan.	Stan.

Tables IV and V cover the data in Tables I and II for men and women separately. At entrance to high school 69.4 per cent of the women had made a decision as to life occupation against 52.8 per cent for the men. During the freshman year in college these percentages were more nearly equal, being 85.1 and 84.0 respectively. Of the women 52 per cent changed their minds against 49 per cent of the men.

The answers to question 3 as tabulated on the right hand half of tables I, II, IV, V speak for themselves. 61 per cent, $(a) + (b)$, had looked forward to going to college with a great degree of certainty while 13.2 per cent, $(e) + (f)$, had not expected to go at all. 25.9 per cent, $(c) + (d)$, had more or less hope of going. These figures vary from institution to institution as might be expected.

TABLE III

Change of Mind by Institution and by Sex

INSTITUTION	Total of Groups 3, 4, 5, 6	Total of Groups 4, 6	Percent of Change	INSTITUTION	Total of Groups 3, 4, 5, 6	Total of Groups 4, 6	Percent of Change
Beloit	34	16	47	Maine	82	44	54
Colorado	30	18	60	Nevada	67	29	43
Dartmouth	216	108	50	Oberlin	184	109	59
Illinois	298	120	40	Oregon	56	38	68
Ill. Summer	76	30	40	Stanford	34	20	59
Kansas	68	40	59	Washington	40	20	50
All Men	833	409	49	Highest			Ore.
All Women	352	183	52	Lowest			Ill.
Men and Women	185	592	50				

Table VI groups the answers to all three questions of the questionnaire. From this table we see that each of the six classes of students of question 3 is divided into six groups with much the same percentages as the total. Foreknowledge of attendance or non-attendance at college has nothing to do with change of mind as to life work. The totals of the columns of Table VI differ somewhat from those of Table I. The wording of some of the answers was such that there could be a difference of opinion as to the group to which it should belong. Table I having been made as the answers came in, and Table VI some months later, the difference in the two tables shows the wavering of the writer's mind as to the classification of doubtful cases.

Summary.—A random sample of 2083 college freshmen from eleven widely scattered states answered the questions:

"If on the day you entered high school you had been asked the question, 'what do you expect to do as your life work,' what would have been the answer."

"What do you now expect to do as your life work?"

TABLE IV

Answers to Questionnaire by Men

INSTITUTION	No. of Ans	QUESTIONS 1 AND 2						QUESTION 3					
		Group 1	Group 2	Group 3	Group 4	Group 5	Group 6						
		1. UNDECIDED 2. UNDECIDED	1. UNDECIDED 2. DECIDED	1. DECIDED 2. SAME DECISION	1. DECIDED 2. DIFFERENT DECISION	1. DECIDED 2. SLIGHTLY DIFFERENT DECISION	1. DECIDED 2. UNDECIDED	(a)	(b)	(c) (1568	(d) Ans	(e) wers)	(f)
Beloit	35	2	11	8	9	2	3	13	8	6	0	4	1
Colorado	11	1	4	2	4	0	0	5	4	0	1	1	0
Dartmouth	405	64	125	91	78	17	30	175	132	44	33	15	6
Illinois	522	37	229	115	91	39	11	116	161	81	63	86	11
Ill. Summer	49	4	16	13	11	3	2	16	17	6	7	4	0
Kansas	117	6	54	16	27	8	6	15	46	19	14	13	7
Maine	135	7	52	16	39	19	2	18	34	23	31	28	2
Nevada	44	2	18	11	9	4	0	8	15	7	5	7	2
Oberlin	119	23	30	21	32	6	7	54	34	14	13	4	1
Oregon	52	7	14	11	16	0	4	14	15	10	5	6	2
Stanford	50	4	17	10	12	3	4	13	17	12	6	2	0
Washington	37	5	11	8	8	1	4	11	14	3	5	3	0
Totals	1576	162	581	322	336	102	73	458	497	225	183	173	32
Percentages	100	10.3	36.9	20.4	21.3	6.5	4.6	29.9	31.7	14.3	11.7	11.0	2.0

Of these students 57 per cent had entered high school with some definite occupation in view. Upon entrance to college about one half of these had changed their minds. Examination of the figures for individual institutions or for sex discovers no more marked difference in percentages than one would expect to find in random samples. Of high school freshmen 57 per cent have made up their minds as to life work while for college freshmen this percentage is 85. Some allowance must be made for the fact that the record is made at the time of the freshman year in college. In high school more girls have made up their minds than boys while in college the percentages for men and women are nearly equal. The women in this investigation are all students in coeducational institutions.

TABLE V

Answers to Questionnaire by Women

INSTITUTION	No of Ans.	QUESTIONS 1 AND 2						QUESTION 3					
		Group 1	Group 2	Group 3	Group 4	Group 5	Group 6	(a)	(b)	(c)	(d)	(e)	(f)
		1. UNDECIDED 2. UNDECIDED	1. UNDECIDED 2. DECIDED	1. DECIDED 2. SAME DECISION	1. DECIDED 2. DIFFERENT DECISION	1. DECIDED 2. SLIGHTLY DIFFERENT DECISION	1. DECIDED 2. UNDECIDED			(501 Answers)			
Beloit	18	1	5	7	1	1	3	3	7	4	1	2	1
Colorado	30	2	4	7	14	3	0	14	10	4	0	1	1
Illinois	65	6	17	14	12	10	6	21	19	10	8	5	1
Ill. Summer	71	3	21	27	13	3	4	21	15	14	11	8	1
Kansas	18	4	3	3	4	1	3	4	5	3	3	2	0
Maine	12	2	4	3	2	0	1	2	4	3	2	0	0
Nevada	59	1	15	18	16	5	4	17	14	10	4	11	3
Oberlin	158	13	27	35	54	13	16	70	39	18	16	7	7
Oregon	41	1	15	4	14	3	4	13	7	2	7	9	2
Stanford	8	1	2	1	4	0	0	6	1	0	1	0	0
Washington	27	2	6	8	4	3	4	6	9	2	4	3	3
Totals	507	36	119	127	138	42	45	177	130	70	57	48	19
Percentages	100	7.1	23.5	25.0	27.2	8.3	8.9	35.3	26.0	14.0	11.4	9.6	3.8

TABLE VI

Answers to Questionnaire by Men and Women

	Group 1	Group 2	Group 3	Group 4	Group 5	Group 6	Totals
(a)	63	167	160	149	47	47	633
(b)	57	202	155	129	44	37	624
(c)	31	115	60	61	19	9	295
(d)	16	93	37	60	17	15	238
(e)	17	102	29	56	13	3	220
(f)	6	15	9	12	8	1	51
Totals	190	694	450	467	148	112	2061

Of 2069 answers to Question 3, regarding foreknowledge of college attendance,

30.7 per cent were certain of going,
30.3 per cent were fairly sure of going,
14.3 per cent thought it probable,
11.6 per cent thought it possible,
10.7 per cent did not expect to go,
2.5 per cent were sure of not going.

These figures vary from institution to institution but do not vary much for sex. There is no connection between foreknowledge of college attendance and change of mind as to occupation.

2. Mathematical Interests of High School Pupils

By Walter F. Downey, *English High School, Boston, Massachusetts*

The purpose of this investigation is to obtain from high school pupils a frank statement regarding their interests and preferences in the study of mathematics.[1] There is no purpose to prove any thesis, but a desire to collect data which, in some small way, may be of service to those interested in the reorganization of this subject.

The schools selected are of different types and sizes, and are geographically widely separated, so that the material should be fairly representative of the attitude and opinions of pupils in this subject of the high school curriculum. In order to obtain an honest expression of opinion from the pupils, the replies were in writing, without signature, and those who conducted the inquiry in each school were asked to impress upon the pupils the purpose of the investigation.

The data presented are based upon the results of a questionnaire sent to high schools in various parts of the country. Replies were received from about seven thousand pupils in fifteen high schools. Seven states representative of the east, the middle west, the south west, and the extreme west, give a selection of localities which may be said to give a fair cross section view, free from conditions peculiar to any one part of the United States.

The following is a list of the schools, and the number of replies:—

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[1] See also the results secured by Mr. Davis in St. Louis on pp. 533 ff.

High School		Number of Replies
1. Santa Barbara High School, Santa Barbara, Cal.		248
2. Chaffey Union High School, Ontario, Cal.		152
3. Ball High School, Galveston, Texas		278
4. Cleveland High School 5. McKinley High School 6. Soldan High School	St. Louis, Mo.	1,685
7. Crane Technical High School, Chicago, Ill.		1,784
8. Canton High School, Canton, Ill.		281
9. Lincoln School, New York, N. Y.		49
10. Asbury Park High School, N. J.		76
11. Belmont High School, Belmont, Mass.		255
12. Hopedale High School, Hopedale, Mass.		39
13. Waltham High School, Waltham, Mass.		220
14. Girls' Latin School, Boston, Mass.		594
15. English High School, Boston, Mass.		1,317
Total replies		6,978

As this is an investigation of the attitude of the pupil and is not a comparative record of schools, the following data will be given without identifying the school. The order is not the same as that given in the previous table, but the list includes the same fifteen schools.[2]

I. *Do you like mathematics?* To this question the replies showed the following results:

School	*Yes*	*No*	*Partially*
A.	87%	5%	8%
B.	84	16	
C.	91	9	
D.	82	15	3
E.	89	11	
F.	79	21	
G.	80	13	7
H.	84	16	
I.	80	16	4
J.	72	16	12
K.	69	31	
L.	66	21	13
M.	90	9	1
Total	84%	14%	2%

The preceding data indicate, in general, that more than four-fifths of the pupils studying mathematics "like" the subject. The highest percentage is 91%; the lowest 66%; the average of all schools is 84%. The variation seems to have no bearing on the size or locality of the school. The four schools with the highest percentages are very widely separated, and are near the two extremes in size, while we have the same thing true of six schools with the lowest percentages.

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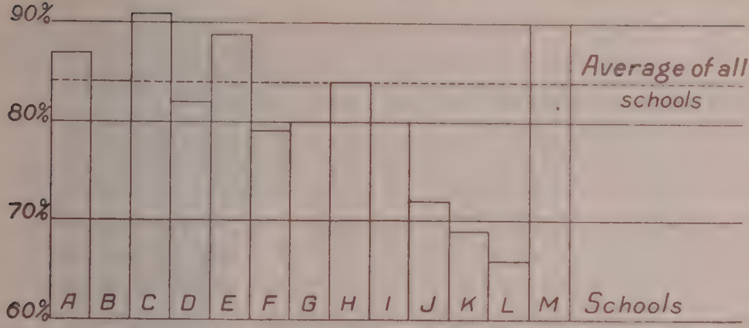
[2] The replies from the three high schools in St. Louis were received as a combined report and are so given above.



GRAPHIC REPRESENTATION OF THE TABLE ON THE PRECEDING PAGE

In only one school was mathematics found to be required of all pupils for four years. The other schools have the subject required from two to four years in some courses, with no requirement in others. In one state represented, which has a state wide school organization, with a state university, there is no mathematical requirement in the high school curriculum. The schools which reported from this state have slightly above the average result. In the school with the highest mathematical requirement the percentage compares favorably with the record of other schools and with the average.

The same inquiry was introduced in a first year college class of 203 students, where mathematics was required in some courses. Of those who were in mathematics courses of the first year, 76% liked it; 23% did not, and 1% were indifferent.

In a few schools a comparison was made on the basis of sex. The totals of these schools showed that 85% of the boys, and 84% of the girls like mathematics. In comparing the coeducational schools with those exclusively for boys or for girls, there does not appear enough variation to conclude that there is any marked difference on the basis of sex.

II. *Why do pupils like or dislike mathematics?* The distribution of reasons for liking the subject, as given up by 5788 pupils is as follows:—

Interesting	*Useful*	*Good mental training*	*Easy*	*Exact and definite*	*Good teacher*	*Miscel. reasons*	*No reason*
41%	22%	19%	12%	2%	1%	1%	2%

The reasons assigned by 1036 pupils for disliking the subject are distributed as follows:—

Dull and uninteresting	*Too difficult*	*Not Useful*	*Requires too great accuracy*	*No reason*
27%	43%	9%	1%	20%

On the basis of interest, there is a wide variation, 57% in one school assigning this as a reason for liking mathematics and 15% in another, a range of 42%. On the basis of usefulness there is a range of 27%; in mental training the range is 33%; on being "easy," the range is 22%. This would seem to indicate a great variety of reasons for motivation in different schools, and, perhaps, a great variation in the emphasis of mathematical values urged by instructors.

The striking feature in the reasons for disliking the subject is that over two-fifths of the pupils who dislike it find it too difficult and one-fifth have no reason for their attitude.

III. *What topics in algebra are considered by pupils* (*a*) *the most popular,* (*b*) *the most unpopular?*

(*a*) On the basis of the returns received, the topics liked best by pupils in the order of the number of pupils who selected them are (1) the equation, (2) the four fundamental operations, (3) factoring, (4) problems. Other topics receiving enough attention to justify notice but selected by a much smaller number of pupils are the graph, square root, and logarithms.

The reasons for selecting the four most popular topics are assigned as follows:—

	Easy	*Interesting*	*Good mental training*	*Exact and definite*	*Useful*	*Like a puzzle*	*No reason*
(1) Equations	30%	26%	10%	1%	24%		9%
(2) Four fundamental operations	35	26	12	3	1		23
(3) Factoring	58	34	4	2	1	1	
(4) Problems	28	26	14	1	18	1	12

(*b*) As may be anticipated, because of the fact that the above topics now probably receive greater emphasis in elementary algebra than others, the same topics also receive the greatest attention by pupils in discussing their dislikes. Symbols of aggregation, fractions and surds also are disliked but by a smaller number.

The reasons for dislike are distributed as follows:—

	Too difficult	*Uninteresting*	*Too much detail*	*Not useful*	*No reason*
(1) Equations	63%	7%	18%		12%
(2) Problems	83	2	9	1	5
(3) Four fundamental operations	66	19	2	13	
(4) Factoring	65	22	5	8	

It seems obvious from the above that the main basis for choice of topics in algebra are (1) facility in doing the work, (2) interest in the topic. As a cause for dislike the question of difficulty outweighs all other reasons combined.

In a few schools pupils who had studied both algebra and plane geometry were asked to express their preference. Algebra proved to be the choice of 20% more pupils than selected plane geometry.

IV. *What topics in plane geometry are considered by pupils* (*a*) ***the most popular,*** (*b*) ***the most unpopular?***

(a) In plane geometry, the four topics liked best, with the reasons are as follows:—

	Interesting	*Useful*	*Easy*	*Good meal training*	*Interested in drawing and measuring*	*Accuracy required*	*Independence*	*Like a puzzle*	*No Reas.*
(1) Constructions	47%	6%	22%	8%	11%	1%	2%	1%	2%
(2) Triangles	38	22	17	11	10	2			
(3) Circles	60	7	18	15					
(4) Originals	37	6	3	30	2		18		4

(b) The topics liked least, with reasons, are:—

	Too difficult	*Indefinite*	*Uninteresting*	*Not Useful*
(1) Originals	75%	3%	22%	
(2) Loci	65	17	18	
(3) Triangles	42		20	38
(4) Circles	50	16	31	3

The number of pupils who selected constructions as the most popular topic was greater than the combined number that selected any other two topics, and practically with the same emphasis the topic, originals, was the most unpopular. As in algebra, topics are disliked, chiefly, because they are "too hard."

V. *Preference for High School Subjects*

Preference	*1st Choice as a Percentage of the Total*	*1st*	*2nd*	*3rd*	*4th*	*5th*
English	26%	1,774	1,604	1,394	1,061	502
History	18%	428	405	565	544	285
Latin	22%	318	293	294	325	156
French	16%	257	341	425	452	174
German	13%	36	74	48	49	27
Spanish	11%	163	231	227	285	229
Mathematics	23%	1,575	1,657	1,465	1,336	1,020
Science	22%	932	842	881	920	572
Commercial Subjects	20%	161	156	158	129	94
Drawing	23%	695	548	565	574	340

In the above table are shown the number of pupils who expressed varying degrees of preference for these subjects, and the percentage of the total of each subject who gave it first preference.

On this basis English was first, with 26%; mathematics and drawing in second place, with 23% each; Latin and science next, with 22% each while German and Spanish occupy last places with 13% and 11%, respectively. Choices beyond the fifth are excluded in this table; also subjects which only a few pupils selected.

It seems clear that mathematics ranks well, when compared with the other common subjects of the curriculum, on the basis of preference.

Conclusions.—1. Mathematics is liked by more than four-fifths of the pupils who study it.[8]

2. Although there is great variation, the result stated above is entirely independent of the size of the school.

3. There appears to be a wide difference in the amount of mathematical requirement in different schools. The average school seems to require it in some courses, and not in others, but there are entire states where there is no requirement whatever.

4. Pupils "like" mathematics mainly on the basis of general interest, usefulness and the mental training involved, while they dislike it mainly because of its difficulty.

5. In algebra, the topics which are favorites, are those selected mainly because they appear the "easiest," while those least favored are so stated on the basis of being "too difficult."

6. At the present time, manipulation appears to be as popular with pupils as the applications of such work. This, possibly, is due to the emphasis still given to the former in most schools.

7. In geometry, constructions are very popular, while original exercises are equally unpopular. The former fact encourages a stronger appeal to the motor minded pupil, while the latter suggests a limitation of difficulty, and proper emphasis on the practical use of geometry with the necessary contact to the life experience of the pupil.

8. In popularity with pupils, mathematics compares well with the other common subjects of the curriculum.

9. From a study of the data upon which this investigation is based, it appears that, in practice, teachers of mathematics have em-

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[8] In 1916 Professor Hanus of Harvard, and Mr. Gaylord made a study of eleven schools in New England cities to determine the pupils' likes and dislikes of this subject. They reported that 18.8% disliked mathematics. It is interesting to note that the two results point to the above conclusion, although one was made on the basis of a study of larger schools in one section of the country, while the present investigation includes schools widely separated, and varying greatly in size.

ployed such a standard of interest as is set forth by Dr. H. H. Horne in his "Philosophy of Education" as follows:—"The securing of interest in the subject is the immediate aim of the work of instruction. The remote aim varies with circumstances; it may be the acquisition of knowledge for its own sake or for utility; it may be the formation of character; it may be the cultivation of the æsthetic sense; it may be social efficiency or the comprehensive aim of complete living. But in any case it is the immediate business of instruction, in justice to its opportunity, to develop an interest in the subject taught as the most efficient means to its remote end. The teacher who has solved the present problem of interesting his class in the subject matter has solved the larger problem of instruction."

3. The Importance of High School Mathematics as Indicated by Certain Questionnaires

By ALFRED DAVIS, *Soldan High School, St. Louis, Missouri.*

In these days when the principles of democracy are gripping the world as never before, it is of increasing importance to know what large groups of people think regarding matters of more than individual interest.

Changes in the accepted order of things, even when necessary, must come slowly. Especial caution is necessary in a matter so supremely important as the education of our youth. For this reason the judgment of men of affairs as to the value of mathematical study may well be considered seriously in connection with proposed changes.

The St. Louis and Chicago Questionnaires. With a view to ascertaining the attitude of the public towards the study of mathematics in our high schools the following letter was addressed by the writer to about two hundred citizens of the City of St. Louis, Missouri:

St. Louis, Mo., May 17, 1919.

Dear Sir:—

The following and similar questions have been submitted to some of the most influential men of the country for the purpose of aiding those who are investigating the reasons for teaching mathematics in our secondary schools. We respectfully submit the questions to you so that we may have the benefit of your practical experience. Furthermore, your name will give to your answers a weight not attached to the opinions of persons less well known.

1. Was the study of high school mathematics worth while to you? If so, why?
2. Did its study contribute to your development anything which could not have been secured in equal degree from some other subject which might have been substituted for it?
3. In employing a young person, or in advising him to follow your profession, what importance would you attach to a good school standing in mathematics?
4. In planning the secondary education of your son or daughter would you include algebra and geometry? Why?
5. Is a knowledge of algebra or geometry of any practical value in your business? If so, in what way?
6. Do you think algebra and geometry should be retained in our secondary schools?

The matter is important and your answer will be of great value to those who are aiming to secure the best possible courses of study for our schools. Your thoughtful response will be greatly appreciated.

Thanking you in advance for your courtesy, I am,

Sincerely yours,

These questions were sent to men in various occupations, excepting teachers. It was thought that teachers might be prejudiced in favor of existing courses of study and so might not give an unbiassed opinion. Again, the purpose was to ascertain the value of mathematics to the man who was playing his part in that world for which the schools are preparing our youth. Hence, the appeal was made to laymen.

The men were college men. Directories at the public library were used to ascertain that this was the case, except that professional men were assumed to have a college education. This, however, did not prove to be true in a few cases, as was learned later. The point was:—Did the person questioned have first hand knowledge of mathematics? Most college men have this.

Prominent men, men recognized as successful in their work, were selected. This seemed desirable because such men have proved the worth of their judgment through winning success. The judgment of such men in matters of education cannot be lightly set aside.

The same questions were submitted by the writer to prominent citizens of the City of Chicago in 1917. The results of this investigation have been published in *School and Society,* November 17, 1917, and in *School Science and Mathematics* for January, 1918. The results of both questionnaires are given here for purposes of comparison.

Replies were received from about fifty per cent. of the letters sent out in each city. This may be considered a high percentage, when we consider how busy these people are, and the fate that must await many of the letters of inquiry which they receive about various topics. In some cases such letters do not get beyond the secretary. Some persons had moved from the city, a few were deceased, and some were absent on extensive business trips. Besides, the times were not normal. Many of these men were at the time overwhelmed by big issues that concerned the immediate welfare of the country and of the world. Surely these men are interested in education, for many of them wrote at length regarding the importance of algebra and geometry in the education of the child.

Tables I to IV give in summary form the results of these questionnaires.

Table I gives a general summary by occupations of the persons from whom replies were received.

Table II gives a general summary of the answers to the several questions.

TABLE I

OCCUPATIONS REPRESENTED	*St. Louis Replies*	*Chicago Replies*	*Both Cities*	
			No.	*P. C.*
Physicians and Surgeons	28	7	35	22.6
Lawyers and Jurists	14	12	26	16.8
Clergymen	9	9	18	11.6
Manufacturers	14	2	16	10.3
Bankers and Brokers	6	8	14	9.0
Merchants	8	4	12	7.7
Printers, Publishers, and Typefounders	5	2	7	4.5
Engineers	3	1	4	2.6
Secretaries	2	2	4	2.6
Insurance Agents	4	0	4	2.6
Authors and Reporters	1	2	3	1.9
Railroad Men	1	2	3	1.9
Economists and Social Workers	0	2	2	1.3
Real Estate Men	1	1	2	1.3
Business Managers	0	1	1	0.6
Meteorologists	1	0	1	0.6
Architects	1	0	1	0.6
Librarians	1	0	1	0.6
Hotel Managers	1	0	1	0.6
Totals	100	55	155	99.7%

P. C. means per cent. taking 155 as the base.

TABLE II

	St. Louis		Chicago		Both Cities	
	No.	*P. C.*	*No.*	*P. C.*	*No.*	*P. C.*
QUESTION ONE (Was the study of high school Mathematics worth while to?)						
Affirmative	88	88.0	43	78.2	131	84.5
Very Emphatic . .	*12*	*12*	*7*	*12.7*	*19*	*12.3*
Negative	3	3	6	10.9	9	5.8
QUESTION TWO (Did its study contribute to your development anything which could not have been secured in equal degree from some other subject which might have been substituted for it?)						
Affirmative	73	73	36	65.5	109	70.3
Very Emphatic . .	*18*	*18*	*9*	*16.4*	*27*	*17.4*
Uncertain	13	13	8	14.5	21	13.5
Negative	5	5	6	10.9	11	7.7
QUESTION THREE (In employing a young person, or in advising him to follow your profession, what importance would you attach to a good school standing in Mathematics?)						
Great Importance . .	56	56	19	34.5	74	48.4
Considerable Importance	26	26	21	38.2	47	30.3
Combining Great and Considerable Importance	*82*	*82*	*40*	*72.7*	*121*	*78.7*
Very Little Importance	8	8	2	3.6	10	6.5
No Importance	2	2	3	5.5	5	3.2
Combining Very Little and No Importance	*10*	*10*	*5*	*9.1*	*15*	*9.7*
QUESTION FOUR In planning the secondary education of your son or daughter would you include Algebra and Geometry?)						
Require Both (*one would have son but not daughter take Mathematics*)	84	84	42	76.4	126	81.3
Algebra at least (*one says Algebra only*) (*one says both for boy and Algebra for girl*) .	6	6	3	5.5	9	5.8
As a Prerequisite for Professional Work . .	5	5	3	5.5	8	5.2
Neither	3	3	2	3.6	5	3.2

TABLE II—*Continued*

	St. Louis		Chicago		Both Cities	
	No.	*P. C.*	*No.*	*P. C.*	*No.*	*P. C.*
QUESTION FIVE (Is the knowledge of algebra and Geometry of any value in your business?)						
Affirmative (*seven say only remote use*) . . .	62	62	23	41.8	84	54.2
Negative	38	38	26	47.3	63	40.6
QUESTION 6 (Do you think Algebra and Geometry should be retained in our secondary schools?)						
Affirmative	89	89	46	83.6	135	87.1
Very Emphatic . .	*25*	*25*	*14*	*25.5*	*32*	*25.2*
As Professional Preparation Only	1	1	2	3.6	3	1.9
Would Make Mathematics Optional . . .	5	5	2	3.6	7	4.5
Negative	4	4	2	3.6	6	3.9

The weight of public opinion in favor of teaching algebra and geometry in high schools, as indicated by these questionnaries, is overwhelming. Many of the replies are of considerable length and of striking emphasis. The following from a prominent lawyer in St. Louis indicates the spirit of most of the replies:

> I am decidedly of the opinion that the study of mathematics in the high school was worth while to me, and I believe it is worth while for anybody. It seems to me just as reasonable that a boy growing into young manhood should continue to feed on pap and expect a perfect physical development, as it is to expect the best mental development without the study of mathematics. Mathematics in my humble opinion is the best mental training offered to the students. The study of mathematics chains the attention and develops a power of concentration in a way which in my judgment no other study can accomplish.
>
> I have had occasion to observe the effect of the study of mathematics in my own family, and that member of the family of six boys who was most proficient in his mathematical studies was unquestionably the best thinker of the lot. He had much greater power of concentration, and a keener and profounder insight into problems of all kinds than any of the other boys.
>
> As to whether a knowledge of algebra and geometry is of any practical value in the profession of law I answer "yes" and "no". It happens frequently that in patent litigation a lawyer has great need of his knowledge of mathematics and of different branches of physical science. As a rule, a lawyer in the general practice

of law has no direct need of algebra or geometry; nor has he any direct need of the plays of Shakespeare, nor of Vergil's Aeneid, nor of the orations of Cicero. The point I am desirous of emphasizing is that algebra and geometry should be studied, not with a view of immediate application of the knowledge in the practical affairs of life, but for the building up of the intellect of the individual to enable him to grasp the problems of life as they present themselves.

I studied Greek, chemistry and botany in the high school. It is not only difficult but impossible to state with any degree of certainty how much benefit may be obtained from any one study, but in my opinion mathematics in the high school was of decidedly more benefit to me than any or all of the other branches mentioned.

The negative vote on Question 1 is negligible, and is no greater than might be expected in a case in which there is room for any difference of opinion. The reasons assigned in the replies to Question 1 are various and do not lend themselves readily to a summary treat-

TABLE III

Answers to Question 3: In employing a young person, or in advising him to follow your profession, what importance would you attach to a good school record in mathematics?

OCCUPATIONS	ST. LOUIS					CHICAGO					BOTH CITIES				
	Number of Replies	Great Importance	Considerable Importance	Little Importance	No Importance	Number of Replies	Great Importance	Considerable Importance	Little Importance	No Importance	Number of Replies	Great Importance	Considerable Importance	Little Importance	No Importance
Bankers and Brokers	5	60	40	0	0	7	57	42	0	0	12	58	42	0	0
Physicians	28	61	32	7	0	7	29	29	14	29	35	54	31	9	6
Lawyers	12	67	0	17	17	11	27	64	9	0	23	49	30	13	9
Merchants	4	100	0	0	0	3	67	0	33	0	7	86	0	14	0
Clergymen	9	78	22	0	0	6	33	67	0	0	15	60	40	0	0
Mfrs.	9	100	0	0	0	2	100	0	0	0	11	100	0	0	0
Miscellaneous	9	78	22	0	0	9	33	44	11	11	18	56	33	5	5

TABLE IV

Answers to Question 5: Is a knowledge of Algebra and Geometry of any value in your business?

The replies in the "yes" and "no" columns are in percentages based on the number of replies received.

OCCUPATIONS	St. Louis			Chicago			Both Cities		
	Number of Replies	Yes	No	Number of Replies	Yes	No	Number of Replies	Yes	No
Bankers and Brokers	7	43	57	5	40	60	12	42	58
Physicians	27	67	33	7	43	57	34	62	38
Lawyers	14	79	21	5	100	0	19	84	16
Merchants	5	40	60	3	0	100	8	25	75
Clergymen	6	33	67	6	17	83	12	25	75
Manufacturers	13	39	61	2	100	0	15	47	53
Miscellaneous	28	75	25	21	48	52	49	63	37
Totals	100	62	38	49	47	53	149	57	43

ment. However, a vote of approximately 90% in favor of some form of mental training removes all possible doubt that one may experience mental exhilaration, confidence, and a sense of power from the mastery even of the elements of this science,—a vitality which is akin to the vigor of body which follows healthful physical exercise. The most of those who said mathematics was of practical value to them claimed that it had given them mental training as well.

Answers to Question 2 show that no other studies can wholly replace algebra and geometry in the high schools. Clearness and virility of thought, ability to reason accurately with a wholesome confidence in this ability, and an effective method for attacking many of the most difficult problems of life are credited to the study of mathematics. And in situations where the subject has practical value nothing can possibly replace it. The negative vote of 7.7% is a negligible one.

The results from Question 3 (see Table III, in addition to Table I) show that the training and the information gained from the study of mathematics are an economic asset. It is reiterated that the world needs men and women who are not only technically trained, but who have a broad education; and that the study of mathematics is an essential part of the intellectual equipment of the person who would serve his fellows best and win the highest success.

The answers to Question 4 indicate that more than 80% would require mathematics of high school students. The students who lack the ability to do this elementary work in the subject are rare.

Question 5 (see Table IV as well as Table I) emphasizes the fact that mathematics is more than a tool. However, St. Louis shows about 20% more than Chicago who find practical applications for the science in their occupations. Some think the information gained is often used unconsciously in many situations. Others find that it may be applied consciously with advantage in situations where it may not be absolutely essential. Yet others consider the mental training gained of practical advantage in many situations.

The answers to Question 6 leave no uncertainty as to whether the public wants algebra and geometry taught in the high schools. Ninety per cent insist that this be done and 25% are very emphatic. It seems to most of these people that, if mathematics is not taught to high school students, there is no strong argument left for teaching anything. Some say "Why have schools?" others say "We cannot turn out one tracked minds," and yet others, "Who on earth thinks otherwise?" Great emphasis is laid on the educational values of the science.

It may be noted that, except for Question 5, the negative vote throughout is negligible. The negative vote in 5 is such as to emphasize the real value and importance of mathematics. The findings of these experiments are, therefore, especially reassuring to the high school teacher of mathematics; and they are valuable to those who are planning courses of study for our youth. It is evident that the public speaks with no uncertainty in the matter. Mathematics must, therefore, continue to be an important part of high school curricula.

Professor Hancock's Investigation. Professor Harris Hancock of the University of Cincinnati recently conducted a questionnaire somewhat similar in results to the one above. The report is given in *School and Society* for June 19, 1915. We shall refer to it briefly here.

The following questions were sent out:

Which of the following courses in a high school would you advise a boy to take?

No. 1

A course where both mathematics and the classics are optional; for example, where history is substituted for mathematics.

No. 2

A course where mathematics is required, the classics being optional.

No. 3

A course where the classics are required, mathematics being optional.

No. 4

A course requiring both the classics and mathematics.

Replies were received from 207 persons, prominent in their various professions. About half of these were residents of Cincinnati, the others were residents of other cities. The results from both groups were approximately the same and as indicated in Table V.

TABLE V

Summary of Professor Hancock's Questionnaire as published in "School and Society" for June 19, 1915.

	Courses Selected							
Persons Outside Cincinnati	No. 1	No. 2	No. 3	No. 4	2 or 4	1, 2, 3 or 4	1 or 4	3 or 4
20 clergymen	1	1	0	17	1	0	0	0
47 business men . .	2	18	1	20	5	1	0	0
15 physicians . . .	1	1	1	10	1	1	0	0
9 lawyers	0	2	0	5	2	0	0	0
8 miscellaneous . .	1	3	0	4	0	0	0	0
3 heads of college faculties	0	0	0	2	0	0	0	1
Persons in Cincinnati						1, 2 or 3		
15 clergymen	1	2	0	9	2	1	0	0
41 business men . .	2	17	0	19	2	0	1	0
17 physicians . . .	2	7	0	6	2	0	0	0
26 lawyers	1	4	0	18	3	0	0	0
6 miscellaneous . .	1	4	0	1	0	0	0	0
207 total	12	59	2	111	18	2+1	1	1

About 91 per cent would require mathematics of boys in the high schools. It is interesting to note that the results are approximately the same as for both girls and boys in the other questionnaires. Evidently there is little need for separate courses for boys and girls. The chief values of the subject seem to be equally accessible to all regardless of sex.

Professor Hancock also proposed the following questions:

Do you believe (A) in a general knowledge of a great number of subjects? or (B) in a through knowledge of a few subjects? The results are shown in Table VI.

TABLE VI

Answers to Questions (A) and (B)

OCCUPATIONS	(A)		(B)	
	Yes	No	Yes	No
Business men	33	31	48	11
Clergymen	20	10	24	5
College faculty	0	3	3	0
Lawyers	14	14	24	5
Physicians	13	14	20	7
Miscellaneous	5	5	7	2
Totals	85	77	126	30

It appears that a large majority of those voting on question (B) favor a thorough knowledge of a few subjects of which mathematics must be one. Many voted for both a thorough knowledge of a few subjects and a general knowledge of a large number of subjects.

The results of these questionnaires justify the conclusion that representative groups of people from every section of the country would give like returns. The few who are opposed to requiring some mathematics of all high school students do not recognize any value in it except the practical utility of the subject in other lines of effort; or they do not recognize its uniqueness in training the mind.

What High School Students Think of Matheamtics.—We are interested, not only in what the person who has studied the subject thinks of mathematics, but also in the opinions of high school students who are actually studying the subject. While the preferences of pupils and their attitude towards their school work vary from term to term with the changes of subjects and of teachers, and because of the variability of adolescents, yet the conscientious teacher is greatly concerned with the attitude of the class as a whole and with the reaction of individual pupils toward the subject he is teaching them.

With the purpose of ascertaining the attitude of high school students towards the study of algebra and geometry the writer recently submitted the following questions to the mathematics classes of the five high schools for white pupils in the city of St. Louis. Replies were received from 3105 in all.

Questions For High School Students

Age........ Grade........ Boy or Girl (underscore).

1 — Do you enjoy the study of mathematics?
(underscore) Arithmetic,—Much — A little — None.
Algebra,— Much — A little — None.
Geometry, —Much — A little — None.

2 — Why do you study mathematics?

3 — What advantages do you expect from its study?

4 — Of all your high school studies, which do you enjoy the most?

5 — Which of your various studies do you value most? Why?

The answers to Question 1 were fairly uniform throughout the city. The replies to this are therefore summarized in Table VII.

The replies to the other questions are shown for the individual schools as well as the combined results for all the schools in Table VIII.

Arithmetic was included in Question 1 for the purpose of determining whether the same proportion would enjoy algebra and geometry in high school as enjoyed arithmetic in the grades. This seems to be the case and is probably a significant fact. It seems that the liking or disliking of mathematics carries over from the grades in most cases. A considerable number did not vote on arithmetic although all had studied the subject.

TABLE VII

The measure of enjoyment realized in their mathematical studies by the students of five St. Louis high schools.

SUBJECTS	Number of Pupils		Boys			Girls			Total		
	Boys	Girls	Much	A Little	None	Much	A Little	None	Much	A Little	None
Arithmetic	1564	785	65.7	30.0	4.2	61.3	30.7	8.0	64.2	30.2	5.6
Algebra	1785	915	59.5	33.2	7.3	67.1	26.1	6.8	62.1	30.7	7.1
Geometry	1106	479	57.6	34.6	7.8	47.6	40.7	11.8	54.5	36.5	9.0

The boys greatly outnumber the girls in the mathematics classes, and there is a considerable decrease in the numbers of both boys and girls who take geometry. These facts are probably due to the fact that mathematics is optional in the St. Louis high schools except in the scientific and manual training courses. However, many of the pupils are preparing for entrance to colleges which require algebra and geometry for admission. Others wish to leave the door as widely open as possible for future choice and opportunity. And many of the pupils indicate that they elect mathematics because they, or their parents, or both, believe the mastery of the subject is a necessary part of a liberal education. In view of the wide range of choice before the pupils it is remarkable that so many should be studying mathematics. Fully half the enrollment of 2,000 at the Soldan High School at the time of the questionnaire were in mathematics classes.

It will be noted that the girls enjoy the study of algebra more than the boys do, while the reverse is true of geometry. A number of thoughts may be suggested to the reader by this fact. It does not appear that election tends to keep in our classes only those who have the greater aptitude for mathematics. The writer has found that some of the brighter students in his classes drop out of mathematics while those who barely pass continue the study. The testimony of his co-workers indicates that this is quite commonly true: Even though it is probable that all of these students need to study mathematics in high school, the above situation suggests that the free election of elementary high school mathematics courses is unwise. The pupil needs

TABLE VIII

Reactions of students in five St. Louis high schools.

NAME OF HIGH SCHOOL	Percent of Students Who Study Mathematics for:						Percent of Students Who Consider Mathematics of all Their Studies to be					
	Mental Training			Its use in vocations, college, etc.,			of Most Value			Most Enjoyable		
	Boys	Girls	Total	Boys	Girls	Total	Boys	Girls	Total	Boys	Girls	Total
Soldan	30	55	36	55	60	56	34	14	29	18	23	19
Yeatman	35	46	41	78	71	74	21	11	16	20	15	17
Central	41	38	40	88	60	80	45	29	40	35	22	31
Cleveland	43	68	51	93	78	89	40	25	39	21	28	23
McKinley	35	40	37	78	57	71	29	9	22	18	19	18
All High Schools	35	41	40	71	64	69	36	17	30	21	22	21

and ought to have the benefit of the experience and the wisdom of those who have preceded him and they ought to require him to take those studies which are of fundamental importance to every one.

Other factors which enter into the above results are: First, the texts used do not give a concrete approach to the subjects, and teachers cannot always arrange courses of study independently of the text. Some have not the ability to do this successfully, and none have the time to do so. The teacher is obliged to use the text whether it be good or bad. Some of the teachers have their attention divided between two or more departments. It does not always happen that mathematics is the chief interest of the teacher even though he be teaching the subject. This administrative problem may not always be possible of solution but the situation must affect the teaching of so difficult and so technical a subject as mathematics in an unfavorable way.

The results indicated in Table VIII will be clearer if we note that the Soldan district contains a larger number of prominent and influential residents than the others. Central may be considered commercial, while the other three are largely industrial.

It is interesting to note that 40% of the pupils expect some sort of mental training from the study of mathematics. The pupils were free to answer as they pleased and were not likely to say things to please the teachers since their names were not called for in these answers. Almost all the answers show thoughtfulness and sincerity. A high school student of ordinary intelligence will, when the subject is properly taught, recognize the disciplinary value of mathematics, without having his attention especially called to the matter. However, there is every reason to call the attention of the pupil to this value, for the idea of gaining mental training can be made to appeal to the pupil as a motive for the study of algebra and geometry. The girls outnumber the boys in giving this as a motive. This is probably due to the fact that few of them are planning to follow a technical course which will have direct use for mathematics.

The use of mathematics in other fields is a predominating motive for its study, although many of the replies indicate a rather vague notion as to what these uses may be. They say, "for use in the future," "for use in business," "to help solve problems," "everybody needs mathematics," and the like. This suggests that we might be more specific in our teaching as to the importance of the study of mathematics as a preparation for one's life and work.

The latter half of Table VIII shows results that are most encouraging to teachers of mathematics. More than one-third of the boys value the mathematics the most of all their studies. This is probably due to the fact that they realize, as one says, "I would make a fine (?) engineer without mathematics." 17 per cent of all the girls value mathematics the most of all their studies, and this is a good representation considering the number of subjects from which choice may be made.

More than one-fifth of all the students enjoy mathematics the most of all their studies. Considering the number of both boys and girls who would study the subject for the pleasure it gives, there seems to be no reason for distinguishing between girls and boys in the matter. Pupils seem to have as strong motives for studying mathematics as for studying any other subject.

Only 5 per cent of the pupils say they expect no advantage from the study of mathematics; but some of these say they study the subject for the pleasure it gives. Only 7 per cent indicate that they study the subject under compulsion; but many of these say that the subject will be useful, or that the effort required in the study is of value. Sixteen per cent say that the chief motive for the study of mathematics is the pleasure they derive from it.

Three years ago Mr. Raleigh Schorling, at that time of the University High School, University of Chicago, gave a questionraire to the students in three of the Chicago high schools, to determine the measure of enjoyment realized in their individual studies. The results were published in *School Science and Mathematics*, October, 1916. The results as far as they concern the study of mathematics are as follows:

The University High School, 318 replies:

Much enjoyment	49.7 per cent
A little enjoyment	40.9 per cent
No enjoyment	9.0 per cent

The Hyde Park High School, 955 replies:

Much enjoyment	51.5 per cent
A little enjoyment	40.2 per cent
No enjoyment	8.3 per cent

The Oak Park and River Forest High School, 745 replies.

Much enjoyment	45.7 per cent
A little enjoyment	42.9 per cent
No enjoyment	11.4 per cent

The Chicago pupils were asked to vote on all their studies. All subjects which were enjoyed more than mathematics were optional studies; whereas, at least one year of mathematics was required of all pupils in all three schools. No attempt was made to compare the attitude of boys and that of girls with respect to their studies, nor to make comparisons between the different mathematical studies, as was done in St. Louis. Out of a total of 2,018 pupils reported as taking mathematics, 1,769, or 87.7 per cent said they got some enjoyment from the study; 991, or 49.1 per cent, claimed to get much enjoyment; and only 229, or 12.3 per cent said they got no enjoyment.

It is well to note that the interests and preferences of pupils vary greatly from time to time, and that they are largely dependent on the skill and personality of the teacher. Some of the pupils say frankly, "I like mathematics when I like the teacher," or, "sometimes I like it and sometimes I don't." Yet we as teachers must measure success or failure largely by the reaction of our pupils. The opinion, therefore, of more than 3100 pupils is of very great importance; and it shows in this case that mathematics can make a strong appeal to and hold the interest of developing boys and girls. The outcome is most encouraging to those of us who believe that the study of mathematics is a necessary part of one's education, and who are striving, and not without success, to make it contribute largely to a stable and intelligent citizenship.

The variability of pupils' interests for both studies and vocations is shown by an investigation of Mr. G. W. Willett, of Hibbing, Minnesota. A questionnaire was given for three successive years to the same pupils of a six year high school. The results have been brought up to date in a report in *School and Society* for March 22, 1919. Some of the questions asked were:

1 — What subject that you have had in high school or seventh or eighth grades do you most prefer?

2 — What is your second choice?

3 — What subject did you most dislike?

4 — What do you expect to make your life work?

Judging from the 488 pupils questioned Mr. Willett concludes that permanence of interest in both studies and vocations is decidedly lacking. It seems, therefore, to be unwise to allow free election of studies, or to decide on a vocation, during the seventh, eighth, or ninth grades. During the period of adolescence interests are likely to be transient, and guidance is imperative. Preparation for life should be broad so that later when the individual has discovered himself he may select his vocation wisely. Studies of fundamental importance should be required of all, with possible rare exceptions, even though the subjects may not at first appeal to the pupil. Mathematics should be one of these required studies.

CHAPTER XVI

Bibliography of the Teaching of Mathematics—1911-1921

BY

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[1] This bibliography intends to include mention of all articles bearing on the teaching of mathematics which have appeared during the years 1911-1921 in the following journals:

The Mathematics Teacher
School Science and Mathematics
The Mathematical Gazette (England)
The Educational Review
Education
The Journal of Education
The Journal of Education (England)
Educational Administration and Supervision
The School Review
The Journal of Educational Psychology
Teachers College Record
School and Society

I. General Topics Relating to the Teaching of Mathematics

1. **Abbott, P.** The Position of Mathematics in Educational Reconstruction. The Mathematical Gazette, vol. 9 (March, 1917), pp. 33-38.

   The effect of the war on conditions in England is discussed. Four specific reforms are suggested: (1) Compulsory attendance at day continuation classes until the age of 18; (2) Readjustment of curricula; (3) Reforms in examinations; (4) Developments in the training of teachers. Recommends that the minimum course in mathematics include algebra through easy equations, the trigonometry of the right triangle, the use of logarithms, and intuitive geometry.

2. **Aitken, Robert.** The language of mathematics. Journal of Education (Brit.), vol. 33 (1911), pp. 307-309.

   Discusses the bearing on modern mathematical education of the idea that the development of mathematical language accompanies closely, and at times even conditions, the development of the science itself.

3. **Babb, Maurice.** Are particular abilities necessary for the pupil to gain an understanding of the elementary and secondary mathematics as usually given at the present time? Mathematics Teacher, vol. 6 (June, 1914), pp. 209-216.

   A real leader rides not too far ahead. A tendency exists to neglect pedagogy in teaching mathematics. Accuracy, mental work and the comprehension of underlying principles are vital. Too much memorizing is a common fault.

4. **Brookman, Thirmuthis.** High school mathematics. School Review, vol. 18, (Jan., 1910), pp. 20-28.

   Suggestions concerning high school mathematics, the outcome of a year's study of various types of schools.

5. **Carmichael, R. D.** Mathematics and life—The vitalizing of secondary mathematics. School Science and Mathematics, vol. 15 (1915), pp. 105-115.

   Discusses the part mathematics plays and should play in the development of the practical, esthetic, and moral phases of life.

6. **Carson, G. St. L.** Some principles of mathematical education. Mathematical Gazette, vol. 7 (January, 1913), No. 103, p. 30.

   The three functions of mathematics: (1) starting from the postulates (truth of which is no concern at present) sets of deductions are evolved by use of the axioms; agreement with experience strengthens the evidence in favor of these postulates; (2) the investigation of the consistence of a set of postulates; (3) redundance of a set of postulates:—redundant if some of its members are logical consequences of others.

7. **Chilcott, C. M.** An experiment in coöperation. Journal of Education, vol. 75 (Feb. 1, 1912), pp. 125-127.

Tells how new pedagogical ideas were worked out by the group method in first year high school mathematics classes. Individual, group and class progress the pupils represent by graphs. They took a marked stand against dishonesty; there was a strong spirit of coöperation, and the making of daily reports cultivated accuracy and helped pupils appreciate relative values.

8. **Curtis, A. M.** Recent criticisms of mathematics teaching and their results. Mathematics Teacher, vol. 9 (Dec., 1916), pp. 94-102.

Results of a questionnaire investigation involving the following questions: 1. Who are the leading critics of mathematics teaching? 2. What are their points of view? 3. What changes have you noted in the time allotted to mathematics in the high school course and in college courses? 4. Have there been appreciable changes in the arrangement of the courses? 5. What changes in length of courses do you suggest? 6. Is the preparation of your entering class improving or declining? 7. To what do you ascribe this change?

9. **Davis, Alfred.** The status of mathematics in secondary schools. School Science and Mathematics, vol. 18 (Jan., 1918), pp. 25-35.

A report of the answers to a questionnaire sent out to prominent doctors, lawyers, merchants, bankers, etc. in Chicago. It shows many interesting facts concerning the feeling entertained by the man of affairs toward high school mathematics.

10. ———— Valid aims and purposes for the study of mathematics in secondary schools. School Science and Mathematics, vol. 18 (1918), pp. 112-123; 208-220; 313-324.

A report of a study by a committee of the Chicago Mathematics Club.

11. **Dobbs, W. J.** Mathematics in secondary schools. Mathematical Gazette, vol. 9 (Mar., 1917), pp. 40-42.

A scheme of study intended to reform the British curriculum. It is based upon the same general line of thought that has characterized the work of the National Committee on Mathematical Requirements. Recommends including mechanics in the secondary school, and an easy introduction to the Calculus.

12. **Durell, Fletcher.** Mathematics and efficiency. School Science and Mathematics, vol. 16 (1916), pp. 25-30.

While this address is not concerned directly with the teaching of mathematics, it carries with it a suggestion in that it points out that mathematics may be an important aid in the "development to its full fruition of the leading thought of the age (efficiency)".

13. **Dyson, F. W.** A plea for astronomy. Mathematical Gazette, vol. 7 (Oct., 1914), No. 113, p. 394.

Astronomy and mathematics have grown up together and astronomical illustrations can be found which will enforce and perhaps enliven lessons in geometry and trigonometry.

14. **Goodwill, G.** Some suggestions for a presentment of mathematics in closer touch with reality. Mathematical Gazette, vol. 9 (Mar., 1918), pp. 225-228.

A plea for the elimination of certain details of secondary mathematics for the purpose of relating the subject more closely to mechanics and other branches in which it is easily applied.

15. **Greenhill, Sir G.** Mathematics in artillery science. Mathematical Gazette, vol. 8 (March, 1915), No. 116, p. 25.

Six months ago there was no such thing. But this is a mathematical war. Many real examples.

16. ———— The use of mathematics. Mathematical Gazette, vol. 7, (Mar., 1914), No. 110, p. 253.

Use not in sense of utilitarian applications but in the cultivation of intellect and correct thinking, in arriving at the right conclusion from data which are certain, and the enlargement generally of interest in life.

17. **Hancock, Harris.** Remarks on certain attacks that have been made upon the teaching of mathematics with counter criticisms. School and Society, vol. 6 (Sept. 22, 1917), pp. 339-344.

Discusses adverse criticisms and deductions. Causes of defective education in our public schools are given as: 1. Pernicious influence of teachers colleges—too much irrelevant and time-wasteful pedagogy; not enough actual and definite training. 2. Little emphasis on efficient scholarship of teachers. 3. Profession over feminized.

18. ———— What course of study should be taken by a boy who is entering high school? School and Society, vol. 1 (June 19, 1915), pp. 893-900.

An account of an investigation which Mr. Hancock carried out in Cincinnati and other places in the United States.

19. **Hawkes, H. E.** Educational values in mathematics training. Mathematics Teacher, vol. 4 (Dec., 1911), pp. 75-78; vol. 5 (March, 1912), pp. 85-89.

Endeavor to find a common standard for all subjects. Certain moral qualities must be the result of our teaching. There should be three results; facts, acquired establishment of mental qualities and moral stamina. Correlation of subjects.

20. **Hixson, Samuel.** The high school boy. Journal of Education, vol. 83 (Feb. 24, 1916), p. 200.

The mathematics of the high school boy should be the real mathematics of practical life. The workshop, store, factory, farm, household and family budgets, water and gas pressure, electricity and scores of other applications in practical life will afford abundant material.

21. **Hobson, E. W.** Extracts from the presidential address before the mathematical section of the British Association at Sheffield, 1910. Educational Review, vol. 40 (Dec., 1910), pp. 524-530.

It is wiser to allow pure mathematics to develop its own course rather than direct it as a useful tool in the application of the physical sciences. Mathematics cannot be defined for it is constantly enlarging its field. The certainty of mathematics depends on certain assumptions which it is the business of the philosopher and not the mathematician to investigate. The discoveries of mathematics were inductive discoveries depending largely on the intuition of the investigator. The teaching of mathematics ought never to descend to rule-of-thumb processes but should emphasize the principles underlying them.

22. Joint meeting of the Mathematical Association and the Public School Science Master's Association. Mathematical Gazette, vol. 5 (March, 1910), No. 85, p. 244.

A discussion of the report of the joint commmittee of the two associations on the "Coördination of the teaching of science and mathematics". The two subjects have numerous points of contact. Energy wasted by unnecessary overlapping; need for increased sympathy and coöperation between teachers of the two subjects.

23. **Koch, E. H., Jr.** Training for efficiency in elementary mathematics. Mathematics Teacher, vol. 3 (March, 1911), pp. 170-184.

An application of efficiency discussion to the special field of mathematics.

24. **Mayo, C. H. P.** The position of mathematics. Educational Review, vol. 57 (March, 1919), pp. 194-204.

The author deprecates the introduction of the purely practical course in mathematics in a course for general education. The mental training that the teaching of mathematics gives is a development of the imagination, a training of the logical faculty, a training in a sense of style, in accuracy and in the power of observation. The actual information imparted in mathematics has little value. The author suggests that under present ideals for the average boy, the study of mathematics should end at a much earlier age than at present. He suggests that lectures on fundamental principles, on mechanics, mechanical powers, astronomy, etc., be made a larger part of school work. For the boy who shows mathematical ability every opportunity and encouragement should be accorded.

25. **Mensenkamp, L. E.** Mathematics and education. School Science and Mathematics, vol. 18 (Nov., 1918), pp. 709-711.

The criticisms directed against mathematics are (1) that it is valueless since its raison d'etre is formal discipline, an exploded theory; (2) that it is abstract and remote from life. The critic emphasizes the lack of application of the elementary courses forgetting that they are introductory to higher courses which are almost entirely "applied". The teacher should be a propagandist and should use every means to bring before the public the importance of mathematics.

26. **Merrill, Helen A.** Why students fail in mathematics. Mathematics Teacher, vol. 11 (December, 1918), pp. 45-56.

(1) Mathematics is difficult as it demands accuracy of thought and statement, definite mental concepts, connected thinking, a fair memory, power of generalization, a willingness to work hard. (2) There is much poor teaching of mathematics due to poor training of teachers, etc. (3) Insufficient drill on fundamentals. (4) Poor textbooks. (5) Too large classes. (6) Too many outside influences.

27. **Milne, W. P.** Mathematics and the pivotal industries. Mathematical Gazette, vol. 9 (March, 1919), pp. 312-316.

An attempt to call attention more specifically to the mathematics needed by the average citizen who goes into ordinary industrial work.

28. **Minnick, J. H.** Our critics and their viewpoints. Mathematics Teacher, vol. 9 (Dec., 1916), pp. 80-84.

Discusses recent criticisms of mathematics by various individuals.

29. **Moore, C. N.** Mathematics and the needs of the hour. School and Society, vol. 8 (Nov. 23, 1918), pp. 622-623.

All the lessons of the war—army, navy, Red Cross activities, etc.—point to the necessity of mathematics. A criticism upon Dr. Snedden's article in the School Review, October, 1918.

30. ———— Mathematics in general education (Reply to Dr. Snedden's article in vol. 8, p. 714); School and Society, vol. 9 (Jan. 4, 1919), pp. 31-33.

"We can legitimately require of all high school pupils only those elements of preparedness which by their contact with a large proportion of possible future activities properly belong in a general education." The claims of mathematics are discussed from this point of view.

31. **Morrison, H. C.** Reconstructed mathematics in the high school. Mathematics Teacher, vol. 7 (June, 1915), pp. 141-153.

Abstract of article in the thirteenth yearbook of the National Society for the Study of Education.

32. **Moritz, R. E.** Mathematics and efficiency. School Science and Mathematics, vol. 16 (1916), pp. 233-245.

Education is efficient in the degree to which it develops the powers of the individual. Mathematics is indispensable to every scientific pursuit and to many occupations and professions. In its reverence for truth, demand for honesty of effort, patience and self-denial, clear thinking—it may claim a share in building character. See also No. 517.

33. National Committee on Mathematical Requirements. Mathematics Teacher, vol. 12 (Sept., 1919), pp. 30-32.

A note concerning the organization and work of the National Committee.

34. **Nunn, T. P.** Mathematics and individuality. Mathematical Gazette, vol. 9, (March, 1918), pp. 188-197.

A statement of ideals with respect to the individuality of the pupil, and of difficulties in attaining these ideals in view of the school organization of the present day.

35. **Pierce, Harriet R.** The value of mathematics as a secondary school subject. School Science and Mathematics, vol. 16 (1916), pp. 780-788.

"Whenever and wherever algebra and geometry are so taught that they give all the possible rich and varied attitudes toward life of which they are capable, their value as secondary school subjects and their value for all pupils will no longer be questioned".

36. **Rabourn, Sara B. F.** Boost mathematics. School Science and Mathematics, vol. 16 (1916), pp. 595-602.

The great need of today is for logical thinkers. Mathematics more than any other subject should produce the student who thinks, who can make a quick, accurate judgment. The subject matter should be humanized by lantern lectures, by historical background, by architectural applications. It "pays to advertise" in the mathematics department as well as elsewhere.

37. **Ransom, W. R.** The mathematical pessimist. Mathematics Teacher, vol. 7 (Sept., 1914), pp. 1-10.

The discouraged student whose only hope is the personality of the teacher. We should be more lenient toward those who grasp it not. If a teacher continues study he will be more sympathetic with pupil.

38. The influence of mathematics on scientific discovery. Educational Review, vol. 39 (April, 1910), pp. 351-356. (Reprinted from the Engineering Supplement of London Times, Jan. 19, 1919).

In the engineering world, by substituting mathematical prediction for trial and error, there is a shortening of the time taken in perfecting invention. In science especially in electricity, physics, biology and chemistry, mathematical work is increasingly important and productive in leading to scientific discovery.

39. Presidential address to the London branch of the Mathematical Association. Mathematical Gazette, vol. 7 (March, 1913), No. 104, p. 87.

The liability to reconditeness is the characteristic coil which is apt to destroy the utility of mathematics in liberal education. The purpose should be not the aimless accumulation of mathematical theorems but the final recognition that the preceding years of work have illustrated those relations of number, quantity and space which are of fundamental importance. History of mathematics helps generalize a student's ideas.

40. **Schorling, Raleigh.** Significant movements in secondary school mathematics. Teachers College Record, vol. 18 (Nov., 1917), pp. 438-457.

A consideration of the criticisms of mathematics and a statement of efforts being made to improve the situation. Discusses the effect of the junior high school, the movement for improved technique, and the standard tests movement, all of which tend to improve the teaching of mathematics.

41. **Schwatt, I. J.** (and others). Is the average high school pupil able to acquire a thorough knowledge of all the mathematics ordinarily given in these schools? Mathematics Teacher, vol. 3 (March, 1911), pp. 101-128.

Secondary mathematics has little application in the daily work of the average man. The purpose of secondary schools is to develop in the young, character and power of mind. A strong mind can be developed only through clear ideas. Very few high school graduates attend college. Have the pupil understand the concept. Select with care the geometry to be taught. Make use of original work for power of mind.

42. **Smith, D. E.** Certain problems in the teaching of secondary mathematics. Mathematics Teacher, vol. 5 (March, 1913), pp. 161-179.

Has the day of secondary mathematics passed? What is the value of algebra and geometry to the democracy of America? The inability of the teacher to judge wisely a child's calling so early makes it unsafe for mathematics to be elective. The nature of general mathematics is yet to be decided. A comparison with foreign schools.

43. ———— The International Commission on the Teaching of Mathematics. Educational Review, vol. 45 (Jan., 1913), pp. 1-7.

The article is a report of the meeting of the International Commission on the Teaching of Mathematics held at Cambridge, England, in August, 1912. 150 reports showing the status of mathematics in the various countries of the world were presented. Comparison of the different reports show that America on the average does not give her teachers the mathematical training that France or Germany does and hence our secondary work is not as progressive as in mid-European countries.

44. ———— What is to be the outcome? Mathematics Teacher, vol. 9 (Dec., 1916), pp. 77-79.

(1) It is likely in a few years that pupils will be admitted to college without mathematics. (2) The attack on mathematics has been of two natures: (a) by educators whose powers of speaking and writing lead them to court applause through cleverness of statement; (b) by serious scholars who try to suggest remedies. (3) The divergent results of investigations do not prove that mathematics has no general disciplinary value. (4) Secondary mathematics will continue to be taught for a long time.

45. **Snedden, David.** Mathematics and the needs of the hour. School and Society, vol. 7 (Dec. 14, 1918), p. 714.

Admits need of mathematics in certain vocations—engineering, artillery, navigators. Require it of some because of its definite use—but why prescribe for all, e.g., girls?

46. ———— Mathematics in the education of youth from 12 to 18 years of age—Some proposals for further study. Educational Administration and Supervision, vol. 5 (March, 1919), pp. 125-145.

"It is the purpose of this paper to analyze in the light of present thought, some of the problems involved as to the place of mathematical studies in secondary schools, and to set forth, as a basis for further inquiry and discussion, certain hypotheses and proposals, especially as regards educational aims".

47. **Steggal, J. E. A.** On practical mathematics in schools. Mathematical Gazette, vol. 7 (March, 1914), No. 110, p. 287.

Outlines work in tests and measurements, graphing, use of logarithms, linkages, use of models of surfaces and curves. However, the primary value of mathematical study is not to be found in such results as skill with the penny ruler, scissors and scale, but in the culture that all genuine mathematical study has been held to give from the days of Plato to the present.

48. **Stratton, W. T.** Mathematics and life (A discussion of Dr. Carmichael's paper). School Science and Mathematics, vol. 15 (1915), pp. 115-120.

The author mentions five points in his discussion. 1. It is of the utmost importance that the teacher have a vision of the value of mathematics and that he have the ability to pass that on to his pupils. 2. Provision must be made for individual differences in pupils. 3. The general public determines the value of mathematics by the ability the pupils show in applying their knowledge to real situations. 4. The proper selection of applied problems will aid in vitalizing mathematics. 5. The teacher is, however, the most important single factor in vitalizing the work.

49. **Sykes, Mabel.** The case against high school mathematics. School Science and Mathematics, vol. 17 (1917), pp. 667-676.

An examination of criticisms of mathematics as a high school subject, and a protest against wholesale criticism based on insufficient data.

50. ———— The mathematical survey of the Chicago school system. School Science and Mathematics, vol. 15 (1915), pp. 120-126.

Discussion of a committee report on the work in arithmetic in the grades and the work in mathematics in the high schools of Chicago.

51. **Wheelock, C. P.** Are particular abilities necessary for pupils to gain an understanding of the elementary and secondary mathematics as usually given at the present time? Mathematics Teacher, vol. 6 (June, 1914), pp. 192-197.

Types of subject matter, (1) technical language, (2) apprehension by senses or imagination, (3) abstraction of mathematics, (4) statistical results of mathematical examinations.

52. **White, C. E.** Mathematics and anti-mathematics. School Science and Mathematics, vol. 19 (1919), pp. 29-37.

Argues for the value of mathematics as a high school subject.

53. **Whitehead, A. N.** The aims of education—A plea for reform. Mathematical Gazette, vol. 8 (Jan., 1916), No. 121, p. 191.

What is proved should be utilized. As to what examples of its use should be given will depend upon the genius of the teacher, the intellectual types of the pupils, their prospects in life and opportunities offered by the immediate surroundings of the school. It is for this reason that uniform outside examinations are so fatal.

54. **Whitehead, A. N.** Technical education and its relation to science and literature. Mathematical Gazette, vol. 9 (March, 1917), pp. 20-33.

An inspiring address by the President of the Mathematical Association of Great Britain, advocating that in mathematics, as in all lines of effort, "work should be transfused by intellectual and moral vision, and thereby turned into joy, triumphing over its weariness and pain".

55. **Wood, W. H.** Some reflections on the current discussion of secondary mathematics. School Science and Mathematics, vol. 17 (1917), pp. 815-818.

There is imperative need for scientific soundness and precision in the choice and use of terms and descriptive language in the present controversy over mathematics as a factor in education. The student of mathematical pedagogy should use every available means and opportunity for analysis, experimentation and psychological demonstration toward a genuinely scientific and universally valid solution of the question.

56. **Yocum, A. D.** Mathematics as a means to culture and discipline. Mathematics Teacher, vol. 6 (March, 1914), pp. 135-157.

Pupil loses when everything is strictly utilitarian. Public opinion is all the product of cumulative impression based on part experience not wholly forgotten. Desirability of vocabulary expansion. What is most useful to those not becoming specialists?

57. **Young, J. R.** Recent tendencies in the teaching of elementary applied mathematics. School Science and Mathematics, vol. 17 (1917), pp. 237-244, 321-328.

A discussion of the changes taking place in mathematics courses in various classes of schools.

58. **Young, J. W. A.** Fifth International Congress of Mathematicians. School Science and Mathematics, vol. 12 (1912), pp. 702-715.

Sketches conditions in America today, noting (1) preparation of teachers; (2) weeding out of obsolete applications in arithmetic; (3) arithmetic to meet child's interests and needs; (4) textbooks to meet local conditions; (5) tendencies to change secondary curriculum in mathematics so as to teach algebra and geometry simultaneously; (6) college entrance requirements; (7) requirements for advanced work in mathematics for the Master's degree.

II. The Curriculum

59. **Aley, R. J.** See No. 67.

60. **Betz, William.** What mathematical subjects should be introduced in the curriculum of secondary schools? Mathematics Teacher, vol. 5 (June, 1913), pp. 218-233.

The need to discover objectives for reorganization. The shift in secondary education. Three distinct ideals: Individualism, vocational training, social efficiency. Attempts at reorganization require agreement on aims of teaching, namely insight and efficiency. The author lists five distinct aims and shows how the present curriculum fails to satisfy.

61. **Breslich, E. R.** Forward movements in secondary mathematics. School Review, vol. 24 (April, 1916), pp. 283-297.

A reply to D. E. Smith's review of the author's "First Year Mathematics for secondary schools" in the Bulletin of the American Mathematical Society for December, 1915.

62. **Breslich, E. R., Schorling, Raleigh, Wright, Horace C., Irwin, Harry N.** Course of study in secondary mathematics in the University High School, University of Chicago, School Review, vol. 24 (Nov., 1916), pp. 648-674.

Describes the course of study being worked out in the University High School of the University of Chicago.

63. **Cobb, H. E.** Current educational movements and general mathematics. School Science and Mathematics, vol. 16 (1916), p. 415.

Only an abstract of this address is printed. Algebra and geometry have been organized as distinct sciences, but there is no reason why, with the broader view and greater enlightenment of the present day, they should not be united and possibly even with other allied subjects make a transformed subject which will be more of an art than a science. The movement is reflected in the modern textbooks, but each teacher should proceed with individual experimentation which will benefit both teacher and pupil.

64. **Cockerell, T. D. A.** Required mathematics. School and Society, vol. 6 (July 21, 1917), pp. 84-85.

Comment on article by A. R. Crathorne in School and Society, vol. 6 (July 7, 1917). Shorthand methods and symbolism lack content. Child needs contact with realities.

65. **Collins, J. V.** The Perry idea in the mathematical curriculum. School Science and Mathematics, vol. 12 (1912), pp. 296-299.

We need the spiral idea in mathematics in high schools no isolated subjects in mathematics. Vitalize mathematics with practical problems from other branches. Use graphs in analytic geometry. Teach elements of calculus in the high school. Unify all mathematics from arithmetic to calculus raising the entrance standards in college and secondary schools.

66. **Crathorne, A. R.** Required mathematics. School and Society, vol. 7 (July 7, 1917), pp. 6-17.

Discusses recent criticisms of mathematics as a required subject and presents arguments in favor of certain reforms.

67. **Crawley, E. S. and Aley, R. J.** What mathematical subjects should be in the curriculum of secondary schools from the point of view of the college? Mathematics Teacher, vol. 5 (Sept., 1913), pp. 234-239, 243-246.

Algebra and geometry. The possibility of simplifying the complex down to a few general principles; insight into beauty and simplicity of process is needed. Teach pupil to use his brain. Seek most of trouble in material rather than in method. Banish memory and teach brain work. Pupils need to know how to study, courage and persistency, mental alertness, accuracy, speed in calculation, short direct methods. Teach him to appreciate logical reasoning.

68. Dooley, W. H. Practical mathematics for the high school. Mathematics Teacher, vol. 10 (June, 1918), pp. 181-184.

There is need to organize the mathematics common to every industry. Teachers may visit the trades and study the materials, handtools, illustrations and then with a well-graded textbook in the subject, be fairly equipped to handle a class. All problems in vocational mathematics should be made to arise in a vital and natural way. Thrift or conservation should be the keynote.

69. Evans, G. W. Reconstruction of mathematical requirements. Mathematics Teacher, vol. 11 (Sept., 1918), pp. 26-33.

Discusses a course in general mathematics, of which the unifying topics are (1) approximate computation; (2) the equation as a means of solving problems; (3) congruence and (4) similarity.

70. ——————. The curriculum of mathematics in secondary schools. Mathematics Teacher, vol. 5 (June, 1913), pp. 214-217.

Why changes are needed: (1) to secure more perfect continuity with elementary work; (2) to secure pupils' spontaneous interest. What work would promote these purposes. In enumerating operations likely to do this the author mentions subjects such as calculus not now taught in high school.

71. Gray, J. C. A general science course of elementary physics and mathematics combined. School Science and Mathematics, vol. 12 (1912), pp. 377-380.

Course of elementary physics and mathematics combined so that each subject will furnish material for the other—to be given in the second year five times a week, following first year algebra—for the purposes of (1) laying a firm foundation for further work in science, and (2) helping the pupil who finds mathematics hard. Algebra and geometry being brought in as they are needed in physics; speed, accuracy and quick methods of solution being emphasized rather than more difficult and less useful principles.

72. Guthrie C. C. and Holden, F. G. What mathematical subjects should be included in the curriculum of the college? Mathematics Teacher, vol. 6, (Dec., 1913), pp. 110-112.

Conditions encountered by students in college. Discussion of schemes of the day for changes in mathematical curriculum. Time and content both have to be considered in college schedule. Let mathematics be elective but show pupils the value of it in preparatory school so they will elect it.

73. **Hawkes, H. E.** Mathematics in the college course. Educational Review, vol. 40 (Sept., 1910), pp. 145-156.

The elective system has forced school subjects to justify their existence. Mathematics can be justified in the course because (1) it gives a training in the precise use of the English language, (2) it develops the faculty of reasoning, (3) it opens the door to an appreciation of truth and beauty. Undergraduate courses fall into two classes; general and advanced. The aim of the advanced courses should be broader than the professional demands of the technical students would require. For the general student the cultural aim should be predominant.

74. **Herrick, C. A.** What high school studies are of most worth? School and Society, vol. 4 (Aug. 26, 1916), pp. 305-309.

The curriculum should contain only those subjects which can be shown to have value because of present day needs. Much geometry and algebra should disappear.

75. **Hester, F. O.** Economics in the course in mathematics from the standpoint of the high school. School Science and Mathematics, vol. 13 (1913), pp. 751-757.

Contrasts the arrangement of the typical American course with that of Germany and France in which the various branches of mathematics are taught simultaneously and are begun much earlier, thus being spread over a longer period of time. Advises a somewhat similar arrangement in our courses and also the elimination of much of the purely mechanical drill in algebraic processes.

76. **Holden, F. G.** See No. 72.

77. **Irwin, Harry N.** See No. 62.

78. **Lennes, N. J.** Mathematics for culture. Educational Review, vol. 47 (May, 1914), pp. 469-477.

The article suggests an outline of general mathematics for cultural purposes in the college. It shows how poorly the traditional college mathematics courses give a broad cultural view of the significance of the subjects taught in the growth of civilization and in the building up of the mental universe of man. It suggests how vital portions of trigonometry, solid geometry, analytical geometry and the calculus could be chosen to make a cultural course in mathematics.

79. Mathematical curriculum for Gymnasia. The Meraner Lehrplan. Translated by E. A. Price. Mathematical Gazette, vol. 6 (Dec., 1911), No. 95, p. 179. See also No. 370.

Gives in detail the mathematical curriculum for gymnasia in Germany recommended by the Meran report.

80. **McKeehan, C. L.** What mathematical subject should be included in the curriculum of the secondary schools? Mathematics Teacher, vol. 5 (June, 1913), pp. 193-196.

Arithmetic, algebra and geometry teach the power of calm, steady attention. Accuracy is only real benefit. Curriculum shows exaggerated estimate of boys' mental capacity. Suggestion of set-up in college entrance requirements to aid secondary school in making boy understand arithmetic thoroughly then take other subjects at more mature years.

81. **Moskowitz, D. H.** A course in the appreciation of mathematics. School Science and Mathematics, vol. 17 (1917), pp. 676-678.

Suggests a tentative outline for a lecture course in the appreciation of mathematics, classes meeting twice a week and composed of those who have had 2½ units of work in the subject.

82. **Myers, G. W.** Educational movements and general mathematics. School Science and Mathematics, vol. 16 (1916), pp. 97-105.

Discusses the advantages of the fusion type of organization of high school mathematics.

83. ———— Report on the unification of mathematics in the University High School of the University of Chicago. School Science and Mathematics, vol. 11 (1911), pp. 777-791.

Claims that unified mathematics material has distinct advantages over the separate types of treatment, and no serious loss in other respects. Improvements noted: (1) Fewer pupils demand exemption from mathematics classes; (2) pupils are more willing to search for reasons for their procedure; (3) improvement in mathematical versatility; (4) greater mathematical self-reliance; (5) diminution in the number of failures; (6) more graphic and vivid modes of presentation. Mathematics is more rational, effective and discriminating.

84. ———— Two years' progress in mathematics in the University High School of the University of Chicago. School Science and Mathematics, vol. 11 (1911), pp. 64-73.

Mixed geometry and algebra in the first two years with primary stress on geometry of second year mathematics. Third and fourth year mathematics consists of two parallel courses, one in plane trigonometry and the other in algebra through and beyond quadratics. During the first half of the third year, trigonometry three hours a week and advanced algebra two hours; in second half year ratio is reversed. For all pupils except candidates for technological schools, the work of third and fourth years is optional excepting a special review course in the fourth year of one hour a week for students seeking University High School Diploma.

85. **Pitcher, A. D.** Reorganization of mathematical curriculum in secondary schools. Mathematics Teacher, vol. 8 (Sept., 1915), pp. 1-15.

A discussion of the mathematical situation, with concrete suggestions for changes in the curriculum.

86. **Price, E. A.** See No. 79.

87. **Reeve, W. D.** Unification of mathematics in the high school. School and Society, vol. 4 (Aug. 5, 1916), pp. 203-212.

Discusses recent criticisms of mathematics teaching by Drs. Morrison, Snedden, and Flexner, and suggests remedies. Gives instances of unification for each of the four years of a high school.

88. Report on vocational mathematics. School Science and Mathematics, vol. 16 (1916), pp. 337-346.

Part 1. The application of algebra, geometry, and trigonometry to the problems of industry. Part II. The application of algebra, geometry, and trigonometry to the problems of agriculture.

89. **Rich, S. G.** Compulsory mathematics. School and Society, vol. 5 (March 10, 1917), pp. 290-291.

Describes mathematical work for normal students among the Zulus at Amanzimtoti Institute, Natal, South Africa. "We should aim to make geometry, algebra and advanced arithmetic keys for understanding our environment".

90. **Schorling, Raleigh.** See No. 62.

91. **Schwatt, I. J.** On the curriculum of mathematics. Mathematics Teacher, vol. 2 (June, 1910), pp. 171-178; vol. 3 (Sept., 1910), pp. 1-8.

There are two tendencies in educational policy today one practical and the other cultural. Judgment and character are necessary. Quality not quantity of subject matter. Intensity must be increased to meet environment. Make the pupil understand concepts. Formulas and logarithmic tables with a pupil of high school age are pernicious. Grade the work, make the course meet needs of pupil. Intellectual power is the aim.

92. **Short, R. L.** Discussion of the unification of secondary mathematics. School Science and Mathematics, vol. 10 (1910), pp. 246-249.

First year—arithmetic, algebra through proportion, all straight line geometry found in Books II, III, IV, V. During the first twelve weeks, teach oral control of numbers in arithmetic, four fundamentals of algebra, geometric construction, four theorems, about a dozen originals, and simultaneous equations of two or three unknowns. Second year—straight line, part of Book IV, trigonometry of the right triangle, evolution, quadratics, the curvilinear parts of Books II, III, IV, V. Make some use of the lever, do something of the proportion of chemistry and simple problems involving gravity.

93. **Smith, D. E.** Introductory course in mathematics. Mathematics Teacher, vol. 11 (March, 1919), pp. 106-114.

Suggestions: (1) Arithmetic should occupy the pupil's attention in the first half of the 7th grade to avoid a too pronounced break in mathematics. (2) The second half year should be given to intuitive geometry. (3) Next the pupil builds up formulas and meets the real equation, the graph, the negative number. (4) Application of algebra to arithmetic. (5) Post-intuitive mathematics should be algebra, trigonometry, demonstrative geometry.

94. ———— Mathematics in the training for citizenship. Teachers College Record, vol. 18 (May, 1917), pp. 211-225.

Presents the reasons for studying mathematics. Contains a suggestion for the revision of the course of study in mathematics for grades 7, 8 and 9.

95. ———— Vocational courses in mathematics for secondary schools. Mathematics Teacher, vol. 10 (June, 1918), pp. 179-180.

The pupils before the tenth grade should know the useful parts of algebra, the significance of intuitive geometry, nature and purpose of trigonometry, the meaning of a mathematical proof. After this the pupil may safely specialize in industrial and commercial lines or in pure mathematics in our secondary school. Two methods of teaching may bring about our mathematical desires, the one of laying a thorough scientific basis in pure mathematics and building the applied mathematics upon this foundation or the Perry idea, that of beginning with applied mathematics and taking only as much theory as might be necessary for a practical working knowledge.

96. **Thomas, F. W.** Mathematics in the secondary schools from point of view of the industrial establishment. Mathematics Teacher, vol. 5 (June, 1913), pp. 206-214.

The importance of thoroughness in elementary schools. The lack shown by experiment of the Santa Fe Railroad. Mathematics most useful branch in trade. Writing and neat figuring are vital. Value of individual instruction and need for more practice in practical fractions. Failure to make changes in curriculum correspond with changes in economic condition.

97. **Von Narchoff, E. R.** Mathematics for service. Mathematics Teacher, vol. 2 (March, 1910), pp. 116-126.

A suggestion to teach additional mathematical subjects in the high school, with remarks as to emphasis and omission. The article closes with a high school curriculum recommended by the author.

98. **Walsh, C. B.** Mathematics in the Ethical Culture High School. Mathematics Teacher, vol. 2 (June, 1910), pp. 155-163.

A report covering the aims of the mathematics course of study at the Ethical Culture High School, the organization and equipment of the department, the course of study, and pupils' activities.

99. **Winslow, I. C.** How much mathematics should be required for graduation from high school? Education, vol. 36 (May, 1916), pp. 581-584.

"....as a universal requirement for graduation from high school, a full course of mathematics for one year should be sufficient."

100. **Wright, H. C.** Discussion of W. D. Reeve's paper "Unification of mathematics in high school." School and Society, vol. 4 (Oct., 1916), pp. 561-563.

Agrees and approves. Suggests modern texts both American and foreign. Unification needs teachers (1) familiar with high school subject matter, (2) familiar with college, technical, etc., matter; (3) free from text domination.

101. ——————— A discussion of the report of the Committee on the unification of mathematics. School Science and Mathematics, vol. 10 (1910), pp. 242-246.

Secondary school pupils should get considerable practice in the fundamental operations of mathematics and some glimpses of the possibilities of the subject. Unification would permit of more time for practice in the technique of the subject, and give pupils an acquaintance with at least four phases of mathematics—arithmetic, algebra, geometry and trigonometry.

102. ——————— See No. 62.

103. **Young, J. W. A.** Concerning systematic exposition of mathematics and the fusion of its various branches in secondary instruction. School Science and Mathematics, vol. 12 (1912), pp. 447-456.

Briefly outlines the mathematical work in French and German schools and concludes that, in general, beginnings should be concrete, and intuition should not be excluded at any stage, the teacher accepting without proof anything that is intuitively sufficiently obvious. Under fusion: (1) algebra and geometry should be taught simultaneously, but not fused; (2) plane and solid geometry (3) plane geometry and trigonometry (4) solid geometry and descriptive geometry (in 2-4 groups, varying degrees of fusion are possible); (5) synthetic and analytic geometry of the conic sections; (6) differential and integral calculus.

III. Arithmetic

104. Arithmetical surgeons (Editorial). Journal of Education, vol. 84 (Dec. 28, 1916), p. 658.

A statement in detail of eliminations in arithmetic recommended by the business men of Connersville, Ind.

105. Arithmetic, new or neutral (Editorial). Journal of Education, vol. 86 (Dec. 6, 1917), p. 574.

Arithmetic today must eliminate some of the traditional applications that no longer apply, must introduce new applications in the concrete, must eliminate definitions that do not define, and introduce statements that are illuminating; but first and last and all the time, the maker of an arithmetic for today must be neutral as to the theories old and new and make a book with which teachers as they are can get adequate results with children as they are.

106. Arithmetic of good roads (Editorial). Journal of Education, vol. 73 (June, 1911), pp. 699-700.

Demonstrates the practical value of a regular school subject in enlightening the public upon a special subject. The pupils in the upper grades of all the schools were furnished with a four-leaf "Arithmetic of Good Roads" in which the pupils took home problems dealing with the population, miles of roads, estimated costs and bond issues, taxes and investments involved and relative per cent of use of existing roads by wagons and automobiles. Prejudice was overcome and the good roads obtained.

107. **Aspinwall, W. B.** Teaching thrift by means of arithmetic. Educational Review, vol. 53 (May, 1917), pp. 512-516.

Developing habits and ideals in school has often been poorly done due to unsatisfactory methods. The article is suggestive of the many ways in which the arithmetic of banking, insurance and investments can be made to function in developing thrift.

108. **Bachman, F. P.** See No. 130.

109. **Bell, J. C.** A class experiment in arithmetic. Journal of Educational Psychology, vol. 5 (1914), pp. 467-470.

Standard Courtis tests in arithmetic, series B, were given to a class of 25 sophomores at the summer session of the University of Texas, and the results compared with scores made by some 20,000 Boston school children. The average accuracy index was low for Texas students.

110. **Boyer, P. A.** The Courtis tests in arithmetic. Mathematics Teacher, vol. 11 (March, 1919), pp. 121-132.

The directions, results, tests and graphic results are given.

111. **Broome, E. C.** Arithmetical emphasis and omissions. Journal of Education, vol. 78 (Nov. 6, 1913), p. 467.

The introduction to the course of study elaborated at Mt. Vernon, New York. This course was so arranged that heaviest work came in the fourth, fifth and sixth grades where pupils are physically best able to handle it, and wherever possible and profitable, there was correlation with other subjects of the curriculum. Omissions were numerous and radical.

112. **Brown, J. C.** An investigation on the value of drill work in the fundamental operations of arithmetic. Journal of Educational Psychology, vol. 2 (1911), pp. 81-88.

Report of an experiment at the Eastern Illinois State Normal School to secure information concerning the value of short drill exercises in the fundamental operations of arithmetic. Five-minute drill periods upon the fundamental number-facts preceding the daily lesson in arithmetic were found to be beneficial in the sixth, seventh, and eighth grades. The benefit was not limited to improved mastery of number habits, but included increased efficiency in arithmetical reasoning.

113. ———— An investigation of value of drill-work in the fundamental operations of arithmetic. Journal of Educational Psychology, vol. 3 (1912), pp. 485-492; 561-570.

An investigation in the sixth grades of several school systems to secure data relative to the value of brief drill periods in arithmetic. The method used was to determine the improvement made by the "drill" class upon its previous record, and the similar improvement in the case of the "non-drill" class. In all the tables it appears that the improvement made by members of the "drill" classes averages from two to three times as much as the improvement made by members of the "non-drill" classes.

114. **Chase, Sara E.** An attempt to make arithmetic lessons more concrete. Journal of Education, vol. 87 (Feb. 14, 1918), p. 187.

A statement of methods and devices used to give children power to use numbers intelligently and accurately.

115. .... Waste in arithmetic. Teachers College Record, vol. 18 (Sept., 1917), pp. 360-370.

116. **Clapp, Amy L.** See No. 155.

Discusses waste in connection with the teaching of mensuration, and as the result of the fact that vocabulary and subject matter of textbooks are unsuited to the capacities of the pupils.

117. **Cobb, Margaret V.** A preliminary study of the inheritance of arithmetical abilities. Journal of Educational Psychology, vol. 8 (1917), pp. 1-10.

The article aims to set forth a method which seems to make possible a quantitative treatment of mental inheritance. An investigation was made in 1912 at the University of Illinois testing the theory by actual measurements of families. The result showed a certain resemblance between parent-and-child ability. The child usually resembled the like parent. It is difficult to avoid the conclusion, says the author, that this likeness is due to heredity.

118. **Cole, L. W.** Adding upward and downward. Journal of Educational Psychology, vol. 3 (1912), pp. 83-94.

Measured by averages, twenty-nine out of a group of thirty persons, selected at random, added the same problems more rapidly and less accurately when adding upward than when adding downward. Counting to the left was also slower and more accurate than counting to the right. In both experiments the factor of habit seemed to produce a saving of time at the expense of accuracy.

119. **Collier, Myrtie.** Practical work in arithmetic. School Science and Mathematics, vol. 16 (1916), pp. 524-529.

In making elementary arithmetic practical two elements enter: (1) the development of number, of the building up of a clear concept of number relations; (2) the application of the number facts to the quantitative side of the pupils' daily experiences. Arithmetic should be treated as a science applied to practical living. The problems should be really practical and suited to the environment of the child. Much of the work may be dramatized, for example, banking.

120. **Collins, J. V.** Accuracy in arithmetic. Education, vol. 32 (Oct., 1911), pp. 81-87.

According to the author, accuracy is to be secured by (1) checks, (2) proofs, and (3) duplicate solutions.

121. **Cosby, Byron.** An experiment in Mound City (Mo.) public schools in arithmetic teaching. School Science and Mathematics, vol. 11 (1911), pp. 629-634.

Shows how to make arithmetic more alluring and effective by the introduction of real problems, giving stimulating illustrations.

122. ———— The purpose and content of high school arithmetic. School Science and Mathematics, vol. 17 (1917), pp. 427-433.

The content of high school arithmetic is a reproduction of those experiences that the race has found valuable or is to find valuable in the near future. The pupil must be made to see the content of arithmetic in terms of his experience, in language that he understands, in the large units of actual every-day happenings (economic, industrial).

123. **Courtis, S. A.** Research work in arithmetic. Educational Administration and Supervision, vol. 3 (Feb., 1917), pp. 61-74.

Discusses the work of the Department of Educational Research of Detroit.

124. **Curtis, A. M.** In what operations in arithmetic should a pupil be required to understand the reasons for the steps taken? In what operation should such a requirement be postponed to a mature age? Mathematics Teacher, vol. 5 (March, 1913), pp. 180-182.

All is dependent on teacher's judgment of individual pupils. A plan of treatment is given for every subject in arithmetic.

125. **Davies, G. R.** Elements of arithmetical ability. Journal of Educational Psychology, vol. 5 (1914), pp. 131-140.

Classes in arithmetic were given certain tests more or less closely connected with the routine work. The tests included computation, reasoning, visualizing, memory span, and mental work. The principal purpose was to discover if there were any simple ways of testing general efficiency in arithmetic, or special ability in any particular department of the work. It was found that while there was a wide diversity in the individual capacities of the pupils, certain tests gave a close approximation to the arithmetical efficiency as measured by the term grade.

126. **Davison, Charles.** On accuracy in arithmetic work. Journal of Education (Brit.), vol. 37 (1915), pp. 209-210.

From a set of examination papers the relative degree of accuracy at various ages was obtained. Conclusion is that accuracy in a given subject cannot be maintained without practice in that particular subject.

127. **Dooley, C. R.** What minimum of practical applications should be included in a course in arithmetic? Mathematics Teacher, vol. 5 (Sept., 1912), pp. 9-17.

Education should help student to fuller life. A few practical problems are needed to show use of mathematical principle. A teacher should have courage to tell a child unqualified for college work not to try to do it.

128. **Dooley, W. H.** Arithmetic in a Massachusetts industrial school. School Science and Mathematics, vol. 11 (1911), pp. 246-249.

A thorough drill in practical problems is given, leaving the logic of mathematics for those better equipped mentally and having an incentive for its study.

129. **Durell, C. V.** The arithmetic syllabus in secondary schools. Mathematical Gazette, vol. 6 (March, 1911), No. 91, p. 28.

The tendency of recent years has been to broaden the course of the average boy. In manipulation, omit unnecessary work in H.C.L., L.C.M., square, and cube root, etc. Given plenty of practical problems, simple areas, volumes, proportion, percentage, etc.

130. **Elson, W. H.** and **Bachman, F. P.** The old vs. the new three r's. Education, vol. 30 (May, 1910), pp. 571-581.

The present arithmetic is regarded not as a science as was the old, but as the science of number applied to business operations; and where the present-day arithmetic differs from that of the past is in the emphasis upon utility and in the particular application made of arithmetical principles and processes.

131. **Evans, J. E.** and **Knoche, F. E.** Effect of special drill in arithmetic as measured by the Woody and the Courtis Arithmetic Tests. Journal of Educational Psychology, vol. 10 (1919), pp. 263-276.

An account of an investigation made to determine the value of systematic drill work. The report gives the manner in which the tests were given and the scores. Concludes that such drill exercises have a great value in increasing the ability of a class in accuracy.

132. **Garfinkel, M. A.**—Effect of the summer vacation on ability in the fundamentals in arithmetic. Journal of Education Psychology, vol. 10 (1919), pp. 44-48.

This investigation was undertaken primarily for the purpose of ascertaining the influence that the summer vacation has on arithmetical ability, with a view to locating weak points needing emphasis on the return to school.

133. **Germann, G. B.** What should be the essentials of a course in mathematics for elementary schools in New York City? Mathematics Teacher, vol. 4 (June, 1912), pp. 127-135.

No course of study can be perfect or final. It should not be the expression of individual opinion. It should be the outgrowth of the experience of many. It should satisfy the conditions of a specific situation. How to select material in New York City. Opinion of experts consulted. Outline of a course.

134. **Gist, A. S.** Errors in fundamentals in arithmetic. School and Society, vol. 6 (Aug. 11, 1917), pp. 175-177.

Standard tests should measure ability of class and individual and definitely locate deficiencies. "To eliminate errors, it is necessary to definitely locate them, that teachers and pupils may know where the greatest possibility of error is, and that problems be specially prepared to give the kind of drill necessary".

135. **Green, Jenny L.** Character-building content of arithmetic. Mathematics Teacher, vol. 11 (Sept., 1918), pp. 36-41.

Society has demanded an education that prepares for life. We need a more socialized content of arithmetic. Society needs a type of ethical appreciations which belong to arithmetic.

136. **Hahn, N. H.** and **Thorndike, E. L.** Some results of practice in addition under school conditions. Journal of Educational Psychology, vol. 5 (1914), pp. 65-84.

One hundred and ninety-two pupils from the 7th, 6th, 5th, and 4th grades were given tests in adding columns, each of ten one place numbers, as rapidly as was consistent with accuracy. A "control group" took only the initial and final tests without intervening practice—and scores were compared. The results were significant with respect to: (1) the amount of improvement; (2) the rate of improvement; (3) changes in rate during course of practice; (4) the effect of period-length upon the rate of improvement; (5) the individual differences in initial efficiency; (6) individual differences in rate of improvement; (7) relation of rate of improvement to initial ability.

137. **Heck, W. H.** The efficiency of the grammar grade pupils in reasoning tests in arithmetic at different periods of the school day. Journal of Educational Psychology, vol. 5 (1914), pp. 92-95.

Reasoning tests in arithmetic were given at morning and afternoon periods of the day to 16 classes in the schools of Lynchburg, Va., to 40 classes in New York City and to 16 classes in the intermediate school in Roanoke, Va. The results showed that the number of examples done in the afternoon was 0.68% greater than in the morning; but the percentage of examples correct in the afternoon was 3.22% less than in the morning.

138. **Hornbrook, Adelia R.** Outline of arithmetic learning. Journal of Education, vol. 80 (Oct. 1, 1914), pp. 292, 297, 319, 324.

Gives course of study in State Normal Training School, San Jose, California, and methods used in carrying it out. The course is given in detail, for years 1 to 8 inclusive.

139. **Hunt, Brenelle.** Selected problems in industrial arithmetic. Journal of Education, vol. 72 and 73 (1910-1911).

A series of 15 articles, including material derived from various trades.

140. **Jackson, Marshall.** First state of arithmetic teaching. Journal of Education (Brit.), vol. 33 (1911), p. 426.

Advocates making arithmetic for pupils from ages seven to ten, wholly mental; no complex processes; problems all simple and within pupil's experience; lessons of from fifteen to twenty minutes at every session; rules first introduced at age of ten; long arithmetical operations seldom introduced.

141. **Johnson, C. W.** See No. 146.

142. **Kennedy, A.** Uses and abuses of "Schools helps" in the teaching of arithmetic. School Science and Mathematics, (1912), vol. 12, pp. 110-114.

Criticizes illustrations from textbooks concluding that a clear line of distinction should be drawn between clearness of reasoning, accuracy and rapidity on the one hand, and oral and written expression, simple, clear, unified on the other. Suggests systematic drills, intensive rather than extensive on the fundamentals in arithmetic, focusing attention, not on results but on thinking.

143. **Knoche, F. E.** See No. 131.

144. **Lindquist, Theodore.** Percentage. Journal of Education, vol. 84 (Oct. 5, 1916), p. 319.

Makes a number of suggestions as to how to introduce the subject of percentage, based on the thesis that the difficulties encountered by the pupil are difficulties of language mainly.

145. **Mackintosh, W. D.** Where the arithmetic fails and why. Journal of Education, vol. 76 (Oct. 31, 1912), p. 456.

For the business office "more arithmetic" in the school, tables, counting machines and checking devices can not eliminate economic waste, because it is due to an undeveloped number-sense. Only when arithmetic recognizes that it is dealing with pure number and confines itself to mental processes, can it succeed as a developer of the number-sense.

146. **Mead, C. D.** and **Johnson, C. W.** Testing practice material in the fundamentals of arithmetic. Journal of Educational Psychology, vol. 9 (1918), pp. 287-297.

An experiment performed with 200 5th and 6th grade pupils in Cincinnati, to obtain results upon which an opinion could be based as to the degree to which pupils were made proficient by the use of two kinds of practice material.

147. ———— and **Sears, I.** Additive subtraction and multiplicative division tested. Journal of Educational Psychology, vol. 7 (1916), pp. 261-270.

An account of a four-months' controlled test in additive subtraction (the so-called "Austrian method") and in multiplacative division.

148. **Morrison, A. H.** Mathematics in the elementary schools. School Science and Mathematics, vol. 12 (1912), pp. 593-605.

Modify the elementary school course in mathematics by confining ourselves largely to those things likely to be of greater use to the children who enter unskilled vocations. Reduce time given to written work and increase that given to oral work, using smaller numbers and more problems of a kind. Teach addition, substraction, movement of decimal point in multiplication and division—confine mensuration to triangles and rectangles and square root to small numbers. Use slide rule. Introduce the simple equation into percentage. Use graphs. First object—accuracy; second—to detect inaccuracy.

149. **Munhall, Ruth.** See No. 155.

150. **Park, L. L.** Applied arithmetic. Mathematics Teacher, vol. 5 (Sept., 1912), pp. 18-25.

How much must we train the pupil in the use of mathematical tools for practical purposes? A pupil should know how to think out problems for himself. Analysis must be a tool for practical results.

151. **Phillips, F. M.** Comparison of the work done in the successive minutes of a ten-minute practice period in the fundamentals of arithmetic. Journal of Educational Psychology, vol. 7 (1916), pp. 271-277.

An investigation made to determine how far the results of a one-minute test in the fundamentals of arithmetic are a fair measure of the pupil's ability. Tests, some Courtis A, were given to representative districts in Iowa in grades 4-8 inclusive, sex division even. Conclusion was: from 6 to 12% more work was accomplished in the first minute. This high level was seldom reached later. Poorest scores occurred frequently in the 2nd minute. Errors less frequent in the first minute.

152. ———— Value of daily drill in arithmetic. Journal of Educational Psychology, vol. 4 (1913), pp. 159-163.

"Stone tests" were given in sixth, seventh and eighth grades of the public schools of Granite Falls, Minnesota. Two groups of as nearly equal ability as possible were made. One drill ten minutes daily for two months; the other had no drill. Then a second test was given and the results compared. Tendencies noted were: (1) Substantial increase in abilities of both groups (2) larger gain made by drill group.

153. **Reed, William.** What minimum of practical applications should be included in a course in arithmetic? Mathematics Teacher, vol. 5 (Sept., 1912), pp. 26-30.

Unfavorable comparison of our youth entering business with the foreign born. Curtail the curriculum to the necessities helpful in business. A discussion of the order in which subjects ought to be taught.

154. **Schlauch, W. S.** See No. 173.

155. **Scoby, F. H., Clapp, Amy L., Munhall, Ruth.** Should arithmetic be taught to all pupils in the high school? Mathematics Teacher, vol. 9 (June, 1917), pp. 196-202.

The three authors discuss the question separately. The first and third answer affirmatively; the second would combine the needed arithmetic with the study of algebra.

156. **Sears, I.** See No. 147.

157. **Shiels, Albert.** City school arithmetic. Journal of Education, vol. 81 (Jan. 14, 1915), pp. 46-47.

Resume of a report by Mr. Shiels based on replies from principals and teachers in 175 elementary schools requesting suggestions for materials for problems. A plea for making problems in arithmetic as real as possible. Discussion of how a problem should be prepared.

158. **Smith, A. G.** The teaching of arithmetic. School Science and Mathematics, vol. 12 (1912), pp. 457-460.

After detailed examination of a modern arithmetic, the writer concludes that the following glaring faults should be remedied at once: lessons too long; a large share of the practical work offered is given to children at an age when they possibly cannot grasp the meaning; reduce variety of methods of handling problems to those actually in use daily; use cancellation, and emphasize fundamentals.

159. **Smith, D. E.** The development of the American arithmetic. Educational Review, vol. 52 (Sept., 1916), pp. 109-118.

Traces the development of the American arithmetic, particularly with reference to the influences exerted by various European countries. Characterizes the larger influences now at work in the writing of arithmetics; states that the next large influence which will bear upon our textbooks in arithmetic will probably be the junior high school.

160. **Springer, I.** Ideational types in arithmetic. Journal of Educational Psychology, vol. 5 (1914), pp. 418-422.

Experiments performed in 1908, on 494 boys in grades 1-8, in New York City schools, to determine: (1) whether there are distinct ideational types in arithmetic, (2) the relation of those types to immediate memory of numbers, (3) the distribution and development of these types in the grades, (4) the relation of intelligence to the types of mental imagery.

161. ———— Teaching denominate numbers. Journal of Educational Psychology, vol. 6 (1915), pp. 630-632.

Describes an experiment planned to determine the best method of making permanent the necessary associations that have to be formed in memorizing the various items in certain tables.

162. **Starch, Daniel.** A scale for measuring ability in arithmetic. Journal of Educational Psychology, vol. 7 (1916), pp. 213-222.

The scale is designed to measure ability in arithmetical reasoning such as is involved in the solution of concrete problems. The values of the problems were determined experimentally by tests given to over 2000 pupils in grades 4 to 8 inclusive in 20 schools in 5 states.

163. **Stone, C. W.** Problems in the scientific study of the teaching of arithmetic. Journal of Educational Psychology, vol. 4 (1913), pp. 1-16.

Two questions: What arithmetic ought to be taught? How can this arithmetic best be taught? are discussed briefly and typical problems in teaching arithmetic are given.

164. Suggestion for mathematics. Three articles. Journal of Education, vol. 74 (Nov. and Dec., 1911), pp. 572, 665, 699.

An account of the working out of the course in elementary grades followed by Superintendent Wilson and his corps of teachers in Connersville, Indiana. After studying standard works and courses of study and consulting business men, the conclusion was reached that time in the grades should be spent on the essentials required by social utility. Eight essential processes of arithmetic are listed and 14 topics suggested to be eliminated.

165. Suzzallo, Henry. The teaching of primary arithmetic. A critical study of recent tendencies in method. Teachers College Record, vol. 12 (March 2, 1911), pp. 65-136.

A study prepared for the purposes of a special report to a subcommittee of the International Commission on the Teaching of Mathematics.

166. Taylor, E. H. A comparison of arithmetical abilities of rural and city school children. Journal of Educational Psychology, vol. 5 (1914), pp. 461-466.

Courtis tests in arithmetic were given to 3rd to 8th grades inclusive in 28 schools in an Illinois county and later to 12 of the same schools. Results were compared with the Courtis table which gives the average scores made by 7008 city children, and showed that the scores in the rural schools were below the scores for city schools.

167. ———— Where is arithmetic well taught? Educational Administration and Supervision, vol. 1 (Feb., 1915), pp. 143-148.

The author quotes from the American report in the Report of the subcommittee on Secondary Commercial Schools to show the complaints about the way arithmetic is taught in the American schools. Then he quotes from the British Report No. 4, from volume 4 of the French reports, from No. 5 of the Swiss reports, from Band IV, Heft 6 of the German Reports showing that practically the same complaints are made in these countries.

168. Taylor, J. S. Omitting arithmetic in the first year. Educational Administration and Supervision, vol. 2 (Feb., 1916), pp. 87-93.

An experiment was conducted in P. S. 16, the Bronx, during the years 1913-1915, to ascertain if it were better to begin arithmetic the first year or to devote the time usually given to arithmetic in the first year to reading. Tests given in arithmetic at the end of the second year show that the classes which omitted number work during the first year of school not only held their own but actually outstripped their competitors.

169. Teachers in Horace Mann School. Curriculum of the Horace Mann Elementary School. Teachers College Record, vol. 14 (March, 1913), pp. 65-77.

The course in arithmetic in grades one to seven, inclusive.

170. **Thorndike, E. L.** See No. 136.

171. **Thorndike, E. L.** Relation between speed and accuracy in addition. Journal of Educational Psychology, vol. 5 (1914), p. 537.

A class experiment made annually for several years gives data leading to the conclusion that the individual who is quick in adding is more accurate than the one who is slow.

172. **Tripp, M. D.** Some simple applications of algebra to arithmetic. See No. 223.

173. **Van Tuyl, G. H.** and **Schlauch, W. S.** Business arithmetic vs. algebra in the high school. Mathematics Teacher, vol. 8 (March, 1915), pp. 101-117.

Discusses the question "Does algebra give one the 'ability to grasp a situation' better than business arithmetic?" He reaches the conclusion that it does not.

174. **Watson, B. M.** Fallacies in arithmetic teaching. Journal of Education, vol. 83 (Feb. 17, 1916), p. 181.

Discusses four alleged fallacies in arithmetic teaching: (1) Only like numbers can be added. (2) The multiplier is always an abstract number. (3) When the dividend and divisor are abstract numbers, the quotient is abstract. (4) Only like numbers can be subtracted. The author gives examples to illustrate his contention.

175. ——————— The aim of arithmetic in elementary schools. Journal of Education, vol. 82 (Sept. 30, 1915), pp. 291-292.

Lists five aims of teaching arithmetic.

176. **Webb, H. E.** Grammar school mathematics. Mathematics Teacher, vol. 7 (Dec., 1914), pp. 37-48.

In this transition period the problem of the grammar school has been neglected for secondary mathematics. Enable the teacher to narrow his field of instruction and place emphasis on vital principles of arithmetic and algebra.

177. ——————— The re-introduction of arithmetic into the high school course. School Science and Mathematics, vol. 13 (June, 1913), pp. 517-524.

A criticism of the traditional high school course in mathematics. The author sees salvation in special emphasis upon the applications of mathematics. Arithmetic is the branch which is most applicative in character and should therefore be restored to its proper place in the high school. The purpose of algebra is to summarize arithmetical principles by means of simple formulas and to give an appreciation of arithmetic processes. Algebra is generalized arithmetic and arithmetic and geometry are fields for applications of algebra.

178. **Wiener, William.** The place of arithmetic in the high school curriculum. Mathematics Teacher, vol. 10 (June, 1918), pp. 174-178.

Arithmetic must be considered a vital and fundamental part of the high school curriculum, despite the general contention that it is taught in the elementary schools. The pupil and teachers must realize that the public demand for arithmetic is based on social necessity. Habits of accuracy, neatness and speed must be fixed.

179. **Wightman, N. J.** What should compose the subject matter of the arithmetic in the elementary schools and in what grades should formal arithmetic be taught? Mathematics Teacher, vol. 4 (June, 1912), pp. 162-172.

The injuries of mechanical teaching. Separate early the mentally unfit. Teach the pupil to think for himself. What must each type of work do for the pupil. Teach economy of effort. Have child give complete statement of problem. Give real problems. Outline of grade work.

180. **Williams, Sherman.** Shortening the course in arithmetic. Mathematics Teacher, vol. 2 (June, 1910), pp. 147-154.

Three divisions or classifications of arithmetical work. (1) Memorizing (2) Acquiring processes (3) Study of principles underlying processes. Nine fundamental principles of pedagogy are enumerated and the repeated violations of these rules are illustrated. The pupil should not be confused with new principles but merely taught a new nomenclature for the four fundamental principles. Practical problems are of benefit. A good method is given in contrast to every poor one mentioned.

181. **Wilson, G. M.** Eliminations. Journal of Education, vol. 83 (March 23, 1916), p. 320.

A statement of recommendations made in a report to the Iowa State Teachers Association, concerning elimination of subject matter. Arithmetic recommendations stated in detail.

182. **Winship, E. O.** Looking about. Journal of Education, vol. 81 (April 1, 1915), pp. 341-342.

A very favorable notice of various activities carried on in the Abraham Lincoln School at Lexington, Kentucky. Special attention given to the "store arithmetic" taught in the third, fourth and fifth grades.

183. **Zook, G. F.** Teaching Thrift in the schools. School and Society, vol. 10 (Nov. 15, 1919), pp. 581-584.

Thrift studied through (1) conversation in geography. (2) biography of successful men. (3) arithmetic—here the matter is organized and definite. Let thrift be the project for the study of much arithmetic. (a) Methods of keeping money records of children's activities; (b) investments—savings banks; government bonds; building and loan associations, mortgages, etc.; (c) place and function of life insurance in one's plan for saving money. (d) methods of buying. (e) personal budgets. (f) household accounts.

IV. Algebra

184. **Allen, Fiske** Some experiments testing the most rapid method of factoring the type $ax^2 + bx + c$. Mathematics Teacher, vol. 4 (Sept., 1911), pp. 32-34.

Tests given in Horace Mann High School. Three classes were formed and each taught a different method.

185. **Babb, M. J.** Some remarks on approximate computation. Mathematics Teacher, vol. 3 (Sept., 1910), pp. 27-36.

Encourage originality. Improve method and sequence of subject matter. Illustrate and discern shorter processes.

186. **Ball, W. W. Rouse** An apology for algebra. Journal of Education (Brit.), vol. 37 (1915), p. 155.

A reply, in the "correspondence" columns, to a headmaster's suggestion that algebra could be excluded from the school course of study.

187. **Borgers, W. B.** Some ideas about high school algebra. School Science and Mathematics, vol. 17 (1917), pp. 603-605.

Suggests that the subject matter of algebra be simplified so that time may be devoted (1) to giving reasons for operations performed (2) to expressing algebraic statements exactly.

188. **Crathorne, A. R.** Algebra from the utilitarian standpoint. School Science and Mathematics, vol. 16 (1916), pp. 418-431.

"The real importance of algebra does not lie in the immediate applications which it is possible to bring before the pupil, but in the fact that algebra is a part of the great material world about us, a part of which must be investigated and understood by all who do not wish to limit their opportunities when the time comes for making a choice of a profession."

189. **Decker, F. F.** The New York State Regents Syllabus in intermediate algebra. Mathematics Teacher, vol. 11 (Sept., 1918), pp. 1-8.

Outlines, and discusses, syllabus of intermediate algebra and the elementary algebra presupposed.

190. Discussion of the Report of the Committee on the Teaching of Algebra and Trigonometry. Mathematical Gazette, vol. 6 (March, 1911), No. 91, p. 42.

This discussion was a result of a paper by Mr. Godfrey on the possibility of dividing algebra into two parts, the educational and the technical. The report proposes to make drudgery tolerable by giving it a meaning by passing on to subjects, such as numerical trigonometry which the boy finds meaningful and by eliminating merely symbolic and abstract manipulation, such as harder fractions and H.C.F.

191. **Dow, E. M.** The teaching of factoring. Mathematics Teacher, vol. 8 (Sept., 1915), pp. 55-58.

Suggests a certain order of types according to which pupils should "size up" a problem in factoring.

192. **Epperson, C. A.** An experiment in teaching algebra. School Science and Mathematics, vol. 16 (1916), pp. 789-790.

Describes the experiment of teaching some of the elements of analytics to a class in the third quarter of algebra. The class completed the required work (pp. 236-365 Slaught and Lennes) thirteen days early and devoted the remaining time to analytics.

193. **Evans, G. W.** In defense of algebra. School and Society, vol. 3 (March 18, 1916), pp. 423-425.

Contends algebra is necessary to geometry; algebra and geometry together enable the student to have certain mental experiences of great value. Algebra has peculiar advantages and contributions to education.

194. **Foraker, F. A.** The relation of the symbols of mathematics to the elements of the problems. Education, vol. 35 (Dec., 1914), pp. 237-240.

"A serious difficulty that we have with our students of the present time in mathematics is that no matter how proficient in problems of technique, we find them deficient in being able to apply their knowledge to unformulated problems." The remedy: "Give proper training in expressing the elements of a problem and their relations to one another in symbols."

195. **Godfrey, C.** The teaching of algebra: What is educational and what is technical? Mathematical Gazette, vol. 5 (Feb., 1910), No. 84, p. 230.

Makes a distinction between ideas of mathematics and technique. Ideas (1) Generalization of an infinity of particular statements into a single universal formula; (2) application of formulas to things in life; (3) functionality and graph; (4) solving problems, etc. Technique: Rapidity, skill in handling the ideas due to practice. Often blots out spirit of ideas. Needed only by engineers, etc., and they are in a minority. Teach algebra then to two distinct classes in which one gets ideas and the other ideas plus skill or technique.

196. **Goodrich, M. T.** The presentation of positive and negative numbers. School Science and Mathematics, vol. 11 (1911), pp. 48-51

Gives in detail a simple method of introducing the subject of positive and negative numbers.

197. **Graham, Effie.** Needed: A funeral of algebraic phraseology. School Science and Mathematics, vol. 14 (1914), pp. 31-33.

A plea for discarding the old phrases "clearing of fractions", "transposing", "simplifying", etc., and substituting in their place the more meaningful expressions "multiply, divide, add or subtract", the only mathematical processes "worth while".

198. **Gray, Ella Durgin.** Freshman algebra and the average freshman. Mathematics Teacher, vol. 6 (March, 1914), pp. 166-181.

What is the nature of the training we claim is so valuable? Discrimination; practice in generalization; flexibility of mind; purposeful thinking; units of achievement. Few pupils succeed. Failures common to freshmen.

199. **Hart, H. J.** Highest common factor. Mathematics Teacher, vol. 3 (Sept., 1910), pp. 167-169.

The finding of the highest common factor requires skill to decide on trial factors. Two methods of attack likely to shorten the process are given. It is a valuable help in teaching the subject.

200. **Hedges, F. M.** The purpose of algebra in the elementary schools. Journal of Education, vol. 81 (June 10, 1915), pp. 625-626.

The general purpose of a course of algebra in the elementary schools, is to make an easier transition to high school work and so lessen the elimination from high school. This general purpose is characterized by the following specific purposes: (1) The algebra course in the elementary school should give a general knowledge of the elementary principles. (2) The subject matter should be developed so as to show the close relation to arithmetic.

201. **Jackson, Dunham.** Variables and limits. Mathematics Teacher, vol. 9 (Sept., 1916), pp. 11-16.

One reason that mathematics is misjudged is that one of the most fundamental notions remains in the background throughout the elementary course. This is the notion of function, the association of two variable quantities so that a value of one corresponds to a value of the other. In solving $x^2 - 3x + 2 = 0$, to different values of x correspond different values of the function. Then the crucial question: With what values of x is the value of zero associated? Thus every problem involving the solution of an equation is a problem involving a functional relation.

202. **Liden, Anna R.** Radicals for the freshman. Mathematics Teacher, vol. 7 (June, 1915), pp. 154-158.

Radicals required for solving the testing equations of quadratic form, solving simple radical equations, evaluating formulas.

203. **Lytle, E. B.** Some suggestions on the teaching of elementary algebra. Education, vol. 31 (Nov., 1910), pp. 175-178.

"In the writer's opinion, it will improve algebra teaching (1) to distinguish carefully between identities and equations, (2) to emphasize that a root must satisfy its equation, and (3) to give clear notions of variable and function and frequently illustrate examples of functionality by means of graphs."

204. **Merritt, H. T.** Reversal equations. Mathematics Teacher, vol. 7 (June, 1915), pp. 135-140.

The idea back of the solution of a reversal is to consider what has been done to the unknown in order to *involve* it in the particular manner indicated by the equation, and then by reversing the things done, *evolve* the unknown from the equation. In the transformation of many formulas of geometry and physics the pupil will find the reversal method helpful.

205. **Milne, W. P.** The teaching of limits and convergence to scholarship candidates. Mathematical Gazette, vol. 6 (May, 1911), No. 82, p. 79.

It is felt by many that: (1) the subject is introduced too academically (2) that its arithmetical foundation is not brought into sufficient prominence (3) that pupils leave school with only a very hazy notion of what it is all about (4) that investigations of convergences filling up pages of our algebras are profitless and barren of ideas. Practical problems are illustrations.

206. **Monroe, W. S.** An experiment in the organization and teaching of first-year algebra. School Science and Mathematics, vol. 12 (1912), pp. 225-232.

After explaining the motive for the study of algebra, contends that problems must precede, not follow the drill on the processes, giving details as to subject matter and arrangement of the problems that have been found satisfactory.

207. **Nunn, T. P.** The aim and methods of school algebra. Mathematical Gazette, vol. 6 (Dec., 1911), No. 95, p. 167.

Aim not to "train power" of reasoning. Direct usefulness not sufficient reason for teaching it, for comparatively few really need it. Should be taught because it is itself a desirable possession, a form of human achievement like literature, art, etc.

208. ———— The aim and methods of school algebra. Mathematical Gazette, vol. 6 (Jan., 1912), No. 96, p. 214.

There are three typical phases in mathematics: (1) the heuristic phase in which the mathematician face to face with a new type of problem devises a new notation or mathematical method to deal with it; (2) the formal phase in which the new notation or method is studied apart from the immediate needs of practical application; (3) the application phase in which the notation and method is the instrument for the solution of problems.

209. **Parker, Elsie G.** Developing ability to solve the verbal problem the basic aim of the ninth-grade course. School Science and Mathematics, vol. 19 (1919), pp. 599-604.

Argues for the use of the verbal problem as offering superior opportunity to develop the ability to think logically.

210. **Ponzer, E. W.** Note on the preparation of college freshmen in elementary algebra. School Science and Mathematics, vol. 11 (1911), pp. 737-740.

Freshmen in college fail in algebra because of definite weak spots in previous instruction notably in exponents and quadratics. The cure lies not in more rigid proof but in more intelligent application of the fundamental principles involved.

211. **Rich, F. M.** Making algebra feed the allies. School Science and Mathematics, vol. 18 (Dec., 1918), pp. 811-819.

It is proposed that problems in balanced rations, nutritive ratio, fertilizer mixtures (of which samples are given in the article) be used to furnish drill in simultaneous equation, graphed or otherwise. Local problems from the neighbors may be collected for class, group or individual solution, and so answers be sought in which someone is vitally interested.

212. **Roe, Jr., E. D.** A generalized definition of limit. Mathematics Teacher, vol. 3 (Sept., 1910), pp. 43-49.

"The object of this paper is to suggest a logical point of view, from which a generalization of the definition of limit may be secured."

213. **Shipley, J. H.** Algebra in the elementary schools. Mathematics Teacher, vol. 4 (June, 1912), pp. 136-143.

Emphasis on examinations are an injury to teacher and pupil. Too much crowding robs teacher of chance to inspire pupil. Subjects that might be omitted to leave room for more vital ones. Algebra should be taught from the start. Teach pupil to think. Use algebraic processes in arithmetic.

214. **Smith, A. W.** What results are we getting from graphic algebra? Mathematics Teacher, vol. 4 (Sept., 1911), pp. 13-20.

There are two methods of expressing thought, graphic and symbolic. The picture should precede the symbol. Introduce the graph into arithmetic. Solution of equations by graphs.

215. **Smith, E. R.** A shortened form of synthetic division and some of its applications. Mathematics Teacher, vol. 3 (Sept., 1910), pp. 18-20.

Illustrates by an example a short method of synthetic division, and advocates the teaching of the subject in first year algebra.

216. ———— Decomposition into partial fractions by means of remainder. Mathematics Teacher, vol. 6 (June, 1914), pp. 203-208.

Extension of method necessary. Illustrations of application to various types.

217. Some common errors in elementary algebra. Report of Committee on Classification. Mathematics Teacher, vol 2 (March, 1910), pp. 84-86.

A list of thirty-two common errors which frequently appear in the first-year work.

218. **Stradler, W. W.** Teaching of first-year algebra. School Science and Mathematics, vol. 19 (1919), pp. 38-44.

A discussion of the aims and content of a course in algebra, with suggestions concerning methods.

219. **Taggert, Anna.** An experiment with drill in algebra. Educational Administration and Supervision, vol. 2 (Nov., 1916), pp. 640-641.

A practice test in algebra following a plan suggested by a study of the Courtis Standard Practice Tests in Arithmetic has been devised and is being applied in the Murphysboro Township High School. The purpose of the test is two-fold: to develop speed and accuracy, and serve as a review drill in the essentials of the subject. The writer concludes that the Review Tests increased the efficiency of the work.

220. **Taylor, J. M.** Equations. Mathematics Teacher, vol. 2 (June, 1910), pp. 135-146.

The axioms of equals are not applicable to equations. Allow the pupil to construct equations himself that he may become familiar with the properties. Factoring is the one perfect method for the solution of a quadratic equation. Other methods are but devices to obtain factors in the end. State the principle of substitution first in the solution of a system of equations. All other methods are but modified forms of substitution.

221. **Terry, H. L.** Algebra in the high school. School Science and Mathematics, vol. 10 (1910), pp. 573-579.

Algebra should be taught as a tool for constant use in mathematical operations. Its great value as a simplifying, generalizing instrument and the advantages given by the ability to think in letters and symbols, rather than in involved English expressions together with a mastery of its universal language will do much to make its purpose definite and inspiring.

222. **Thompson, O. S.** The algebra situation. Journal of Education, vol. 83 (March 23, 1916), pp. 315-317.

Results of an investigation, by the questionnaire method, made in order to discover the chief causes of the great number of annual failures in algebra, to present the best educational thought and practice in meeting the problem, to learn just how far the grades should be held responsible for the trouble, and what kind of preparation in arithmetic is needed for work in algebra.

223. **Tripp, M. D.** Some simple applications of elementary algebra to arithmetic. School Science and Mathematics, vol. 15 (1915), pp. 496-500.

Gives several illustrations of ways in which first work in algebra can be made more concrete and interesting and more vitally connected with the previous work in arithmetic.

224. **Wheeler, A. H.** Blank forms in algebra. Mathematics Teacher, vol. 7, (Dec., 1914), pp. 58-68.

This is a personal method for presenting topics to oblige pupils to do original work. Illustrations of blank forms and how they may be used.

225. **White, Elizabeth G.** A connection between algebra and life. Mathematics Teacher, vol. 10 (Dec., 1917), pp. 104-109.

Use interesting facts from daily papers, etc., if possible, to make real problems within the experience of the students, to furnish a natural starting point for more difficult problems, to arouse student initiative in making problems, to give facility in translating rapidly and correctly the words of a problem into the symbols of an algebraic equation.

226. **Whitford, E. E.** Application of the Pell equation to the problem of extraction of the cube root of a binomial surd. Mathematics Teacher, vol. 7 (Dec., 1914), pp. 68-71.

Illustrative problems.

V. Geometry

227. **Auerbach, Matilda.** A syllabus of solid geometry used in the Ethical Culture School. School Science and Mathematics, vol. 12 (Dec., 1912), pp. 743-754.

Groups the theorems usually studied so that none of the logical sequence shall be sacrificed, while they will be psychologically better adapted to the use of the pupils.

228. **Austin, W. A.** A "Flu" dream in mathematics. School Science and Mathematics, vol. 19 (1919), pp. 701-708.

A Utopian vision of a high school administered by a business man as a work-shop with an eight-hour day (halftime on Saturday). No home-work, periods of an hour and twenty minutes each. Individual progress through supervised study, or rather supervised work, was the rule. First they think through their particular drawings and measurements to a general truth; second they state the general proposition; third they demonstrate the truth of this general proposition; and lastly they solve problems by means of the proposition just proved.

229. **Baker, A. L.** J'accuse Geometry teachers. School Science and Mathematics, vol. 12, (1912), pp. 300-306.

Shows in detail with illustrations how to make geometry more intelligible to the students by training them in the use of the three W's. What figure is expected to be secured by the construction? What use is to be made of the figure? What theorem is to secure this use?

230. ——————— The syllabus in geometry. School Science and Mathematics, vol. 10 (1910), pp. 392-398.

Suggests amalgamation and compromise of syllabus and text book methods by introducing a systematized course of procedure in grouping theorems and corresponding diagrams.

231. **Barnes, H. O.** Geometry by analysis. School Review, vol. 27 (Oct., 1919), pp. 612-618.

The aim of the article is to show that the analytic method as taught in the Springfield, Illinois, High School (1) leads to greater efficiency in attacking originals, (2) makes pupils depend on their own initiative, and (3) makes them more enthusiastic because of the constructive nature of the work. The method is illustrated by the detailed consideration of a demonstration.

232. **Bell, J. C.** See No. 288.

233. **Breslich, E. R., Lytle, E. B., Miller, O. M.** Report of Committee on Geometry. School and Society, vol. 5 (Jan. 13, 1917), pp. 53-59.

Suggests reforms and courses of study beginning with intuitive geometry in sixth grade and extending through demonstrative geometry.

234. **Carson, G. St. L.** Intuition. Mathematical Gazette, vol. 7 (Mar., 1913), No. 104, p. 60.

Intuitions are no more than working hypotheses or assumptions, formed unconsciously as a result of universal experience rather than conscious experiment. Intuition plays an important part in mathematical education.

235. **Clayforth, E. W.** The spirit of mathematics and the teaching of elementary geometry in secondary schools. Journal of Education (Brit.), vol. 39 (1917), pp. 221-223.

Complains of the "incubus of the examinations", and the multiplicity of theorems. Recommends a short sequence of propositions, including the law of sines.

236. **Colleum, A. R.** The study of the Pythagorean Theorem and its proofs. Mathematics Teacher, vol. 4 (Dec., 1911), pp. 45-47.

Indicates a number of methods of proof.

237. **Collins, J. W.** Notes on the teaching of geometry. School Science and Mathematics, vol. 17 (1917), pp. 583-589.

A discussion of various questions relating to methods and content of geometry teaching.

238. **Coolidge, J. L.** What is a ratio? School Science and Mathematics, vol. 10 (1910), pp. 406-410.

In order to gain a clear idea of the process of measurement, observational geometry should be the preparatory work, converging algebra and geometry so that irrational numbers could appear in geometry and algebra at the same time.

239. **Cosby, Byron.** Efficiency in geometry teaching. School Science and Mathematics, vol. 12 (1912), pp. 406-415.

Present geometric principles in a concrete way at first, followed by the logical demonstration. No need of purely utilitarian or purely cultural problems. Many of the elementary principles of constructional geometry should be taught in eighth grade mechanical drawing. The child, as well as the teacher, should always know the objective point. The work should be grouped about types and co-ordinated with other branches. A plan is outlined, giving advantages and disadvantages as compared with the usual method.

240. **Davison, Charles.** A note on the teaching of geometry. Journal of Education, (Brit.), vol. 34 (1912), pp. 674 and 676.

The teaching of geometry in England has been disarranged by the substitution of more modern texts for Euclid. In some instances the effect has been deplorable. However, this is probably due to the rapidity with which the change was made. The same thing had happened in Italy some years back.

241. **Decker, F. F.** Educational values of geometry. Mathematics Teacher, vol. 5 (Sept., 1912), pp. 32-35; (Dec., 1912), pp. 41-45.

The nature of the subject creates difficulties that the honest teacher cannot deny and makes it a questionable subject for high school pupils. The logic of geometry conflicts in some instances with good pedagogy. There is danger in too much practical application. A laboratory period and manual are needed. Give student methods of attack, make pupil appreciate truth and beauty of geometry.

242. Discussion of the Board of Education Circular (Geometry). Mathematical Gazette, vol. 5 (July, 1910), No. 7, p. 289.

Discussion of such points as (1) aims of geometry teaching; space knowledge, logical power; (2) absurd to begin logical geometry before child has geometrical knowledge; (3) definitions with the exception of those that are a matter of convention, are a luxury to be included or omitted at discretion; (4) let child learn geometry inductively first, then deductively; (5) third stage in geometry should be scientifically conducted (a) experiment, (b) statement of facts observed, (c) experimental verification of facts stated, (d) logical proof of facts.

243. **Dobbin, Emily E.** Analytic methods in elementary geometry. School Science and Mathematics, vol. 16 (1916), p. 602-606.

The general questions as to the relative importance of education viewed as (1) the acquisition of knowledge, (2) the development of the natural powers are discussed with reference to the study of mathematics. The author feels that it is more important to prepare for the solution of financial, commercial, moral and ethical questions that arise in adult life, than to give technical training in the subject matter peculiar to one's profession or business.

244. **Durrell, Fletcher.** The locus problem in geometry with some discussion of the utilities in geometric study. School Science and Mathematics, vol. 11 (1911), pp. 40-46.

The loci arising in Book I are so simple and obvious as not to justify in the pupil's mind the effort necessary to master this new concept. At the outset, broader illustrations should be supplied. A natural transition to conic sections may be made by using the word "path" instead of "locus" and citing cases of the ellipse and parabola in ordinary life.

245. **Fishback, W. M.** Some experiments in geometry examinations. School Science and Mathematics, vol. 17 (1917), pp. 678-679.

Describe an experiment in examining (1) classes having completed one semester of geometry by asking them to pick out seven propositions from Book I and three from Book II which they liked best, giving their reasons for their preference (2) classes having completed one year of geometry by asking them to give their reasons for liking or disliking the subject and any helpful results which they felt had come from studying it.

246. **Fletcher, W. H.** Concrete geometry in the junior high school. School Review, vol. 27 (June, 1919), pp. 441-457.

Describes content and method of presentation of a course in concrete geometry in the seventh grade of the training school of the State Normal School at Oshkosh, Wisconsin.

247. **Ford, W. B.** The future of geometry. School Science and Mathematics, vol. 14 (1914), pp. 485-490.

Reviews briefly the characteristics of the geometry of the Egyptians and of the Greeks. Emphasizes the fact that the chief value of geometry does not lie in its bare formulas and results but in the training which it affords in well-sustained connected thought and expression. Something should be done to render geometry less artificial to the beginner. As steps in the right direction notes (1) increased attention to the applied problem (2) the introduction of constructive or intuitional geometry earlier in the course as a forerunner to the more formal deductive work.

248. **Francis, W. A.** Some changes in teaching geometry. School Science and Mathematics, vol. 10 (1910), pp. 399-406.

As too much emphasis has been given to the logical, and too little to the practical side of geometry, we should introduce more number work and better drawing, according to scale, using the protractor and T-square. Suggested high school course in mathematics: 1st year, Elementary algebra and concrete geometry; 2nd year, Algebra and beginning demonstrative geometry; 3rd year, Review algebra and complete plane geometry. Also contains suggestions regarding college entrance examinations.

249. Geometry (Editorial). Journal of Education, vol. 88 (July 4, 1918), p. 15.

An editorial notice of J. H. Minnick's "Investigation of certain abilities fundamental to the study of geometry".

250. **Godfrey, C.** Board of education circular on the teaching of geometry. Mathematical Gazette, vol. 5, part 1 (Feb., 1910), No. 84, p. 199.

Early geometry teaching seemed to be in three stages: (1) teach primary concepts and meaning of geometric terms; use instruments not essentially with accuracy but merely to help ideas; the meaning of geometric terms not taught by definition but by a series of experiments and questions bearing on everyday objects. (2) Teach informally four groups of fundamental facts (angles at a point, parallel lines, angles of triangle and polygon, congruence of triangles). Should be able to give each fact in good set terms for the deductive reasoning to follow. (3) Euclid revised. Arrive at Euclid's goal but not by his road.

251. **Goff, R. R.** Plane geometry for the ninth and tenth grades. School Science and Mathematics, vol. 19 (1919), pp. 357-358.

If the two important aims of geometry are information and good habits of thought, the writer thinks that proofs (1) of first principles, (2) by superposition, (3) by the indirect method, take too much time, and a study of these groups should be made by drawing and observation.

252. **Hansen, Lena B.** Group recitation in geometry. School Science and Mathematics, vol. 18 (Feb., 1918), pp. 103-108.

Two plans are suggested — a division of the class into three groups according to ability, or a division into small groups, the selection in the latter groups being based on congenialty as workers. The teaching is done by the group leaders, the main steps being put on the board so that the teacher may make sure that the work is satisfactory when she visits the different groups. In outdoor work it is better to divide into two larger groups. More interest is aroused in the students and the teacher becomes more of a counsellor and less of a pedagogue.

253. **Hardwick, Rose S.** Concerning loci. School Science and Mathematics, vol. 12 (1912), pp. 605-609.

Detailed outline of improved work in this subject.

254. **Hart, C. A.** Doing our bit in the teaching of geometry. School Science and Mathematics, vol. 18 (Dec., 1918), pp. 804-810.

Correct fundamental principles should be given to the pupil. Definitions should be given with great care, and the theory following should be consistent with the definitions adopted. The pupil should be trained to look for truth, to be patient and courageous.

255. **Hart, H. F.** The way to begin solid geometry. Mathematics Teacher, vol. 4 (Dec., 1911), pp. 54-57.

Indicates aims and methods of a plan for an introduction to solid geometry by five studies in relative positions of lines and planes in space.

256. ——————— Solid geometry. Mathematics Teacher, vol. 3 (Sept., 1910), pp. 24-26.

Illustrates the benefit of the author's original notation in the proof of certain theorems.

257. **Hart, W. W.** Genetic instruction in geometry. School Science and Mathematics, vol. 11 (1911), pp. 708-713.

In order to prevent memorizing of proofs and to develop methods of attack on originals, emphasize from the very beginning of the term the necessity of having some plan of proof. To do this, a few general principles should be utilized in the whole course. As the course progresses, group the theorems by means of which certain facts may be proved. In constructions have analysis first.

258. **Hobson, E. W.** On geometrical construction by means of the compass. Mathematical Gazette, vol 7 (March, 1913), No. 104, p. 49.

In connection with the theory of the constructions of Euclidean geometry it is possible to show that essential elements of all constructions can be obtained by compass alone, without aid of the ruler. Use of the properties of inversion.

259. **Hopkins, G. I.** Some class room experiences in teaching geometry. Mathematics Teacher, vol. 8 (Sept., 1915), pp. 21-30.

Discusses various methods of teaching geometry and states a number of reasons for preferring the syllabus method.

260. **Howard, B. A.** The teaching of geometry to first-year pupils. Mathematical Gazette, vol. 9 (March, 1919), pp. 317-332.

Argues for the development of the reasoning faculty; insists on the necessity for sweeping away the abuses of the Euclidean tradition; recommends an increase of intuitive geometry.

261. **Jackson, N. A.** Grading papers in geometry. School Science and Mathematics, vol. 17 (1917), pp. 483-487.

Mr. Jackson mentions the well-known lack of uniformity shown in grading geometry papers. He does not agree with the standards given by the Regent's Board, but suggests certain nine standards which he considers necessary.

262. **Kelly, D. J.** Should formal geometry be taught in the elementary schools? If so, to what extent? Mathematics Teacher, vol. 4 (June, 1912), pp. 144-150.

Experiments to date are unsatisfactory except those of a constructional and practical nature. A comparison of opinions leads to the conclusion that practical and constructive goemetry only is suitable for grade work.

263. **Kennedy, A.** Geometry—positive vs. negative. School Science and Mathematics, vol. 11 (1911), pp. 635-642.

Offers an open-minded, positive, exhaustive method of studying geometry as against Euclid's method.

264. **Lasher, W. R.** Some suggestions on decreasing the mortality in our geometry classes. Mathematics Teacher, vol. 4 (Sept., 1911), pp. 26-31.

Get down to the level of the pupil. Give interesting and striking facts first. Review and review. Use numerical examples and illustrations, drawings and simple constructions. Form a special class for dull students not as a penalty but as an opportunity for advancement.

265. **Lennes, N. J.** An attempt to adapt elementary geometry to the understanding and interests of young people. School Science and Mathematics, vol. 10 (1910), pp. 521-528.

After describing mathematical teaching in Europe and England, the author suggests changes in our work. Let formal augumentation be approached gradually by first introducing simple propositions informally. Then make the work as concrete as possible. Let the concrete and practical on the one hand and the abstract on the other illuminate each other and lend mutual interest.

266. **Lytle, E. B.** Introduction of demonstrative geometry. School Science and Mathematics, vol. 18 (March, 1918), pp. 221-227.

Discussion desirable methods of approach to the study of geometry and pitfalls to be avoided by the teacher.

267. ———— Limits in elementary geometry. School Science and Mathematics, vol. 10 (1910), pp. 530-533.

The notion of limit is too important and too useful to be dropped out of elementary geometry. It can be presented in a form sufficiently simple to be understood by high school pupils if we (1) teach a correct, but not the most precise definition; (2) prove the existence of limits only in simple cases of monotone variables; (3) eliminate many difficult "proofs" by defining length of curves, areas bounded by curves, areas of curved surfaces, and volumes bounded by curved surfaces as particular limits.

268. ———— See No. 233.

269. **Manning, H. P.** Geometry of four dimensions. Mathematics Teacher, vol. 7 (Dec., 1914), pp. 49-58.

Illustrative theorems and proofs.

270. **Marsh, H. B.** Originals in geometry. Mathematics Teacher, vol. 6 (Sept., 1913), pp. 17-21.

Eagerness for individual achievement is natural to all. Explain the application of propositions as a tool for original work. It is necessary to create interest in originals. Value of good figures and clear hypotheses.

271. **Metzler, W. H.** Problems in the experimental pedagogy of geometry. Journal of Educational Psychology, vol. 3 (1912), pp. 545-560.

Lists results to be secured in the teaching geometry, and offers suggested tests in (1) fundamentals; (2) reasoning; (3) concentration.

272. **Miller, O. M.** See No. 233.

273. **Millis, J. F.** Real applied problems in algebra and geometry handed to a committee on investigation. School Science and Mathematics, vol. 10 (1910), pp. 74-77.

The purpose is to reform the teaching of mathematics in secondary schools by teaching the subject more in relation to its practical uses: (1) contribute real applied problems, (2) use the material printed in the classroom with a view to determining its adaptability to the interests and needs of secondary school pupils.

274. **Minnick, J. H.** Certain abilities fundamental to the study of geometry. Journal of Educational Psychology, vol. 9 (1918), pp. 83-90.

The author states that the success of a formal demonstration of a theorem of geometry depends upon at least four abilities: (1) ability to draw a figure; (2) ability to state accurately the hypothesis and conclusion; (3) ability to recall additional facts about the figure when one or more are given; (4) ability to select from the available facts those necessary for proof so as to arrive at the desired conclusion. It was sought to show by investigation the relation of these abilities to the teachers' marks, and the extent to which the teacher valued these abilities. Tests for this purpose were given in sixty-three schools.

275. **Parker, M. J.** Notes on Board of Education Circular, No. 851; on the teaching of geometry. Mathematical Gazette, vol. 8 (May, 1915), No. 117, p. 86.

Recommends that certain propositions which have hitherto come at the beginning of deductive geometry should be taken for granted as a direct outcome of intuition and the subsequent structure should be based upon them instead of on ultimate postulates to which they could be logically reduced.

276. **Parsons, Ada M.** Colored crayons as an aid in teaching mathematics. School Science and Mathematics, vol. 14 (1914), pp. 33-35.

Recommends the use of colored crayons in the teaching of geometry, and gives reasons therefor.

277. **Peck, P. N.** Special devices in teaching geometry. Mathematics Teacher, vol. 3 (Dec., 1910), pp. 49-55.

A statement of the deficiencies of the average college freshman class. Faults of teachers that cause these difficiencies and a few devices for remedying the situation.

278. Provisional report of the National Committee of Fifteen on geometry syllabus. School Science and Mathematics, vol. 11 (1911), pp. 434-460, 509-532.

Recommends axioms, postulates, definitions, informal proofs, theorems and constructions, — and recommends time and place in the curriculum, special courses, exercises and problems. Includes a syllabus for plane and solid geometry.

279. **Putney, Edith N.** On the teaching of geometry. Educational Review, vol. 41 (Feb., 1911), pp. 200-204.

An outline of the first two lessons in the formal course in demonstrative goemetry is given to emphasize the needs of the beginner.

280. **Schreiber, E. W.** Construction work in solid geometry. School Science and Mathematics, vol. 19 (1919), pp. 407-413.

The idea of this method of presenting geometry was suggested by the words of Kepler "To measure is to know." Real models are made by the pupils of at least eight polyhedrons. This arouses their interest, fixes in their minds the meaning of geometric terms, trains pupils' minds in space relations, and permits allowance to be made for individual differences, since the more difficult model can be assigned to the more advanced student.

281. **Schwartz, A. J.** The evaluation of Pi in elementary geometry. School Science and Mathematics, vol. 11 (1911), pp. 791-793.

The numerical evaluation of Pi presents the best opportunity we have in geometry for developing in the pupils' minds a clear conception of limits as applied in mathematical investigations. The simpler computation should be made by the pupils themselves.

282. **Schwatt, I. J.** Some suggestions on the teaching of geometry. Mathematics Teacher, vol. 2 (March, 1910), pp. 94-115.

A pupil derives greater mental power from the solution of original problems in geometry than from memorizing propositions. Create a thirst for knowledge. Increase the intensity and efficiency of knowledge of the student rather than the scope and the quantity of subject matter. Power so gained carries over into the problems of real life. Details of a course planned on this basis.

283. **Sherk, W. H.** Approximate values of π. Mathematics Teacher, vol. 2 (March, 1910), pp. 87-93.

A discussion of different methods for finding the value of π; the difficulties of these methods are indicated. The author concludes that it is best to teach children to compute the value of π till a shorter method has been found.

284. **Slocum, S. E.** Geometry in the elementary school. Educational Review, vol. 54 (Oct., 1917), pp. 266-273.

The idea of form is as fundamental with a child as the idea of number—it is merely accidental that arithmetic and algebra have preceded geometry in the curriculum. We can recognize these stages in the unfolding of the geometric concept; First, by object teaching, paper folding, etc., to amplify the ordinary space concept; second, abstraction of ideas including the locus and function. Constructional drawing may help in this period of intuitional geometry. In the third or high school stage we come to geometry as a logical process. By beginning geometry earlier and laying a good foundation, modern geometry can be introduced into the high school.

285. **Smith, T. W.** The definition in geometry. School Science and Mathematics, vol. 11 (1911), pp. 794-801.

Much of the confusion and dislike for geometry is due to unscientific definitions and inconsistencies in textbooks. A thorough discussion of definitions by pupils and the standardizing of them by teachers should be made.

286. **Spencer, Miss M. C.** A color scheme for figures in solid geometry. School Science and Mathematics, vol. 10 (1910), p. 549.

Outlines a plan for using colors in drawing figures lying in more than one plane.

287. **Stark, W. E.** Measuring instruments of long ago. School Science and Mathematics, vol. 10 (1910), pp. 48-67, 126-139.

Description of simple instruments of 16th and 17th centuries which may be used to vitalize high school work in algebra and geometry: Instruments for measuring distance, for measuring angles, for measuring distance by proportion, the geometric square.

288. **Stockard, V. L.** and **Bell, J. C.** A preliminary study of the measurement of abilities in geometry. Journal of Educational Psychology, vol. 7 (1916), pp. 567-580.

Gives a test devised by the authors. First, a preliminary set of questions was submitted to the criticisms of professors of mathematics high school teachers, and then tried upon four classes in a Texas high school. The authors present graphs showing the results of the test, and consider these results in detail.

289. **Sykes, Mabel.** Chicago geometry syllabus. School Science and Mathematics, vol. 13 (1913), pp. 587-598.

The syllabus is intended to be suggestive only and has two distinct aims (1) to show the possibility of emphasis in presentation, (2) to show the possibility of applications with concrete settings. The theorems of plane geometry are divided into a number of groups, each group centering about a key-theorem or theorems with others placed in subordinate positions. In each group is given (a) the preliminary work necessary to lead up to the principal theorem; (b) the fundamental theorems; (c) important dependent theorems; (d) applications, with references to supplementary problems.

290. **Taylor, C. K.** Teaching geometry: A criticism of the common method. Education, vol. 30 (Jan., 1910), pp. 297-299.

Advocates teaching the class without the use of a textbook until four "books" have been covered.

291. **Wagstaff, C. J. L.** Should we continue to teach geometry? Mathematical Gazette, vol. 9 (March, 1917), pp. 38-40.

A plea for greatly reducing the amount of demonstrative geometry, mechanics, trigonometry, and other subjects in required mathematics.

292. **Walsh, C. B.** First year mathematics for high schools. School Science and Mathematics, vol. 17 (1917), pp. 787-794.

Opportunity is the keynote of the junior high school and so the ninth year work in mathematics should be "a course in demonstrational geometry with a limited syllabus, borrowing freely from algebra and trigonometry, rich in applications — a course which might be described by calling it a 'First Study of Plane Geometry.'" The author gives some practical hints as to the method of introducing the course.

293. **Walsh, Sara C.** Some experiences with a class of reviewers in geometry. Mathematics Teacher, vol. 5 (June, 1914), pp. 198-202.

Attitude of pupil helps failure or success. Reasons why pupils fail and how they may be corrected.

294. **Woodard, D. W.** The teaching of geometry at Tuskegee. School Science and Mathematics, vol. 13 (1913), pp. 400-410.

At Tuskegee an attempt is made to build up geometry about situations which arise in daily work. Students instead of feeling that problems are given for the purpose of illustrating principles already learned, see in the principle an economical method of adjusting themselves to a new situation.

VI. Trigonometry

295. **Baker, A. L.** Pedagogy in trigonometry. School Science and Mathematics, vol. 13 (1913), pp. 118-120.

First the author criticizes textbooks in trigonometry for unpedagogical treatment of subject matter. As an illustration he chooses the synthetic proof for the formula sin $(M + N) = \sin M \cos N + \cos M \sin N$ which is usually found in textbooks. He shows how this may be presented as a problem sin $(M + N) = ?$ and worked out analytically.

296. ——————— What is trigonometry? Education, vol. 34 (Nov., 1913), pp. 169-170.

"Trigonometry is usually regarded and taught as a separate subject, but when we come to seek its peculiar characteristics, we fail to find a clear cut central idea which would serve to give the subject its own individuality." (Young, The Teaching of Mathematics.) Dr. Baker answers this by saying that the individuality of trigonometry is marked by the scale and the consequent ability to measure all angles.

297. **Jackson, W. H.** A simplification in elementary trigonometry. Mathematics Teacher, vol. 3 (Sept., 1910), pp. 21-23.

The relation between the trigonometric ratios of a single angle. The author translates each trigonometric problem into an algebraic one.

298. **Mercer, J. W.** The teaching of numerical trigonometry. Mathematical Gazette, vol. 7 (Dec., 1913), No. 108, p. 193.

No reason why trigonometry should not be taught to boys of 13 or 14 years of age, provided of course unnecessary work in arithmetic, algebra and geometry are omitted. The plan proposed begins and ends with the solution of triangles, and the tools are only provided as they are needed. Many of the fundamental ideas are introduced by scale drawing.

299. ——————— The teaching of numerical trigonometry. Mathematical Gazette, vol. 7 (Jan., 1914), No. 109, p. 240.

Explains, in detail, with specific problems, what the course should contain. The work centers chiefly around the solution of triangles.

300. **Mitchell, S. C.** A method of demonstrating and teaching the trigonometric functions. School Science and Mathematics, vol. 17 (1917), pp. 245-247.

Describe a trigonometric function indicator, devised to make clear the changes of values and relations between sides and angles of a right triangle which take place as the acute angles vary between 0° and 90°.

301. **Rice, L. H.** A mnemonic for certain trigonometric identities. Mathematics Teacher, vol. 5 (March, 1913), pp. 145-146.

As the title indicates, a device for aiding the memory of certain identities.

VII. Logarithms and Slide Rule

302. **Breckenridge, W. E.** Mathematics teacher use of the slide rule. Mathematics Teacher, vol. 9 (Sept., 1916), p. 1 and 2.

An editorial concerning the use of the slide rule in mathematics classes.

303. **Counselman, T. S.** Logarithms and some of their applications for high school pupils. School Science and Mathematics, vol. 18 (Jan., 1918), pp. 21-24.

Complex cases of removal of parenthesis, cube root, euclidean method of finding the H. C. F. and the more difficult problems in factoring should give place to the theory and practice of logarithms. They may be profitably used in the manipulation of some of the interest formulas. Of special interest is the problem solved by the Farm Loan Bank in determining the equal, annual, semi-annual, quarterly and monthly payment necessary to pay off a loan with interest.

304. **Dobbs, W. J.** The teaching of indices and logarithms. Mathematical Gazette, vol. 8 (July, 1915), p. 119.

A very detailed account of classroom procedure.

305. **Lodge, A.** As to the completeness of mathematical tables required in school and examination work. Mathematical Gazette, vol. 8 (Jan., 1916), No. 21, p. 209.

Discusses the arrangement of logarithmic tables (numbers and functions) with respect to easy handling and reading. The slide rule should be used for the solution of equations. In the discussion it was suggested that compound interest be taught with a compound interest table from 16-50 years at a number of usual rates of percent.

306. **Ponzer, E. W.** A slide rule for class room use. School Science and Mathematics, vol. 10 (1910), pp. 776-779.

The slide rule should be introduced for checking purposes as soon as the student has studied the subject of exponents. A method of making one for classroom use is given.

VIII. Graphs

307. **Carson, G. St. L.** The various uses of graphs. Mathematical Gazette, vol. 7 (March, 1914), No. 110, p. 265.

Graphs provide the best introduction to the ideas of functionality and elementary analysis and further show the relation of algebra to these ideas in a clear and convincing manner. A graph is a vivid and enlightening method of seeing things. We may regard graphical expression as a language just as algebra is a language.

308. **Killam, S. D.** A few graphical methods. Mathematics Teacher, vol. 6 (Sept., 1913), pp. 10-16.

Value of a new point of view to the student and illustrative cases.

309. **March, M. C.** The use of statistics and graphs in the teaching of geography. Journal of Education (Brit.), vol. 36 (1914), p. 317.

Recommends graphic statistics, and the use of graphs in connection with studies of rainfall, temperature, etc.

310. **Mensenkamp, L. E.** What graphical and statistical material should be included in the ninth grade mathematics course? School Science and Mathematics, vol. 19 (1919), pp. 595-598.

The first graphs to be considered should deal with statistical material of a type which will appeal to the pupil. A number of simple mathematical graphs based on concrete problems should then be taken up. The relation between the graph and the formula may be emphasized here. Graphs of simultaneous linear equations should be considered because of their value in giving a meaning to the solutions. Graphs of quadratics are not essential to the first course.

311. **Pahner, Emily G.** History of the graph in elementary algebra in the United States, School Science and Mathematics, vol. 12 (1912), pp. 692-694.

Outlines history of the graph and indicates its uses today.

312. **Smith, A. W.** What results are we getting from graphic algebra? See No. 214.

313. **Upton, C. B.** Stenographic report of a lesson in graphs. From one of the high schools of Teachers College. Teachers College Record, vol. 2 (Sept., 1910), pp. 238-251.

Everything said by teacher and pupils is reported. A temperature graph was worked out and the graphs for two sets of linear equations. The pupils had studied algebra through quadratics. Seven reasons for studying graphs are given.

IX. The Metric System

314. American Institute of Weights and Measures, 115 Broadway, New York City.

The activities of this organization are specifically devoted to securing the defeat of the compulsory adoption of the metric system by legislative enactment. Through its publicity service, the Institute issues numerous newsletters for wide distribution, and prepares articles for publication in magazines, trade journals, newspapers, et cetera. Its material may be had upon application.

315. **Collins, J. W.** Metric reform in the United States. Educational Review, vol. 52 (Oct., 1916), p. 265.

Discusses advantages of metric system. The two chief things standing in the way of the early adoption of the French system are (1) the absence of knowledge by the American people of what is involved in the question and the advantages of the new system; (2) the difficulty of changing to the system, especially in metal manufactures.

316. **Jones, H. I.** Why Not Now? School Science and Mathematics, vol. 19 (1919), pp. 512-519.

Claims that the adoption of the metric system is coming. "A few uninformed senators may be able to delay its arrival but they cannot change the final outcome. Everything points to the present as the supremely advantageous hour to make the move. Economy, science and commerce all demand it."

X. Advanced Mathematics in Secondary Schools

317. **Carson, G. St. L.** On some unrealized possibilities in mathematical education. Mathematical Gazette, vol. 6 (March, 1912), No. 97, p. 246.

Mathematics formerly regarded as utterly unpractical now recognized as an important factor in science and engineering. Now the theories of pure number, of space, function, etc., are considered unpractical. But who can say? Theory of point sets or theory of groups may find application in economics. Why not teach older pupils about non-euclidean geometry, etc?

318. **Godfrey, C.** The teaching of calculus in public and secondary schools. Mathematical Gazette, vol. 7 (Jan., 1914), No. 109, p. 235.

"Calculus for the average boy." It is not decided as yet what or how far to go in the teaching of mathematics. Some teach Taylor's theorem and others omit it. The pupil is introduced to it by the graphing of simple functions. Broadly speaking the movement has received general support. Engineers and physicists give it a firm support.

319. **Jackson, C. S.** The calculus as an item in school mathematics. Mathematical Gazette, vol. 7 (Dec., 1913), No. 108, p. 197.

It is believed that a little calculus will be a valuable asset to a boy and it is possible for the high school to give it to him.

320. **Knowles, W.** The teaching of easy calculus to boys. Mathematical Gazette, vol. 7 (Dec., 1913), No. 108, p. 201.

The boys of the technical day school of Borough Polytechnic Institute at the end of the second year are ready to take up calculus. The third year's work deals with the parabola, cubic curve, sine curve, logarithmic curve, each taken graphically and numerically. The calculus work which is the simplest is taken as it is naturally required in connection with the above curves, and has a very strong place on the graphical side. It consists of differentiation and integration of equations of these curves, finding areas, volumes of solids of revolution, maximum and minimum values and sometimes finding the center of gravity.

321. ———————— The teaching of easy calculus to boys. Mathematical Gazette, vol. 7 (May, 1914), No. 111, p. 322.

Application to the parabola $y = kw^2$ as representing the path of a projectile. Velocity. Maxima and minima. Find the value of ratio $\Delta y/\Delta x$ for any value of Δx. The pupil is now ready for many problems (area of circle, value of sine and cosine of any given angle, value of π).

322. **Milne, Wm. P.** The teaching of modern analysis in secondary schools. Mathematical Gazette, vol. 8 (March, 1915), No. 116, p. 30.

Eliminate existing useless matter of higher algebra and with time gained study modern analysis. Scheme suggested begins with elements numerically treated, a numerical discussion of approximations and limits, dealing numerically with polynomials and power series, and concluding with the differentiation and integration of power series.

XI. The Junior High School

323. **Betz, William.** The teaching of mathematics in the junior high school. The Mathematics Teacher, vol. 10 (Dec., 1917), pp. 58-84.

An extended discussion under (1) the pedagogic elements of the question; (2) subject matter; (3) the administrative elements of the question.

324. **Evans, G. W.** Mathematics for the junior high school. Mathematics Teacher, vol. 9 (Dec., 1916), pp. 73-76.

Considers criticisms of mathematics, and suggests the following reconstructions: (1) approximate computation; (2) use of formulas; (3) use of problems to obtain formulas; (4) introduction to geometry.

325. **Gentlemen, F. W.** The content of a mathematical course for the junior high school. Mathematics Teacher, vol. 9 (June, 1917), pp. 209-218.

Discusses and outlines a course in mathematics for the seventh, eighth and ninth school years.

326. **Northwood, Louise.** Junior high school mathematics in Trenton. Mathematics Teacher, vol. 10 (Dec., 1917), pp. 100-103.

Describes the mathematics course in the Trenton High School.

327. **Renshaw, Emily.** A junior high school course in mathematics. Mathematics Teacher, vol. 12 (Sept., 1919), pp. 23-27.

Describes the mathematics course of the Holmes Junior High School, Philadelphia.

328. Report of the subcommittee on content of course in first year mathematics. School Science and Mathematics, vol. 19 (1919), pp. 259-264.

The report is divided into five parts (1) In general the work should have a unifying thread throughout (the algebraic equation), while including arithmetic, observational geometry and elementary algebra. (2) The method of treatment should be introductory rather than conclusive, stressing principles. (3) The minimum algebra requirements are listed. (4) The topics to be omitted are listed. (5) The geometry material available for use is given.

329. **Smith, D. E.** Mathematics in the junior high school. Educational Review, vol. 53 (April, 1917), pp. 391-396.

Granting the value of the junior high school movement, what mathematics should it include? Many current tendencies to abolish secondary mathematics; to fuse mathematics, and to make all school work play. But mathematics is too deeply rooted in our existence to have it disappear. A tentative curriculum for the mathematics in the junior high school would be: Grade 7: arithmetic, intuitive geometry; Grade 8: algebra, arithmetic; Grade 9: formal algebra, formal geometry.

330. **Stewart, J. A.** Light on junior high school courses. Journal of Education, vol. 89 (March 6, 1919), p. 271.

A notice of a group of pamphlets devoted to the course of study in the junior high school at Neodesha, Kansas, published by the Board of Education in that town.

331. **Taylor, E. H.** The course in mathematics in the junior high school. Educational Administration and Supervision, vol. 2 (Sept., 1916), pp. 460-465.

Discusses mathematics for years seven, eight and nine. Recommends inclusion of elements of commercial arithmetic, algebra, geometry with much practice in computation, and applications of the mathematics studied to problems.

332. **Walsh, C. B.** Tentative program of junior high school mathematics. Mathematics Teacher, vol. 10 (Dec., 1917), pp. 85-93.

Outline. Seventh year: arithmetic and intuitive geometry; Eighth year: algebra; Ninth year: geometry. This work is required of all students. It is to be of an extensive nature rather than intensive; the mathematics should be practical, stressing the equation, the graph, and the formula. In geometry a limited syllabus with a large amount of applied exercises should be used.

XII. Needs of Special Groups

333. **Babb, M. J.** See No. 338.

334. **Ball, Katherine F.** Mathematics applied and domestic arts. Mathematics Teacher, vol. 9 (March, 1917), pp. 158-164.

Describes the course in household mathematics at the Plainfield High School.

335. **Burstall, Miss.** The place of mathematics in girls' education. Mathematical Gazette, vol. 6 (Jan., 1912), No. 96, p. 203.

The author does not feel that girls should have the same amount of mathematics that boys have. She proposes these reforms: No compulsory mathematics on matriculation examinations; an elementary course of reformed mathematical teaching for girls in the middle school, emphasis on geometry rather than algebra, the whole subject to be taken in close relationship with its practical aspects; exemption at '15-16 for the average girl from any mathematical study; continuation of the existing system for the exceptional minority who will go on to college and take honors in mathematics.

336. **Carver, W. B.** Some topics of school mathematics of special importance to those students who expect to study analytic geometry and calculus. Mathematics Teacher, vol. 4 (Sept., 1911), pp. 21-25.

College students lack thorough drill in rules for areas, volumes, etc.; also manipulation. They should seek short methods and do mental work. Pupils without mathematical ability should not be encouraged to further study.

337. **Evans, Mary A.** How should secondary mathematics for girls differ from that for boys? Mathematics Teacher, vol. 7 (Sept., 1914), pp. 17-23.

For those entering higher institutions the only differences needed are the requirements of the institution. In a general course, if the curriculum is elastic, it should be elective. A discussion of industrial subjects and their benefits.

338. **Gale, A. S.** and **Babb, M. J.** What high school mathematics is of the most importance as a preparation for analytic geometry and calculus in college? Mathematics Teacher, vol. 4 (Dec., 1911), pp. 58-64.

Conflict between requirements of college professors and aims of secondary schools. Importance of four fundamental operations in algebra. What is necessary in each topic. The parts to be emphasized in algebra. A general knowledge of geometry and the regular course in trigonometry is needed. Comparison of results with those of other countries.

339. **Gentlemen, F. W.** The purpose and content of a course of mathematics for technical and manual training high schools. Mathematics Teacher, vol. 6 (Sept., 1913), pp. 22-40.

Limitation of field and results of questionnaire. Dividing of pupils into three classes: short time, possible graduates, and college preparatory. Discussion of aims for each group and how each subject may be made to meet the aims of each class. Errors common in different attempts and suggestions for a plan.

340. **Griswold, C. S.** Mathematical preparation desired for high school physics. Mathematics Teacher, vol. 8 (Sept., 1915), pp. 16-20.

The physics teacher finds that the pupil's former mathematics does not function. The pupils show a decided lack of concrete experiences. References to candle power, amperes, velocities, temperature, horse power, machines, lenses, mirrors, etc., contain materials for countless problems. Graphs should play an important part in mathematical instruction.

341. **Hart, J. N.** What mathematical knowledge and ability may reasonably be expected of the student entering college. Mathematics Teacher, vol. 6 (March, 1914), pp. 158-166.

In comparison with languages, mathematics shows a double percentage of failures. Pupils have a tendency to elect courses for which they have no ability, especially engineering. Tendency for professor to forget immaturity of student's mind.

342. **Hathaway, A. S.** What mathematical topics should be in the curriculum of the secondary school from the point of view of the technical school. Mathematics Teacher, vol. 5 (June, 1913), pp. 240-242.

Expertness and accuracy in the use of tools. Elementary algebra, arithmetic and geometry are the fundamental basis of all higher mathematical work. Narrow gap between elementary and higher mathematics by strengthening mathematical conception.

343. **Holmes, M. T.** Elementary mathematics in evening schools. Mathematical Gazette, vol. 5, Part 1 (Feb., 1910), No. 84, p. 200.

Discusses mathematics as it should be given in the evening schools. Making mathematics practical is the chief theme. "Practical mathematics and practical drawing should, where possible be taught as one subject and by one teacher. . . . . In examination only rules and topics of real importance to the technical student will be dealt with. For this reason complicated fractions, recurring decimals, G. C. D. and L. C. M. will be excluded."

344. **Holroyd, Ina E.** Mathematics in the education for girls. School Science and Mathematics, vol. 14 (1914), pp. 490-494.

Argues for inclusion of mathematics in courses of study for girls.

345. **Milne, W. P.** The teaching of scholarship mathematics in secondary schools. Mathematical Gazette, vol. 7 (March, 1913), No. 104, p. 80.

The tendency today is to make teaching suitable to minds of average ability and as a result pupils have a better understanding of the subject, but the clever boy loses time by keeping pace with the slow boy. Special classes should be formed for the bright student.

346. **Moritz, R. E.** On the relations of mathematics to commerce. School Science and Mathematics, vol. 19 (1919), 350-357.

The direct relations between mathematics and commerce and allied fields are mentioned in detail; the inter-relation in historical development, the present dependence of commerce and business on higher mathematics. The habits of thought developed by the study of mathematics are shown to be essential to the business world.

347. **Moss, S. A.** Mathematics in the secondary schools from the point of view of industrial establishment. Mathematics Teacher, vol. 5 (June, 1913), pp. 197-206.

Benefit of mental training, culture, detailed knowledge of mathematical processes, memory training are the chief results. Pupils taking course divide into three classes: Mathematical minded, passing, and those doing nothing. How best to secure to the first two classes of pupils the greatest possible benefit in different directions.

348. **Piaggio, H. T. H.** Mathematics in business and industry. Mathematical Gazette, vol. 9 (July, 1919), pp. 361-364.

A summary of investigations made to determine the mathematical needs of bookkeepers, insurance officials, financiers, small investors, manufacturers, artillery officers, opticians, and aviation officers.

349. **Russell, W. B.** Mathematics required by industrial workers as applied in high schools. School Science and Mathematics, vol. 11 (1911), pp. 704-708.

Mathematics for the industries is a different problem from mathematics for college entrance, varying according to locality and predominant industries, and not readily classified into subjects. Possible changes in high school mathematics: greater flexibility in the amount of ground covered in stated periods; closer connection between mathematics and its applications; tie algebra, geometry and trigonometry together, concrete examples; teachers should develop the inborn common sense of the pupils.

350. **Smith, K. G.** The application of mathematics to problems of the shop. School Science and Mathematics, vol. 12 (1912), pp. 306-311.

Lays down practical principles which should govern the application to the shop of all grades of mathematics from simple arithmetic to trigonometry.

XIII. College Entrance Examinations and Requirements

351. **Bowers, R. E.** How should the high school prepare for college? Journal of Education, vol. 87 (Jan. 31, 1918), pp. 117-118.

Foreign languages, mathematics, science, and history are so intricately connected with life's activities and so thoroughly grounded as subjects of study that every high school should give opportunity for at least three units in each. These with English should be recognized by all colleges as majors.

352. **Colebank, G. H.** Rational college entrance requirements. Journal of Education, vol. 89 (Feb. 6, 1919), pp. 149-150.

An argument for freedom of the high school to work out its own way of producing the qualities - - abilities to read and write English, ability to organize and present data, etc. - which are desired by the college, rather than have the college prescribe studies through which these qualities are to be manifested.

353. **Cowley, Elizabeth B.** Comprehensive examinations. Mathematics Teacher, vol. 10 (Sept., 1917), pp. 30-34.

Suggestions concerning relations of comprehensive examinations to the teaching of mathematics.

354. **English, Harry.** College preparation: what is its effect on what you teach and how you teach it? Mathematics Teacher, vol. 9 (Sept., 1916), pp. 21-32.

I. Colleges should agree to use the same language to designate the same thing. II. Standardization of requirements.

355. **Hart, H. F.** See No. 357.

356. **Keller, S. S.** Entrance requirements again. Mathematics Teacher, vol. 9 (March, 1917), pp. 149-153.

Favors "personal interview" as substitute for examinations, and intimates belief that ultimately the examination test will be left wholly to the psychologist.

357. **Koch, E. H., Jr., Wilson, A. H., Smith, F. E.** and **Hart, H. F.** Comprehensive examinations in mathematics. Mathematics Teacher, vol. 6 (Dec., 1913), pp. 64-105.

These papers attack the subject by three roads, from the general point of view, as an entrance test, interrelation of subjects. Preparation and grading of examinations, decision of aim in teaching, certain attempts and suggestions already made. The faults of present entrance examinations with suggestion to combine tests in all branches of mathematics into one general test. Illustrations of waste of time in geometry and algebra. If change is needed, let it be tested by comprehensive examinations.

358. **Lewis, E. E.** Foreign languages and mathematics as requirement for admission to, and graduation from, American colleges and universities. School Review, vol. 25 (Jan., 1917), pp. 1-6.

The results of this study indicate a tendency toward granting one degree for all liberal studies, and the elimination of specific requirements in mathematics and foreign languages both for entrance to, and graduation from American colleges and universities.

359. **Rapeer, L. W.** College entrance requirements. School and Society, vol. 3 (April 15, 1916), pp. 549-556.

Argues for the elimination of the requirements of the variously stated number of "units" in algebra, geometry, Latin, Greek, French, German, etc., from college entrance requirements.

360. **Shipley, J. H.** How can we minimize the effect of examinations on secondary education? Mathematics Teacher, vol. 10 (March, 1918), pp. 125-138.

Discusses injurious effects of examinations, and proposes a remedial plan.

361. **Smith, F. E.** See No. 357.

362. **Wilson, A. H.** See No. 357.

363. **Wright, Amelia C.** College preparation: what is its effect on what you teach and how you teach it? Mathematics Teacher, vol. 9 (Sept., 1916), pp. 17-20.

Miss Wright emphasizes her ideal of seeking first the physical, mental and moral welfare of her girls, teaching them their geometry and algebra in such a fashion as to contribute the utmost to their intellectual training — and the college preparation will be added.

XIV. Mathematics in Foreign Countries

364. **Archibald, R. C.** Mathematical instruction and the professor of mathematics in the French Lycées for boys. School Science and Mathematics, vol. 13 (1913), pp. 43-56; 105-117.

The author discusses the course of study of the French lycée noting particularly the mathematical phases and the preparation of teachers.

365. **Austin, W. A.** Mathematics in normal schools of Germany and Scotland. School Science and Mathematics, vol. 12 (1912), pp. 107-110.

As an aid in solving the problem of mathematics in American normal schools, the author outlines subject matter and methods in German and Scotch normal schools.

366. **Bartholemy, A.** Progressive science and mathematics courses and teaching in France. School Science and Mathematics, vol. 19 (March, 1919), pp. 199-204.

The paper gives a short description of the mathematics instruction in France stressing essentially its practical character.

367. **Butz, Carrie.** Mathematical teaching of girls in Germany. Mathematics Teacher, vol. 7 (Sept., 1914), pp. 27-33.

This is a description of secondary schools in Prussia by classes.

368. **Myers, G. W.** Secondary mathematics in France. School Review, vol. 19 (Sept., 1911), pp. 433-453.

An account of the characteristic features of secondary mathematics in France, with an outline of the course and percent of time devoted to each subject.

369. **Packard, J. C.** A proposed reform in mathematics. Educational Review, vol. 39 (May, 1910), pp. 455-458.

Tells of the movement in England under Professor Perry's enthusiastic leadership to reform mathematical education by linking it up with the work in science and industry. He tells of an interview he had with Professor Perry and gives a list of the new type of textbooks the movement has produced.

370. **Price, E. A.** Reform of mathematical teaching in Germany. Mathematical Gazette, vol. 7 (Dec., 1912), No. 101, p. 394. vol. 7 (Jan., 1913), No. 102, p. 7.

The essence of the spirit of reform in mathematical teaching is the attempt to introduce at an early stage the idea of variable quantities and functions and the permeation of the whole mathematical course by these notions treated in a common sense manner with concrete quantities. That is to say x and y should at introduction not only be regarded as unknown quantities to be determined but also as variable quantities whose course as their values continually alter should be studied as a whole. The "Meraner Lehrplan" crystallized these reforms into a tangible form and around their proposals has since centered the subsequent discussion of mathematical teaching.

371. **Rose, J. J.** The teaching of mathematics in Belgium. L'Enseignement Mathematique, vol. 12 (1910), p. 20.

A part of the report of the International Commission on the Teaching of Mathematics.

372. **Skinner, E. B.** The arrangement of mathematical studies in the high school course. A comparison with the French system. School Science and Mathematics, vol. 10 (1910), pp. 772-775.

After contrasting the work in American schools with work along the same line in France, the author advocates two reforms in American schools. The earlier introduction of the simpler parts of geometry in connection with drawing and the postponement of the formal algebra.

373. **Smith, D. E.** and students. Present teaching of mathematics in Germany. Teachers College Record, vol. 13 (March, 1912), pp. 67-190.

Abstracts of reports issued by the German branch of the International Commission on the Teaching of Mathematics.

374. **Taylor, E. H.** Teaching of mathematics in the continuation schools in Germany. School and Society, vol. 1 (Feb. 6, 1915), pp. 208-213.

Arithmetic is taught in the commercial and industrial continuation schools. In many schools courses in algebra, geometry, trigonometry and elementary mechanics also are open to apprentices. Courses are organized and the methods of instruction planned from the point of view that mathematics is a tool. In each case the data and problems worked with would be met in practice.

XV. Teachers and Teaching

375. **Barnes, H. O.** Geometry by Analysis. See No. 231.

376. **Betz, William** The teaching of mathematics in the junior high school. See No. 323.

377. **Colleum, A. R.** The Study of the Pythagorean theorem and its proof. See No. 236.

378. **Collins, J. W.** Notes on the teaching of geometry, No. 237.

379. **Cosby, Byron.** Efficiency in geometry teaching See No. 239.

380. **Davison, Charles.** A note on the teaching of geometry. See No. 240.

381. **Dobbs, W. F.** The teaching of indices and logarithms. See No. 304.

382. **Dow, E. M.** The teaching of factoring. See No. 191.

383. **Epperson, C. A.** An experiment in teaching algebra. See No. 192.

384. **Ferry, F. C.** Mathematics. The subject and the teacher. Mathematics Teacher, vol. 6 (June, 1914), pp. 217-228.

Mathematics is the source of exact reasoning. Use of mathematical tools in other sciences. Subject needs the continued service of able minds as new problems constantly rise. Sane criticisms of teachers are just, but frequently the reformer is confident because of inexperience.

385. **Francis, W. A.** Some changes in teaching geometry. See No. 248.

386. **Godfrey, C.** The teaching of algebra. See No. 195.

387. ——————— Board of Education Circular on the teaching of geometry. See No. 250.

388. ——————— The teaching of calculus in public and secondary schools. See No. 318.

389. **Hancock, Harris.** Defective scholarship and colleges for teachers. School and Society, vol. 9 (May 10, 1919), pp. 552-556. Part I.

The author argues that most of the education in the United States is defective, especially in the secondary schools, and holds the training in teachers' colleges and the teachings of educational theorists as mainly responsible.

390. ——————— The defective scholarship of our schools. The pernicious influence of the colleges for teachers. Part II. School and Society, vol. 10 (Sept. 20, 1919), pp. 336-343.

Discussion of the relative values of the education which amounts to a smattering of many things and that education which is founded on a few good things well learned.

391. **Hansen, Lena B.** Group recitation in geometry. See No. 252.

392. **Hardwick, Rose S.** Concerning Loci. See No. 253.

393. **Hart, C. A.** Doing our bit in teaching geometry. See No. 254.

394. **Hart, H. F.** The way to begin solid geometry. See No. 255.

395. **Hart, W. W.** Genetic instruction in geometry. See No. 257.

396. **Hartsough, R. C.** Some faults in pedagogy. School Science and Mathematics, vol. 18 (Nov., 1918), pp. 725-727.

Criticizes present day pedagogy as superficial; recommends higher standards for passing, and introduction of new subjects.

397. **Hawkes, H. E.** Educational values in mathematical teaching. Educational Review, vol. 43 (March, 1912), pp. 267-273.

Specific results to be obtained from the study of mathematics are: (1) The acquisition of subject matter; (2) the development of mental processes; (3) the evoking of moral qualities, such as (1) the respect for truth, (2) the use of intuition, (3) the appreciation of unity and harmony.

398. **Henderson, R. H.** Recent advances in the teaching of mathematics. Mathematics Teacher, vol. 9 (March, 1917), pp. 141-147.

Discusses recent advances: (1) algebra centered around the equation; (2) determinants for their use in the solution of equations; (3) correlation of algebra and physics; (4) introduction of graphs and their many uses; (5) improvement in trigonometry; (6) use of the laboratory method; (7) real problems are limited by pupils' individual interests; (8) higher personnel of teachers.

399. Hopkins, G. I. Some class room experiences in teaching geometry. See No. 259.

400. Howard, B. A. The Teaching of Geometry to First-year Pupils. See No. 260.

401. Karpinski, L. E. The teaching of elementary mathematics. School and Society, vol. 5 (Jan. 20, 1917), pp. 78-86.

After a survey of world conditions, the author discusses the progress in the teaching of mathematics during the past decade under the following topics: (1) Correlation between various branches of mathematics; (2) Correlation between mathematics and life; (3) Historical development; (4) The study of teaching; (5) Societies; (6) Journals; (7) Better textbooks. The article closes with a brief bibliography of books of value to teachers of mathematics.

402. Keyser, C. J. The humanization of the teaching of mathematics. Educational Review, vol. 44 (Sept., 1912), pp. 140-156.

Mathematics penetrates to the roots of existence; the concepts of constant and variable, of equation, of limit, of function, of transformation, of invariance, of infinitude, of group, of discreteness, of continuity, penetrate to the foundation of human life. It is our privilege, in teaching the elements, to avail ourselves of the higher conceptions that are present in them.

403. Knowles, W. The teaching of easy calculus to boys. See Nos. 320, 321.

404. Lindquist, Theodore. Training teachers of mathematics. Journal of Education, vol. 79 (Feb. 5, 1914), pp. 151-152.

Instructors of future teachers of mathematics should present their courses from the standpoint of professional training, even though these courses may appear purely educational. Two illustrations are given, one of a phase in the presentation of the theory of quadratic equations, and one of the application to arithmetic of some principles of algebra.

405. Lytle, E. B. Some suggestions on the teaching of elementary algebra. See No. 203.

406. ———— Introduction of demonstrative geometry. See No. 266.

407. Mercer, J. W. The teaching of numerical trigonometry. See Nos. 298, 299.

408. **Miller, G. A.** Some group theory. School Science and Mathematics, vol. 18, pp. 675-680.

A discussion of such portions of group theory as it is desirable to include in the preparation of teachers.

409. ——————— Training of mathematics teachers. School Science and Mathematics, vol. 15 (1915), pp. 1-12.

Discusses chiefly the training of teachers of secondary mathematics.

410. **Milne, W. P.** The teaching of limits and convergence. See No. 205.

411. **Monroe, W. S.** An experiment in the organization and teaching of first-year algebra. See No. 206.

412. **Morris, J. V. L.** Why Teachers' Colleges? School and Society, vol. 10 (Nov. 1, 1919), pp. 522-524.

A discussion of No. 388 and No. 389. Some suggestions for improvement of the colleges and therefore of teaching. 1. Teachers' colleges should not be considered an alternative to normal school for elementary teachers unless practice teaching and specialized work are offered. 2. If it rivals the liberal arts college, it must provide for general culture, and also for specialized study.

413. **Myers, G. W.** Progressive teaching of mathematics—what is it? School Science and Mathematics, vol. 18 (May, 1918), pp. 387-396.

The progressive teacher should not neglect four aspects of educational view (1) The educational (considering both the scientific and the utilitarian value) (2) Mathematical (3) Social (4) Rational (training the students to want to know why); and he should accept and try to secure with his pupils (1) the formal value of the subject (2) the literary value (3) the instrumental value (4) the scientific value (5) the transfer value.

414. **Nunn, T. P.** The aim and method of school algebra. See Nos. 207 and 208.

415. **Olds, G. D.** The opportunity of the teachers in the classroom. School Science and Mathematics, vol. 12 (1912), pp. 355-364.

Teachers should have broad, deep preparation, involving years of study and thought, also immediate preparation for the work of the day; also clear understanding of each pupil; also boundless enthusiasm for the subject and appreciation of its significance. Then gives detailed outline of how to arouse and keep the interest of the pupils during the period.

416. **Ponzer, E. W.** On the teaching of mathematics to freshman engineering students. School Science and Mathematics, vol. 12 (1912), pp. 99-106.

After analyzing the various types of freshmen, he suggests many aids toward efficiency, as figures drawn to scale, checking up, use of tables, use of well-chosen English.

417. **Putney, Edith N.** On the teaching of geometry. See No. 279.

418. **Reeve, W. D.** Courses in special methods for the teaching of mathematics. School Review, vol. 24 (Feb., 1916), pp. 89-106.

Teachers are not trained for their careers in the same definite way that physicians are. Four ways in which prospective teachers may profit: (1) by reading the published experience of others; (2) by consultation with experienced teachers; (3) by observation; (4) by actual teaching. A course is recommended, concluding with a list of reference books and magazines for teachers of mathematics in high schools.

419. **Schwatt, I. J.** Some suggestions on the teaching of Geometry. See No. 282.

420. **Smith, A. G.** The teaching of arithmetic. See No. 158.

421. The teaching of mathematics in the normal schools of the United States. Preliminary report of Committee appointed by the American Section of the International Commission on the teaching of Mathematics. School Science and Mathematics, vol. 12 (1912), pp. 213-224.

After a detailed account of work actually done in normal schools now, much reorganization of subject matter in arithmetic, algebra and geometry is suggested, the possible dangers in recent movements is discussed.

422. **Smith, E. R.** The shortened form of synthetic division and some appliances. See No. 215.

423. **Springer, I.** Teaching denominate numbers. See No. 161.

424. **Stradler, W. W.** Teaching of first-year algebra. See No. 218.

425. **Taggert, Anna.** An experiment with drill in algebra. See No. 219.

426. **Tripp, M. D.** Some simple applications of elementary algebra to arithmetic. See No. 223.

427. **Upton, C. B.** The training of teachers of mathematics in professional schools of collegiate grade. Educational Review, vol. 41 (April, 1911), pp. 282-294.

Consider the development of schools and colleges of education that have offered courses preparing teachers of mathematics for the secondary and elementary schools. Following is a brief statement of the graduate courses offered at Columbia and Chicago in the teaching and history of mathematics. It continues by setting forth the present tendencies and concludes with a presentation of an ideal preparation for teachers of mathematics.

428. ——————— Stenographic report of a lesson in graphs. See No. 313.

429. **Wagstaff, C. J. L.** Should we continue to teach geometry? See No. 291.

XVI. Methods and Motivation

430. **Baker, A. L.** Pedagogy in trigonometry. See No. 295.

431. **Bird, J. M.** The mathematics of warfare. Mathematics Teacher, vol. 10 (Sept., 1917), pp. 35-51.

A discussion of the applications of mathematics to warfare, with detailed information useful to the teacher in motivating the study of mathematics.

432. **Bouden, Joseph.** Russian peasant method of multiplication. Mathematics Teacher, vol. 5 (Sept., 1912), pp. 4-8.

Statement of rule and illustrative examples.

433. **Breckenridge, W. E.** Applied mathematics in high schools — some lessons from the war. Mathematics Teacher, vol. 12 (Sept., 1919), pp. 17-22.

The real problem should be used for purposes of motivation, illustration and vocational training. The author suggests several means for improving the teaching of mathematics by the use of real applied problems.

434. **Briggs, T. H.** Aims and purposes of instruction. High School Journal, vol. 3 (Jan., 1920), No. 1, p. 3.

Teachers, experienced, as well as inexperienced, need the guidance and stimulus which come from a supervision that will urge each one to formulate or to accept statements of purpose for his subject that are specific, definite and worthy, that will insist on such purposes being sought in each recitation unit, and that will show by measurements the results of such purposeful instruction.

435. **Collins, J. V.** Rational vs. mechanical methods in teaching mathematics. School Science and Mathematics, vol. 15 (1915), pp. 600-605.

Rational teaching lays stress on the whys and wherefores, mechanical on the whats and hows. A balanced proportion of these is needed in good teaching. The mechanical method results in skill which makes a good showing on examination but the rational gives the greater breadth of view and initiative on the part of the pupils.

436. **Curtis, A. M.** Study supervision. Its needs in the mathematics of the elementary and secondary schools. Mathematics Teacher, vol. 4 (Dec., 1911), pp. 50-53.

A teacher should see the pupil's original work not the copied. Is efficiency manifested by taking the best and shortest method? Is the work slovenly or accurate? Makes the pupil independent in thought.

437. **Doss, Gladys.** Toothpick geometry. School Science and Mathematics, vol. 17 (1917), p. 850.

A device for making figures from toothpicks, stickers and thread.

438. **Durell, C. V.** The earlier introduction of the calculus. Mathematical Gazette, vol. 6 (March, 1912), No. 97, p. 259.

Just as graphs were introduced into algebra from higher mathematics thereby introducing the idea of functionality so it is possible to introduce calculus into algebra and the author thinks it should be done. Starting out with a clear and sound foundation of limit, he would gradually build up until he reaches the definition of differential.

439. Education after the war. School and Society, vol. 14 (Aug. 26, 1916), pp. 332-334.

Mathematical courses should include arithmetic "for practical utility" and algebra "to initiate pupils' minds into symbolic handling of problems". Geometry is not mentioned.

440. **Fawdry, R. C.** Laboratory work in connection with mathematics. Mathematical Gazette, vol. 8 (March, 1915), No. 116, p. 36.

Practical mathematics needs a laboratory — it is the application of mathematical processes to data which have been obtained by the pupil as results of experiments he has performed. Instead of giving the student a list of the corresponding values of x and y he is given a stop-watch and a pendulum and told to find the relation between the length of the pendulum and the time of oscillation. The experiments performed in a physical laboratory are most frequently used.

441. **Gheury, M. E. J.** Mathematics or drudgery? Mathematical Gazette, vol. 6 (July, 1911), No. 93, p. 131.

When manipulation is drudgery it is because it is aimless. Drudgery can never claim to be related to mathematical science, while the right of manipulation to be reckoned within its sphere may be tested by the criteria of method, utility and elegance.

442. **Glazier, Harriet E.** Mathematics of common things. School Science and Mathematics, vol. 16 (1916), pp. 667-674.

A plea for the use of illustrative material taken from the pupils' every day environment.

443. **Godfrey, C.** Board of Education Circular on Teaching of Geometry: London, England.

Boys should be set to using tools in drawing to find out what truths they can discover. The foundation should be practical work and then recourse to this as often as possible in the theoretical part. This way of doing the work will not tend to help the boy to write propositions well but his deductions will be better. It is building up geometry deductively.

444. ——————— On the work of the International Commission on the Teaching of Mathematics. Mathematical Gazette, vol. 6 (March, 1912), No. 97, p. 243.

Question: Degree of rigour the teachers of the different countries expect and wish to develop in the treatment of mathematical problems as compared with the amount of intuition allowed. Italy most rigour, Germany most intuition. (All of this chiefly in geometry).

445. **Harper, G. A.** Discussion of the report of the committee on real applied problems in algebra and geometry. School Science and Mathematics, vol. 10 (1910), pp. 249-253.

Instead of devoting the first year in high school mathematics to the abstract science of algebra, we should teach principles to solve practical problems. Introduce more concrete material. A large number of practical problems solvable by the equations may be found by introducing a little geometry and some of the elementary principles of physics. But the problems should be chosen to suit the locality, and the teachers should be enthusiastic about them.

446. **Hobson, E. W.** The democratization of mathematical education. Mathematical Gazette, vol. 6 (March, 1912), No. 97, p. 232.

The practical side of mathematics affords the proper approach to the subject and is also an equipment for various departments of practical life; but this may be over emphasized. Must not lose sight of the paramount importance of mathematics as part of a real education of the intellect.

447. **Jackson, C. S.** The International Commission on the Teaching of Mathematics (abstract). Mathematical Gazette, vol. 6 (Dec., 1912), No. 101, p. 384.

The Perry method. Perry said introduce more concrete material in the work for we have judged the boy capable of more abstract reasoning than he really is. But this is being overdone and in many cases sound elements have been swept away in too hasty reform. As to formal discipline. American warns against the hasty acceptance of the thesis that "formal discipline is exploded". America also warns against allowing the attempt to bring out the concrete side of mathematics to degenerate into mere utilitarianism.

448. **Jackson, N. A.** Grading papers in geometry. See No. 261.

449. **Kennedy, A.** Geometry—positive vs. negative. See No. 263.

450. **Lasher, W. R.** Some suggestions on decreasing mortality in geometry classes. See No. 264.

451. **Marsh, H. B.** Originals in geometry. See No. 270.

452. **Metzler, W. H.** Problems in the experimental pedagogy of geometry. See No. 271.

453. **Miller, G. A.** Case method of teaching mathematics. School Science and Mathematics, vol. 19 (1919), pp. 344-349.

Compares the problem method of teaching mathematics with the "case method" of teaching law.

454. ———— Reform in the teaching of mathematics. School Science and Mathematics, vol. 16 (Jan., 1916), pp. 35-39.

Most of the proposed mathematical reforms deal with subject matter, but another reform is necessary especially on the part of the teachers who return from their summer session work enthusiastic over a broader view point which is inspiring only to the maturer mind. The reform should lead the teacher to present his subject matter from the student's point of view, reviving difficulties which no longer exist for him, adapting his presentation to the individual difficulties presented by his class.

455. ———— School mathematics and the war. School and Society, vol. 4 (Oct. 14, 1916), pp. 588-590.

School mathematics owes modern standing to military, naval, civil and railroad engineering. Training and needs of soldiers in the late war puts mathematics on firm basis. Some interesting examples of a soldier's mathematical needs.

456. **Minnick, J. H.** Supervised study of mathematics. School Review, vol. 21 (Dec., 1913), pp. 670-675.

The author gives details of conducting two classes, one supervised and the other not. His results and conclusions are: (1) the increased expense is little; (2)the students master the text more thoroughly; and (3) show more ability in taking the initiative in solving originals.

457. **Mitchell, S. C.** A method of demonstrating and teaching the trigonometric functions. See No. 300.

458. **Newhall, C. W.** The real problem in secondary mathematics. Educational Review, vol. 43 (May, 1912), pp. 472-482.

Our traditional mathematics curriculum has in the past emphasized the formal side—a new movement (the Perry movement) is now advocating the practical side. The real problem has some direct value in teaching the use of mathematics as a tool, but its significance lies in its use to motivate the study.

459. **Nunn, T. P.** The aim and methods of school algebra. See No. 207.

460. **Peck, P. N.** Special devices in teaching geometry. See No. 277.

461. Real applied problems in algebra. Second report of the Committee of the Mathematics Section of the Central Association of Science and Mathematics Teachers. School Science and Mathematcs, vol. 10 (1910), pp. 673-683.

Problems encountered in fields of activity that enter into the experience of the average person everywhere should be worked into textbooks, and used in the most economical way in all schools. The great majority of applied problems are of interest only in a given locality; hence in each school supplementary problems of local interest should be added.

462. **Rice, L. H.** A mnemonic for certain trigonometric identities. See No. 301.

463. **Schlauch, W. S.** An experiment in motivation. Mathematics Teacher, vol. 12 (Sept., 1919), pp. 1-9.

An account of results secured by teaching trigonometry and algebra as collateral and necessary tools of investigation in the fields of artillery practice and navigation. The conclusion of the author is that teachers should make themselves thoroughly familiar with the applications of elementary mathematics and use these to arouse interest.

464. **Schorling, Raleigh.** The problem of individual differences in the teaching of secondary school mathematics. School Review, vol. 23 (Oct., 1915), pp. 535-549.

The writer lists current practices in meeting individual differences of high school pupils; studies the validity of these practices and discusses the technique of dealing with the slow student. Includes a copy of "Study Helps" for students in the University High School, University of Chicago.

465. Should concrete multipliers and divisors be allowed? School Science and Mathematics, vol. 11 (1911), pp. 295-296.

Mensuration should introduce the subject.

466. **Schwartz, A. J.** The evaluation of Pi in elementary geometry. See No. 281.

467. **Walsh, Sara C.** Some experiences with a class of reviewers in geometry. See No. 293.

468. **Webb, H. E.** The cumulative examination in mathematics. School Review, vol. 21 (Nov., 1913), pp. 627-636.

The author proposes a series of cumulative examinations, extending over the three or four years of high school mathematics. Such examinations, while distinctly characteristic of the subjects just covered in the classroom, should each offer a considerable opportunity for the application of principles brought to light in the preceding term's work. The writer submits a cumulative examination paper in algebra, and in geometry.

469. **Whitney, A. W.** Constructive suggestions for high school mathematics. School Science and Mathematics, vol. 12 (1912), pp. 461-465.

Adapt mathematics to the particular group taught: e. g. in solid geometry much perspective drawing, and in descriptive geometry much construction work from the trades may be given. In the College of Commerce, the nominal subject matter may be economic, but the method mathematical.

470. **Woodard, D. W.** The teaching of geometry at Tuskeegee. See No. 294.

471. **Wright, H. C.** Mathematical equipment and its uses. School Science and Mathematics, vol. 15 (1915), pp. 500-504.

The author points out the deplorable lack of mathematical equipment usually found in schools, and lists the equipment found in the University High School, Chicago.

472. **Yokum, G. D.** Mechanical drill as a means to intellectual freedom. Journal of Education, vol. 78 (July 17, 1913), pp. 65-66.

"Three steps, therefore, must be taken in American education before citizenship is effectively taught: The relatively most useful ideas, ideals and habits must be determined, they must be built up into a system more definite and complex than that of mathematics, and the mastery of that system must be required of every pupil before he is permitted to leave school."

XVII. Educational Measurements and Tests

473. **Allen, Fiske.** Some experiments testing most rapid method of factoring the type $ax^2 + bx + c$. See No. 184.

474. **Bell, J. C.** A class experiment in arithmetic. See No. 109.

475. **Boyer, P. A.** The Courtis tests in arithmetic. See No. 110.

476. **Brown, J. C.** Investigation of value of drill work. See No. 112.

477. **Chambers, G. G.** A study of the reliability of test questions. Mathematics Teacher, vol. 7 (March, 1915), pp. 90-100.

The paper is a study of a test on deductive reasoning and a comparison of the results of that test with the teachers' marks in plane geomtry. Only a partial report is given covering the work to date. The questions which were used are given and a tabulation of the answers received. The questions were based on non-mathematical subject matter and were used to see if these questions had any real value in measuring reasoning ability. The report seems to justify that there is some value.

478. **Clark, J. R.** See No. 500.

479. **Courtis, S. A.** Standard tests in arithmetic. Journal of Educational Psychology, vol. 2 (1911), pp. 272-274.

A short article explaining the Courtis tests which attempt to standardize the work in arithmetic.

480. ——————— The measurement of high school mathematics. School Science and Mathematics, vol. 18 (June, 1918), pp. 507-526.

Positive educational standards can be set up by measurement only. Recommends as a program the following three items: (1) Formulate aims of teaching; (2) Make sure that method is the one best adapted to bring about the desired results; (3) Devise tests to measure the product.

481. **Davies, G. R.** Elements of arithmetic ability. See No. 125.

482. **Elliott, E. C.** See No. 506.

483. **Evans, J. E.** and **Knoche, F. E.** Effect of special drill in arithmetic. See No. 131.

484. **Gaylord, H. C.** See No. 485.

485. **Hanus, P. H. Gaylord, H. C.** Courtis arithmetic tests applied to employees in business houses. Educational Administration and Supervison, vol. 3 (Nov., 1917), pp. 505-520.

This report represents the results obtained with the Courtis Research Tests in arithmetic applied to 446 employees of one of the largest trust companies and one of the largest department stores in Boston. Compares scores made by these employees with scores made by eighth grade pupils.

486. **Heck, W. H.** Efficiency of grammar grade pupils in reasoning tests in arithmetic at different periods of the day. See No. 137.

487. **Ingels, Nelle L.** A statistical study in correlation of efficiency in secondary mathematics and efficiency in other high school branches. Mathematics Teacher, vol. 11 (June, 1919), pp. 172-176.

From the study one is justified in saying that there is a very high degree of correlation in efficiency in the study of mathematics and other secondary subjects. The results obtained by quantitative analysis differ widely from those obtained from impressions which are often obtained from isolated cases.

488. **Irwin, H. N.** A preliminary attempt to devise a test of the ability of high school pupils in the mental manipulation of space relations. School Review, vol. 26 (Oct., 1918), pp. 600-605; 654-670; 759-772.

The writer points out that much of the student's success in mathematics depends on his ability to visualize space relationships. This experiment is an attempt to find the best ways of improving such visualization.

489. **Mead, C. D.** and **Johnson, C. W.** Testing practice material in fundamentals of arithmetic. See No. 146.

490. **Minnick, J. H.** Scale for measuring a pupil's ability to demonstrate geometric theorems. School Review, vol. 27 (Feb., 1919), pp. 101-109.

Reports an attempt to construct a scale that shall require no writing on the part of the pupil, and that shall be easily scored.

491. ———— A comparative study of the mathematical abilities of boys and girls. School Review, vol. 23 (Feb., 1915), pp. 73-84.

An experiment based on comparative grades of 150 boys and 243 girls at the Bloomington, Indiana, High School. The investigation includes comparisons of achievements in language, English, history and science with achievements in mathematics.

492. ———— Mathematical tests:—Their relation to the mathematics teacher. Mathematics Teacher, vol. 11 (June, 1919), pp. 199-205.

Dangers: Most tests deal with the mechanical or tool phases of the subject and fail to measure the more important products of mathematical education. Value: (1) Furnish an impersonal standard. (2) Determine whether the class has attained a satisfactory standard before passing to a new phase of the subject. (3) Tests and scales may be used for purposes of diagnosis.

493. **Monroe, W. S.** Measurements of certain algebraic abilities. School and Society, vol. 1 (March 13, 1915), pp. 393-395.

Describes six tests made for algebra, given to 275 first year students. The results showed that the accurate students were the ones who worked rapidly.

494. ———— The derivation of reasoning tests in arithmetic. School and Society, vol. 8 (Sept. 7 and 14, 1918), pp. 295-299; pp. 324-329.

An extended discussion of reasoning tests in arithmetic now on the market with the conclusion that they are inadequate and of methods for improvement under the following divisions: Analysis of process of solving printed problems, three types of tests required to measure ability to solve problems, steps in devising an arithmetic test, proposed plan for arithmetic reasoning test.

495. ———— A test of the attainment of first-year high school students in algebra. School Review, vol. 23 (March, 1915), pp. 159-171.

The writer purposes to devise a set of tests which could be used to measure algebraic abilities, and to determine the standards which should be attained in these abilities. Tabulates the results of tests given to classes of first year high school pupils. Concludes that improvement may be made in two ways: (1) by getting pupils to take an interest in the work, and (2) by making automatic each type of manipulation that is required.

496. **Myers, G. W.** A plan for testing methods of teaching secondary mathematics. School Review, vol. 22 (Feb., 1914), pp. 91-97.

An account of systematic testing of the heuristic as against the expository mode of teaching algebra to first-year high school classes.

497. **Phillips, F. M.** Comparison of work done in successive minutes of practice period in arithmetic. See No. 151.

498. **Rogers, A. L.** Tests of mathematical ability—their scope and significance. Mathematics Teacher, vol. 11 (June, 1919), pp. 145-164.

Discusses measurements in the field of algebra and geometry.

499. **Rugg, H. O.** The experimental determination of standards in first-year algebra. School Review, vol. 24 (Jan., 1916), pp. 37-66.

A report of a preliminary investigation for the purpose of (1) establishing certain methods by which standards for measuring the outcomes of a year's instruction in high school algebra may be constructed; (2) stating tentative results obtained in eight Illinois high schools; (3) making certain criticisms of the learning and teaching process in algebra.

500. ———— and **Clark, J. R.** A coöperative investigation in the testing and experimental teaching of first year algebra. School Review, vol. 25 (May, 1917), pp. 346-349.

Describes "standardized tests in first year algebra", and announces coöperative experiment in teaching algebra.

501. ———— Standardized tests and the improvement of teaching in first year algebra. School Review, vol. 25 (Feb. and March, 1917), pp. 113-132 and pp. 196-213.

A final report on "Experimental determination of standards in first year algebra".

502. ———— The improvement of ability in the use of the formal operations of algebra by means of formal practice exercises. School Review, vol. 25 (Oct., 1917), pp. 546-554.

Suggestions and directions for the use of such material in a coöperative experiment in the teaching of first year algebra.

503. **Rutherford, George.** Some observations on the use of the "reckoning test" in school children. Journal of Education (Brit.), vol. 36 (1914), pp. 363-364.

Maloney's modification of Professor Kraepelin's Reckoning Test was given. The test consisted in adding ten columns of figures each column containing 21 units. The children tested were eleven years of age and were tested for fifteen minutes on five consecutive days. All showed improvement from day to day.

504. **Smith, E. R.** Scales for the study of children's characteristics. Mathematics Teacher, vol. 12 (Sept., 1919), pp. 10-16.

An account of pupil's records kept at the Park School, Baltimore, Md. A card is kept for each pupil analyzing the child from the following points of view: Intelligence, physical, social, and moral characteristics, school work, standardized tests, general information, home conditions, supplementary remarks, with many subheads such as public spirit, honesty, sociability, initiative, leadership, etc., under social and moral; and similar subheads under the other topics.

505. **Starch, Daniel.** A scale for measuring ability in arithmetic. See No. 162.

506. **Starch, Daniel** and **Elliot, E. C.** Reliability of grading work in mathematics. School Review, vol. 21 (April, 1913), pp. 254-259.

This is an investigation of the grading of mathematics papers. The same test paper was graded by 155 different teachers and the grade varied from 25% to 90%. The reasons for variation in marking are (1) the different deductions made for form, make up and appearance of the paper, (2) the fact that some teachers deduct for spelling, (3) that the schools which did the marking are scattered over a wide area and have different standards, (4) that there are countless ways of considering the demonstration given.

507. **Stockard, V. L.** and **Bell, J. C.** A preliminary study of the measurement of abilities in geometry. See No. 288.

508. **Suzzallo, Henry.** The teaching of primary arithmetic. See No. 165.

509. **Taylor, C. K.** Teaching geometry. See No. 290.

510. **Taylor, E. H.** Comparison of arithmetic abilities of rural and city children. See No. 166.

511. **Taylor, J. F.** Classification of pupils in elementary algebra. Journal of Educational Psychology, vol. 9 (1918), pp. 361-380.

The author attempts to answer these questions: (1) Is there any advantage in learning rapidly? (2) What effect has the slow learner on the class? (3) How much benefit does the slow pupil receive from class work? (4) How much the bright pupil? (5) Effect of the slow pupils on the bright? (6) Conditions most favorable to best work?

Tests given in Minnesota. Summary: (a) Rapid performers are most accurate; (b) Slower pupils tend to become fixed in the group; (c) The brighter are retarded by the slower; (d) Best retainers are also the most rapid workers; (e) The poorer derive no advantage from being with brighter. Proposes: (1) Letting more capable carry extra work; (2) Excusing more capable from examinations; (3) Organizing classes for repeaters; (4) Giving supplementary work; (5) Letting brighter pupils finish earlier; (6) Individual help to poorer pupils; (7) Having textbooks for minimum and maximum work.

512. **Thorndike, E. L.** An experiment in grading problems. Mathematics Teacher, vol. 6 (March, 1914), pp. 123-134.

A test of problems with directions for grading and the results obtained.

513. ———— Relation between speed and accuracy in addition. See No. 171.

514. **Watson, E. E.** The college freshman and mathematics. Journal of Educational Psychology, vol. 7 (1916), pp. 223-226; pp. 607-609.

Describes an attempt to test the mathematical efficiency of college freshmen at Parsons College, Fairfield, Iowa, in 1914, and as far as possible to test their growth during two years. The uniformity of the average record of the different sections was striking. Two sections, of thirty students each, selected at random, showed practically the same ability on series A and B Courtis standard tests. Different texts were studied by each section, and results compared at the end of the year.

515. **Werremeyer, D. W.** Reliability of grades of testpapers in mathematics. School Science and Mathematics, vol. 14 (1914), pp. 422-429.

A report of an investigation in which six teachers graded test papers in geometry, algebra and arithmetic. The tests show that teachers differ greatly as to the value that is placed upon geometry, algebra and arithmetic papers and that they do not place the same value upon the same paper twice. Describes conditions necessary to improvement.

516. **Young, J. W. A.** Concerning psychological tests in the army and their meaning for the teacher. School Science and Mathematics, vol. 19 (1919), pp. 544-548.

Shows that familiarity with the most elementary notations of the mathematical subjects in elementary and high school increases the probability of high rating in general intelligence.

XVIII. Disciplinary Values. Psychology

517. **Asker, William.** Mathematics and efficieny in secondary school work (a reply). School Science and Mathematics, vol. 17 (1917), pp. 147-150.

A reply to Professor Moritz' article, vol. 16, p. 233; see No. 32.

518. **Davis, Alfred** and committee of the Chicago Mathematics Club. The status of mathematics in secondary schools. School and Society, vol. 6 (Nov. 17, 1919), pp. 576-582.

Consists of answers to a questionnaire as to the value of mathematics and the deductions therefrom as follows. Value lies in (1) Utility—increasing use of graphs, formula, etc. (2) Culture. (3) Makes one face facts, attack problems, carry on to end. (4) Mental training,—develops reasoning; analytic, logical and abstract thinking; concentration; speed; accuracy. All of these are invaluable in directing modern business which is becoming very complex.

519. **Grove, C. C.** A note on formal discipline and mathematics. Journal of Education, vol. 84 (Nov. 9, 1916), p. 459.

A plea for the existence of "transfer" in the habits of mind which the study of mathematics develops.

520. **Harvey, N. A.** The doctrine of formal discipline. School Science and Mathematics, vol. 18 (June, 1918), pp. 536-538.

An answer to some of the statements made by Professor Young in his article in the January and February numbers of School Science and Mathematics 1918 "concerning experiments to test the transfer of training".

521. **Kruse, P. J.** A strategic retreat. School and Society, vol. 7 (May 4, 1918), pp. 531-532.

A discussion of Professor Moritz' paper "Mathematics as a test of mental efficiency" (School and Society, vol. 7).

522. Mathematics and Psychology. Mathematics Teacher, vol. 8 (June, 1916), pp. 182; vol. 8 (Sept., 1916), pp. 3-10; vol. 9 (Dec., 1916), pp. 103-124.

Indicates an attitude towards attacks upon disciplinary value of mathematics, and the proper reaction of teachers of mathematics to those attacks. Deals with "mental measurement", and criticizes the use made of statistics and statistical theory by psychologists.

523. **Moore, C. N.** Disciplinary values. Educational Review, vol. 54 (Oct., 1917), pp. 245-255.

The paper undertakes to show the fallacies in the arguments raised against formal discipline and the errors in the psychological experiments and the results deduced therefrom. It then shows that mathematics is a subject "par excellence" for furnishing a training in logical reasoning and logical organization.

524. ———— Mathematics in secondary schools. School and Society, vol. 7 (Jan. 12, 1918), pp. 54-55.

Upholds the methods of investigation and conclusions drawn by the Committee of the Chicago Mathematics Club and attacks the psychology, pedagogy and scientific spirit of certain educational theorists as shown by Dr. Snedden in School and Society, vol. 6, p. 651.

525. ———— On correlation and disciplinary values. School and Society, vol. 2 (Sept. 11, 1915), pp. 378-385.

Considers the significance of the literature on correlation between abilities in different school subjects. Final conclusions are not yet justified, but that such work as has been done furnishes considerable support for the contentions in favor of disciplinary values.

526. ———— On the disciplinary and applied values of mathematical study. Education, vol. 39 (Dec., 1918), pp. 209-216.

Gives results of a careful examination of the literature of the controversy on formal discipline. Takes a position advocating the high value of the study of mathematics as training in deductive reasoning, and in inculcating scientific ideals and respect for truth.

527. ———— The disciplinary value of the study of mathematics. School and Society, vol. 7 (March 9, 1918), pp. 290-294.

Replies to the arguments advanced in certain preceding papers.

528. **Moore, E. C.** Does the study of mathematics train the mind specifically or universally? School and Society, vol. 6 (Oct., 27, 1917), pp. 481-491.

Argues that universal training is negligible in amount and that "we must build upon the solid rock of specific education".

529. ———— Does the study of mathematics train the mind specifically or universally? Mathematics Teacher, vol. 10 (Sept., 1917). pp. 1-18.

Education is a conscious process and a purposive undertaking. Three reasons verify the need of study; namely, (1) because we reverence certain subjects; (2) because we can not get along without them; (3) to perfect or improve the mind.

530. ———— Does the study of mathematics train the mind specifically or universally? A reply to a reply. School and Society, vol. 7 (June 29, 1918), pp. 754-764.

An extended refutation of an article by R. E. Moritz (School and Society, vol. 6, p. 481).

531. ———— Mathematics and formal discipline—again. School and Society, vol. 7 (Feb. 2, 1918), pp. 137-140.

Discusses certain preceding papers and letters. Concludes: "the contention that particular studies have special and greatly to be preferred formal disciplinary effects has not been proven".

532. **Moritz, R. E.** Does the study of mathematics train the mind specifically or universally? A reply. School and Society, vol. 7 (April 27, 1918), pp. 485-492.

A refutation of the arguments made by E. C. Moore in an article with the same title (School and Society, vol. 6, p. 481).

533. ———— Mathematics as a test of mental efficiency. School and Society, vol. 7 (Jan. 12, 1918), pp. 59-60.

The opinions of certain psychologists and critics in regard to specific discipline, mental transfer and intrinsic value of mathematics to the contrary notwithstanding, facts prove: (1) "Efficiency in mathematics is a concomitant of strong intellects; conversely inability to master mathematics is indicative of low general mentality". (2) "No other study takes the place of mathematics since it has to do with time, space, matter and motion which are the ultimate terms in our three-dimensional world".

534. ———— The disciplinary value of mathematics. School and Society, vol. 7 (June 22, 1918), pp. 739-740.

A letter in reply to P. J. Krause article "A strategic retreat". Maintains his position and claims that casual relation and concomitancy are not only compatible but that the former concept implies the latter.

535. **Rogers, Agnes L.** Established results of the new psychology as it bears upon the teaching of mathematics. Mathematics Teacher, vol. 9 (Dec., 1916), pp. 85-93.

The new psychology is based upon experiment and measurement; the old phychology was based upon introspection and observation. Little has been done in algebra and geometry except by Thorndike, Monroe, and Rugg. Outcome to date: (1) High correlation exists between efficiency in mathematics and efficiency in other subjects. (2) There is transfer — the amount varies with the degree of difference between the functions in question. (3) Suggestions for transfer: proper attitudes obtained first, focus attention upon the art of learning and upon the methods of procedure; direct attention upon related ideas; keep attention at high level.

536. ——————— The bearing of the new psychology upon the teaching of mathematics. Teachers College Record, vol. 17 (Sept., 1916), pp. 344-352.

A discussion of the experimental work that has been done in recent years to determine the relation of mathematics ability to other ability and to determine the amount of transfer of learning, concluding with the four general factors that the author found in such transfer. These are attitudes, ideals, concepts of method, and a high level of attention.

537. **Snedden, David.** Mathematics in secondary schools. School and Society, vol. 6 (Dec. 1, 1917), pp. 651-652.

Criticism of a preceding paper. What say the "life failures" who studied the same mathematics.

538. **Starch, Daniel.** Transfer of training in arithmetical operation. Journal of Educational Psychology, vol. 2 (1911), pp. 306-310.

Eight observers practiced for fourteen days on mental multiplication. Before and after the practice they were given six tests in arithmetical operations, and two in auditory memory span. For comparison, seven other observers were given the preliminary and final tests without the practice series. The practiced observers showed from twenty to forty percent more improvement in the arithmetical tests than the unpracticed observers. There was little change in memory span for either group.

539. **Young, J. W. A.** Concerning experiments to test the transfer of training. School Science and Mathematics, vol. 18 (Jan. and Feb., 1918), pp. 1-19; 130-138.

In the first paper is presented a brief sketch of the experiments listed in Rugg's Monograph (The experimental determination of mental discipline in school studies; Warwick and York, Baltimore, Maryland, 1916). The second paper is given up to the author's own theories on the disciplinary value of studies, especially of mathematics.

XIX. Historical Material

540. **Ball, W. W. R.** A school course in mathematics in the seventeenth century. Mathematical Gazette, vol. 5, Part I (Feb., 1910), No. 84, p. 202.

Discusses mathematical teaching at Christ's Hospital, London, in the seventeenth century. Used a book which contained sections on arithmetic, algebra, practical geometry, together with the substance of Euclid's Elements I-VI, XI, XII, plane and spherical trigonometry, cosmography, navigation, the sphere, astronomical and logical tables and geography.

541. **Barwell, Miss M. E.** The advisability of including some instruction in the school course on the history of mathematics. Mathematical Gazette, vol. 7 (March, 1913), No. 104, p. 72.

Students are interested in knowing how a subject grew. It is a gain if he can look upon mathematics as a living and growing thing rather than as a crystallized thing from a textbook. Such historical facts as those concerning the introduction of zero, imaginary roots, etc., are invaluable to a beginner.

542. **Carmichael, R. D.** A thread of mathematical history and some lessons. School Science and Mathematics, vol. 13 (1912), pp. 684-694.

The author treats the history of conic sections from the time of the Greeks, and from it draws lessons for teachers of mathematics of the present day.

543. **Kimmel, Herbert.** The status of mathematics and mathematical instruction during the colonial period. School and Society, vol. 9 (Feb. 15, 1919), pp. 195-202.

Arithmetic, only, was generally taught below the college. Probably only a small percent of pupils studied arithmetic. Twenty-seven books were published on arithmetic between 1662 and 1776, and used in the Colonies. A burden was placed upon memory; no attempt was made to stimulate the powers of reasoning and independence.

544. **Miller, G. A.** Historical notes in mathematical textbooks. School Science and Mathematics, vol. 19 (1919), pp. 414-416.

The main objects of the article are to emphasize the need of greater accuracy as regards the historical notes in the textbooks on elementary mathematics and to point out how improvements may easily be made.

545. ——————— History and use of mathematical text-books. School and Society, vol. 4 (Dec. 15, 1916), pp. 918-924.

A bad textbook is a great menace. Two ideals: (1) as a reference book; (2) as a guide and teacher. Should be accurate and clear, not necessarily detailed. Should contain historical notes to add human interest.

546. ——————— Modern developments in elementary and secondary mathematics. School Science and Mathematics, vol. 17 (1917), pp. 32-42.

The modern movements in elementary mathematics have been largely influenced by an emphasis on applications, and this may serve to explain the great progress made in the last few decades toward the earlier use of differential and integral calculus, and the introduction of the elementary of trigonometry into our high school courses in geometry.

547. ———— Recent changes of view as regards some points in the history of elementary mathematics. Educational Review, vol. 39 (April, 1910), pp. 403-406.

The article deals with changes of view due to recent discoveries in the history of mathematics. It is now believed we can ascribe the discovery of zero to the Babylonians instead of to the Hindus. The choice of 360 to measure the circle is explained as the merger of two systems of counting — one on the base ten and the other on the base six. Recent discoveries of a knowledge of mathematics higher than that given in the books of Ahmes leads to the conclusion that the book of Ahmes is the work of a student.

548. **Pahner, Emily G.** History of the graph in elementary algebra in the United States. See No. 311.

549. **Smith, D. E.** A glimpse at early colonial algebra. School and Society, vol. 7 (Jan. 5, 1918), pp. 8-11.

A short study of an algebra notebook made by a sixteen year old boy under the direction of a Harvard professor in 1789. "The little work has also a further suggestion to us in the disturbed condition of our present education, namely, that this boy of fifteen or sixteen — a high school youth as he would be today, was not bothering with hundreds of examples in factoring and complicated fractions, but was giving his energies to the study of those cubic and numerical higher equations needed in his work in what was known as natural philosophy. Might not a certain part of the objection to our algebra of the present be removed if we gave less attention to various topics of a purely mechanical nature and more to the real applications of the science?"

550. **Vivian, Roxana H.** Mathematics: A great inheritance. Educational Review, vol. 53 (Jan., 1917), pp. 30-43.

A résumé of the development of mathematics in history with special emphasis on its connection with the institutions of civilization. Goes on to show that for the individual of today a study of mathematics on the mechanical side gives a command of short cuts in computation and other symbolic devices; and on the spiritual side it forces him to develop a sense of proportion, a grasp of details and power to marshal them, constructive ability, a feeling for relations, the power to distinguish the large from the small, permanent from transitory values, essential qualities from superficial. Finally mathematics can help to put one's intellectual domain in order — to order one's experiences in various realms with due proportion, harmonize one's tastes, and organize one's ideals and principles into a well-related whole.

551. **Wilson, J. M.** On two fragments of geometrical treatises found in Worcester Cathedral Library. Mathematical Gazette, vol. 6 (March, 1911), No. 91, p. 19.

One fragment was of Gerbert's treatise of the tenth century, the second fragment, written in the fourteenth century, was really a copy of a work of the twelfth century, the work of Adelhard who translated the Arabic version of Euclid into Latin. In discussing these fragments the question was brought up as to whether the statement on parallels was to be regarded as an axiom or a postulate.

XX. Clubs and Recreations

552. **Boughn, E. T.** A mathematical contest. School Science and Mathematics, vol. 17 (1917), pp. 329-330.

The article gives the rules for a contest in manipulative skill between high schools.

553. **Brown, I. M.** A mathematics club in a girls school. Journal of Education, (Brit.), vol. 38 (1916), p. 556.

The fact that girls dislike mathematics may be accounted for by the fact that the useful or practical side of it is held too steadily before them. It was to counteract this that the club was formed. Its purpose was to bring human interest into the subject. Models of the solid figures were made and graphs of the rarer curves. A play based on Abbott's Flatland was given before the school. Mathematical puzzles and stories of great mathematicians were collected. Old cumbersome methods of multiplication and division were demonstrated. Logarithms and the slide rule were taken up. Suggests courses to be used for extra study; astronomy, mechanics, architecture, projectiles, and various others.

554. **Ferguson, Zoe** A fairy tale. School Science and Mathematics, vol. 17 (1917), pp. 152-153.

A little story which the author has found "useful in vitalizing high school mathematics".

555. **Gegenheimer, F. C.** Mathematics clubs. School Science and Mathematics, vol. 16 (1916), pp. 791-792.

An account of a mathematics club as answer to the question "what shall we do for the bright students?"

556. **Hatcher, Frances B.** A living theorem. School Science and Mathematics, vol. 16 (1916), pp. 39-40.

A mathematical recreation.

557. **Koch, E. H., Jr.** Mathematics contests. Mathematics Teacher, vol. 9, (June, 1917), pp. 179-186.

Describes the "mathematics relay" as it has been developed at the High School of Commerce in New York City.

558. ——————— and **McCormick, T. H.** Mathematics relays for high schools. School Science and Mathematics, vol. 16 (1916), pp. 530-536.

This article states that an interest in acquiring speed and accuracy in the arithmetic operations may be stimulated by forming relay teams of four each (or possibly five allowing an alternate). Each member of the team performs one of the four operations, each one using the result of the student who preceded him as a starting point in his calculations. This drill should prove very beneficial in the ninth grade work.

559. **Newhall, C. W.** "Recreations" in secondary mathematics. School Science and Mathematics, vol. 15 (1915), pp. 277-291.

Discusses the pedagogic value of recreations, and traces the historical development of mathematical recreations. Discusses methods of using recreations: (1) The mathematics club. (2) Informal meetings. (3) Assigned readings. (4) Debates. (5) In the classroom. Groups the possible material under three heads. (a) Recreations with numbers. (b) Recreations in elementary algebra. (c) Recreations in geometry. Gives rather detailed suggestions under each topic and selected bibliography.

560. **Reeve, W. D.** Exhibit of high school mathematics—its history and educational value. School and Society, vol. 2 (Aug. 7, 1915), pp. 191-197.

A brief account of the author's experience in organizing material for an exhibit at the University High School of the University of Chicago. The latter part of the paper deals with the relations of the library to the school.

561. **Rorer, J. T.** A new form of school contest. Educational Review, vol. 57 (April, 1919), pp. 339-345.

Describes a number of contests in mathematics suitable for group participation, as successfully used in the William Penn High School, Philadelphia, Pa.

562. **Schorling, Raleigh** A mathematical contest—its relation to the general problem of individual differences. School Science and Mathematics, vol. 15 (1916), pp. 794-797.

Describes a mathematical contest between first year classes in the Hyde Park School and the University High School, Chicago. The real significance of such a contest lies in the fact that it is an effective method of providing for individual differences. It is a powerful stimulus to the best students in the class.

563. **Shoesmith, Beulah I.** Mathematics clubs in secondary schools. School Science and Mathematics, vol. 16 (1916), pp. 106-113.

Clubs offer an opportunity to add to the pupils' fund of mathematical knowledge, to inspire pupils to do original work, and to lead them to realize that the subject is rich in interest.

564. **Smith, D. E.** and students. Number games and number rhymes. Teachers College Record, vol. 13 (Nov., 1912), pp. 385-495.

Seven papers, prepared independently of each other, having to do with various phases of the subject of recreations in mathematics.

565. **Snell, C. A.** Mathematics clubs in the high school. Mathematics Teacher, vol. 8 (Dec., 1915), pp. 73-78.

Mathematics clubs should be based upon scholarship, proficiency and interest in mathematics. Clubs stimulate further individual development, furnish an opportunity for expression of individual ideas and opinions and for applications of theory. The complete organization of a club is given as a suggestion for carrying out the plan in a high school.

566. **Webster, Louisa M.** Mathematics clubs. Mathematics Teacher, vol. 9 (June, 1917), pp. 203-208.

(1) Many points of a crowded curriculum must be treated lightly and the mathematics club provides time for such discussion and research. (2) Offers social opportunities. (3) Stimulates interest in mathematics. (4) Suggestions for topics for discussion are given.

567. A mathematical recreation. Mathematics Teacher, vol. 11 (June, 1919), pp. 177-181.

Several problems are given such as: (1) Can a right triangle be geometrically constructed, having one of its angles exactly one degree? (2) How near to an isosceles commensurable right triangle can a construction approach?

568. A mathematical victory. School Science and Mathematics, vol. 17 (1917), pp. 475-482.

A play in two acts written under the auspices of the Mathematics Club of the Los Angeles State Normal School of California contains suggestions of the possibility of working out a similar play for high school students.

569. Mock trail of B vs. A—solving a personal equation by the judicial process. School Science and Mathematics, vol 18 (Oct., 1918), pp. 611-621. A play.

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APPENDIX

List of Coöperating Organizations

The organizations marked with an asterisk (*) appointed special co-operating committees. Officers given are for the year 1921-1922.

National Organizations

The American Mathematical Society
President: G. A. Bliss, University of Chicago, Chicago, Ill.
Secretary: R. G. D. Richardson, Brown University, Providence, R. I.

The Mathematical Association of America
President: R. C. Archibald, Brown University, Providence, R. I.
Secretary: W. D. Cairns, Oberlin College, Oberlin, Ohio.

**Iowa Section*
Chairman: W. J. Rusk, Grinnell College, Grinnell, Iowa.
Secretary: John F. Reilly, University of Iowa, Iowa City, Iowa.

Kansas Section
Chairman: J. A. G. Shirk, Kansas Manual Training Normal School, Pittsburg, Kansas.
Secretary: E. B. Stouffer, University of Kansas, Lawrence, Kansas.

Kentucky Section
Chairman: C. H. Richardson, Georgetown College, Georgetown, Ky.
Secretary: Flora LeStourgeon, University of Kentucky, Lexington, Ky.

**Maryland, District of Columbia, Virginia Section*
Chairman: Oscar Adams, U. S. Coast & Geodetic Survey, Washington, D. C.
Secretary: G. R. Clements, U. S. Naval Academy, Annapolis, Md.

Minnesota Section
Chairman: Arvid Reuterdahl, College of St. Thomas, St. Paul, Minn.
Secretary: R. W. Barton, University of Minnesota, Minneapolis, Minn.

**Missouri Section*
Chairman: Louis Ingold, University of Missouri, Columbia, Mo.
Secretary: Paul R. Rider, Washington University, St. Louis, Mo.

Ohio Section
Chairman: B. F. Yanney, University of Wooster, Wooster, Ohio.
Secretary: G. N. Armstrong, Ohio Wesleyan University, Delaware, Ohio.

Rocky Mountain Section
Chairman: G. W. Finley, State Teachers College, Greeley, Colo.
Secretary: G. H. Light, University of Colorado, Boulder, Colo.

Texas Section
Chairman: H. J. Ettlinger, University of Texas Austin, Texas.
Secretary: J. L. Riley, Junior Agricultural & Mechanics College, Stephenville, Tex.

The National Council of Teachers of Mathematics
President: J. H. Minnick, University of Pennsylvania, Pa.
Secretary: J. A. Foberg, Department of Public Instruction, Harrisburg, Pa.

**Society for the Promotion of Engineering Education*
President: C. F. Scott, Yale University, New Haven, Conn.
Secretary: F. L. Bishop, University of Pittsburgh, Pittsburgh, Pa.

New England

**The Association of Teachers of Mathematics in New England*
President: Walter F. Downey, English High School, Boston, Mass.
Secretary: Harry D. Gaylord, Brown & Nichols School, Cambridge, Mass.
Connecticut Valley Section
Chairman: Percy F. Smith, Sheffield Scientific School, New Haven, Conn.
Secretary: Eleanor C. Doak, Mt. Holyoke College, South Hadley, Mass.

MAINE

Mathematical Association of Eastern Maine
President: Vivian Skinner, High School, Bangor, Me.
Secretary: Harvard L. Carter, Junior High School, Old Town, Me.

Maine Teachers Association, Mathematics Section
Chairman: Fannie Harlow Robinson, High School, Bangor, Me.

Maine State Committee on Courses in Mathematics
Chairman: L. E. Moulton, Principal, Edward Little High School, Auburn, Me.

MASSACHUSETTS

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